Connections on Lie algebroids and the Weil - Kostant theorem [*,†,‡]

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Abstract

The Weil-Kostant theorem characterises those alternating (real-valued) 2-forms which are curvature forms of connections in $U(1)$ - bundles. We present in this talk an overview of a corresponding result for arbitrary Lie groups which was proved by K.C.H.Mackenzie in 1987, using the notion of Lie algebroids. A Lie algebroid is a vector bundle whose module of sections has a Lie algebra bracket and a vector bundle morphism to the tangent space of the base manifold which preserves the Lie brackets. The main aims of this talk are two: First, to demonstrate that the theory of Lie algebroids is a suitable enviroment in which one can do connection theory (which is this speaker's main research interest), and second to show how one can use Lie algebroids to tackle non-abelian problems which often arise in the process of quantization and elsewhere.

1 Introduction

The "Weil-Kostant theorem" is a result proven by Weil [8] in 1958, and by Kostant [3] in 1970. It was also proven independently by Kobayashi (see [2]). In [3] we find it in the form of a characterisation of those alternating differential 2-forms with values in $\mathbb{R}$ which are the curvature of some connection in some principal bundle with structural group the circle. The integrality condition implies that the element $[\omega] \in H^2_{DR}(M, \mathbb{R})$ defined by such a differential form ω should lie in $\overset{\vee}{H}{}^2(M, \mathbb{Z})$.

The circle is a connected and abelian Lie group with Lie algebra $\mathbb{R}$, which is at the same time its universal cover, because the exponential map is a homomorphism of Lie groups. $\mathbb{Z}$ is nothing else than the kernel of the exponential map in this case.

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Every connected and abelian Lie group has its Lie algebra as universal cover, and it is possible to generalise the Weil-Kostant theorem in this case using the same technique as Kostant in [3].

Theorem 1.1 *Let G be a connected and abelian Lie group and ω a closed alternating differential 2-form which takes values in the Lie algebra $\mathfrak{g}$ of G. Then ω is the curvature of some connection in some principal bundle $P(M,G)$ if and only if $[\omega] \in \overset{\vee}{H}{}^2(M,\mathfrak{g})$ lies in $\overset{\vee}{H}{}^2(M,\ker(exp))$.*

This of course is not a very big step, as the Lie algebra of any connected and abelian Lie group is $\mathbb{R}^n$ for some non negative n, although it has not been treated explicitly before. In this talk we are going to discuss a generalisation of the Weil-Kostant theorem in the non-abelian case. This generalisation involves the theory of connections in Lie algebroids and is due to K.C.H.Mackenzie. The complete proof can be found in [4], and a survey on Lie algebroids can be found in [7]. We do not intend to give full details here. It is this speaker's view that a short summary of particular results is more useful, as one can go through the details in [4] and the references listed there.

2 Connection theory on Lie algebroids

Let us now concentrate on Lie algebroids. These are defined as:

Definition 2.1 *A Lie algebroid is a vector bundle $A \to M$ over a manifold M, such that its module of sections is equipped with a Lie bracket $[\ ,\ ] : \Gamma A \times \Gamma A \to \Gamma A$ which makes ΓA a Lie algebra over $\mathbb{R}$, and a vector bundle morphism $\alpha : A \to TM$ (anchor) such that:*

$$[X, f \cdot Y] = f \cdot [X,Y] + \alpha(X)(f) \cdot Y,$$

and

$$\alpha([X,Y]) = [\alpha(X), \alpha(Y)],$$

$\forall\ X,Y \ \in\ \Gamma A,\ f\ \in\ C^\infty(M)$.

Some examples of Lie algebroids are the following:

1. $A = \mathfrak{g},\ M = \{pt\}$ for any Lie algebra $\mathfrak{g}$
2. $A = TM,\ \alpha = id_{TM}$
3. Given a principal bundle $P(M,G,\pi)$ and regarding as $A := \frac{TP}{G}$ the space of orbits of the differentiable action of G on TP defined as $(X,g) \mapsto TR_g(X)$, then the module of sections $\Gamma(\frac{TP}{G})$ is isomorphic to the module of right-inveriant vector fields $\Gamma^G(TP)$. Therefore, $\Gamma(\frac{TP}{G})$ borrows the Lie bracket of $\Gamma^G(TP)$. So, we have the Lie algebroid

$$\pi_* : \frac{TP}{G} \to TP,$$

where the anchor is $\langle X \rangle \mapsto T\pi(X)$.

Since the projection π is a submersion, the anchor in this example is surjective, consequently we can write this Lie algebroid as an exact sequence:

$$\ker \pi_* \rightarrowtail \frac{TP}{G} \twoheadrightarrow TM.$$

It is a fact that $\ker \pi_*$ is isomorphic to $\frac{P\times \mathfrak{g}}{G}$, the orbit space of the action $((u,V),g) \mapsto (ug, Ad_{g^{-1}}(V))$. This is a Lie algebra bundle, namely a fibre bundle with fibre type the Lie algebra $\mathfrak{g}$. The Lie algebroid

$$\frac{P\times \mathfrak{g}}{G} \rightarrowtail \frac{TP}{G} \twoheadrightarrow TM$$

is known as the *Atiyah sequence.* In general, Lie algebroids with surjective anchors are called *transitive* and they are written in the form of an exact sequence, namely

$$L \rightarrowtail A \twoheadrightarrow TM,$$

where L is the kernel of the anchor map. This is proven to be a Lie algebra bundle in [4], and it is called the *adjoint bundle.*

Moreover, suppose given a connection, namely a differential 1-form $\omega : TP \to P \times \mathfrak{g}$ with the properties:

$$\omega \circ TR_g = Ad_{g^{-1}} \circ \omega$$

and

$$\omega(X^*) = X,$$

where X^* is the fundamental vector field induced by $X \in \mathfrak{g}$. Then the first of these two properties allows quotienting the connection to a map $\delta_\omega : \frac{TP}{G} \to \frac{P\times\mathfrak{g}}{G}$ and the second makes it a left-split of the Atiyah sequence. Therefore, δ_ω induces a right split of the Atiyah sequence,

$$\gamma_\omega : TM \to \frac{TP}{G}.$$

This is the induced connection of ω on the Atiyah sequence.

The curvature of ω is a differential 2-form Ω_ω which is horizontal and has the property:

$$\Omega_\omega \circ (TR_g \times TR_g) = Ad_{g^{-1}} \circ \Omega_\omega$$

This property allowes us to quotient the curvature to a map $\frac{TP\times TP}{G} \to P \times \mathfrak{g}$, which in turn, due to the horizontality of Ω_ω quotients to $R_\omega : TM \times TM \to \frac{P\times\mathfrak{g}}{G}$. It is proven ([4], A 4.19) that R_ω is given from the formula:

$$R_\omega(X,Y) = \gamma_\omega([X,Y]) - [\gamma_\omega(X), \gamma_\omega(Y)].$$

Connections on a general transitive Lie algebroid, as well as their curvature, are defined in the following way:

Definition 2.2 *Let $\alpha : A \to TM$ be a transitive Lie algebroid. A* connection *on A is a vector bundle morphism $\gamma : TM \to A$ over M such that $\alpha \circ \gamma = id_{TM}$. The* curvature *of γ is the alternating vector bundle morphism $R_\gamma : TM \oplus TM \to L$ which is defined as:*

$$j(R_\gamma(X,Y)) := \gamma([X,Y]) - [\gamma(X), \gamma(Y)],$$

$\forall\ X, Y\ \in\ \Gamma TM$, where $j : L \rightarrowtail A$ is the canonical embedding.

3 Integrability of transitive Lie algebroids

Mackenzie [4] proved that if the base M of a transitive Lie algebroid is contractible, then it has a flat connection. One can compare this with the classical result that every principal bundle with contractible base is trivializable. Now, for any transitive Lie algebroid $L \rightarrowtail A \twoheadrightarrow TM$, if we choose a simple open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M, then there are flat connections $\theta_i : TU_i \to A_{U_i}$. We can also choose charts $\{\psi_i : U_i \times \mathfrak{g} \to L_{Ui}\}_{i\in I}$ over $\mathcal{U}$, which are compatible with θ_i in the following sense:

$$[\theta_i(X), \psi_i(V)] = \psi_i(X(V)),$$

$\forall X \in TU_i, V : U_i \to \mathfrak{g}$.

Now, considering the differential forms

$$\chi_{ij} := \psi_i^{-1} \circ (\theta_i - \theta_j) : TU_{ij} \to U_{ij} \times \mathfrak{g}$$

then these are related to the transition functions $\{g_{ij} : U_{ij} \to Aut(\mathfrak{g})\}_{i,j\in I}$ of the adjoint bundle L in the following way:

$$d\chi_{ij} + [\chi_{ij}, \chi_{ij}] = 0\ (Maurer - Cartan)$$

$$\chi_{ik} = \chi_{ij} + \omega_{ij}(\chi_{jk})$$

$$\Delta(g_{ij}) = ad \circ \chi_{ij},$$

where Δ denotes the right invariant derivative, also known as Darboux derivative. The first of these equations shows that the χ_{ij} can be integrated to smooth functions $s_{ij} : U_{ij} \to G$ such that $\chi_{ij} = \Delta(s_{ij})$, where G is the simply connected Lie group that corresponds to the Lie algebra $\mathfrak{g}$. If the s_{ij} formed a cocycle, then they would define a principal bundle. So, we must look at the element $e_{ijk} := s_{jk} \cdot s_{ik}^{-1} \cdot s_{ij}$. All we have about this element is $\Delta(e_{ijk}) = 0$ which implies that the e_{ijk} are constant and $Ad \circ e_{ijk} = 1$ which implies that the e_{ijk} are central. So, the e_{ijk} define an element $[e] \in \check{H}^2(M, ZG)$. This element is the integrability obstruction. In [4] we can find the following theorem:

Theorem 3.1 *The Lie algebroid $\alpha : A \to TM$ over the simply connected manifold M is integrable to a principal bundle $P(M, G, \pi)$ if and only if $e \in \overset{\vee}{H}{}^2(M, D)$ for some discrete subgroup $D \subseteq ZG$.*

For example, Lie algebroids with $\mathfrak{su}(2)$ as the fibre type of the adjoint bundle, are integrable because the center of the corresponding Lie group is $\{I, -I\} \equiv \mathbb{Z}_2$. Moreover, if the adjoint bundle has fibre type $\mathfrak{su}(n)$ for arbitrary $n \in \mathbb{N}$ then the centre of its corresponding Lie group is $\mathbb{Z}_n$, therefore any Lie algebroid with fibre type of the adjoint bundle $\mathfrak{su}(n)$ is integrable too. Still moreover, any Lie algebroid whose adjoint bundle has fibre type with corresponding Lie group compact and semisimple is integrable.

Examples of non integrable Lie algebroids arise when $\mathbb{R}$ is the fibre type of their adjoint bundle as in the classical Weil-Kostant theorem. The corresponding Lie group is also $\mathbb{R}$ which has itself as centre.

4 Generalisation of the Weil-Kostant theorem

Now it is time to talk about the generalisation of the Weil-Kostant theorem. This will be done in the following sense: Consider an alternating differential 2-form $R : TM \oplus TM \to L$ which takes values in a Lie algebra bundle L with fiber type $\mathfrak{g}$. We must wonder first whether R is the curvature of some connection in a Lie algebroid, and second if this Lie algebroid can be integrated to a principal bundle.

As far as the second question is concerned, necessary and sufficient conditions have already been given in the previous section. Now, about the first, we must further wonder what $d\omega = 0$ means in this case. For example, take $L = M \times \mathbb{R}$ and an $\omega \in \Lambda(M, \mathbb{R})$ such that $d\omega = 0$. Then, we can form the Lie algebroid

$$M \times \mathbb{R} \rightarrowtail TM \oplus (M \times \mathbb{R}) \twoheadrightarrow TM$$

where the Lie bracket is defined by the formula:

$$[X \oplus f, Y \oplus g] := [X, Y] \oplus \{X(g) - Y(f) - \omega(X, Y)\}$$

for all $X, Y \in \Gamma A$, and $f, g \in C^\infty(M, \mathbb{R})$. This bracket satisfies indeed the Jacobi identity because ω is closed. Moreover, ω here is the curvature of the canonical connection $X \mapsto X \oplus 0$.

In the case of a general Lie algebra bundle L, in order to form the Lie algebroid $L \rightarrowtail TM \oplus L \twoheadrightarrow TM$ we need to replace the Lie derivative in the previous formula with the value of some connection on the adjoint bundle L. But we need to choose a connection which will be compatible with the Lie bracket of L. We must also bear in mind that L need not be abelian, therefore a term $[V, W]$ should be introduced in the above formula, which will now become:

$$[X \oplus V, Y \oplus W] := [X, Y] \oplus \{\nabla_X(W) - \nabla_Y(V) + [V, W] - \omega(X, Y)\},$$

for $X, Y \in \mathcal{X}(M)$ and $V, W \in \Gamma L$. For this bracket to satisfy the Jacobi identity, we need to suppose that $\nabla(\omega) = 0$. So, we have the following theorem:

Theorem 4.1 *Let M be a simply connected manifold and L a Lie algebra bundle over M. Suppose $R : TM \oplus TM \to L$ is an alternating 2-form. Then R is the curvature of a connection on a principal bundle $P(M, G, \pi)$ such that $L \equiv \frac{P \times \mathfrak{g}}{G}$, if and only if:*

1. *There is a connection ∇ on L such that:*

$$\nabla_X([V, W]) = [\nabla_X(V), W] + [V, \nabla_X(W)]$$

$$R_\nabla = ad \circ R$$

$$\nabla(R) = 0$$

   and

2. *The integrability obstruction $e \in \overset{\vee}{H}{}^2(M, Z\tilde{G})$ defined by the transitive Lie algebroid which corresponds to ∇ and R lies in $\overset{\vee}{H}{}^2(M, D)$ for some discrete subgroup D of the centre $Z\tilde{G}$ of the simply-connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$.*

The case in which the base manifold M is multiply connected has also been dealt with by Mackenzie. In [1], he and P.J. Higgins discuss pullback Lie algebroids, and in [5] he uses the notion of PBG-Lie algebroid to pull back Lie algebroids with multiply connected base to the corresponding Lie algebroid over the universal cover of the base, which is a PBG-Lie algebroid. In general, a PBG-Lie algebroid is a Lie algebroid whose base is the total space of a principal bundle whose structure group acts on the Lie algebroid by automorphisms. However, it is not the aim of this speaker to discuss these ideas in this talk.

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