CROSSED MODULES AND THE INTEGRABILITY OF LIE BRACKETS [*]

Iakovos Androulidakis
C.N.R.S. - Université Pierre et Marie Curie (Paris VI)
Institut de Mathématiques de Jussieu - UMR 7586
175, Rue du Chevaleret
F-75013 PARIS
France
androulidakis@math.jussieu.fr

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Abstract

We prove that in the transitive case the obstruction to the integrability of a Lie algebroid coincides with the lifting obstruction of a crossed module of Lie groupoids associated naturally with the given Lie algebroid. Then we investigate the generalisation of this result for extensions of transitive Lie algebroids, giving explicitly the corresponding lifting obstruction and classifying the lifts in case the obstruction vanishes.

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1 Introduction

Lie algebroids can be thought of as generalisations of the notion of the tangent bundle of a manifold. More formally, they are vector bundles whose module of sections is equipped with a Lie bracket, preserved by a certain morphism of vector bundles with values in the tangent bundle of the base, which is called the *anchor*. Their global counterparts are Lie groupoids, and roughly speaking these can be thought of as categories where every arrow is invertible, plus a suitable smooth structure. Formal definitions to both notions can be easily found in the literature, for example in [11]. They both have profound importance in geometry, as they provide the framework for the development of various areas, such as symplectic and Poisson geometry, quantization, foliation theory, and non commutative geometry.

Many of the results on the above areas of geometry stem from the relation between Lie algebroids and Lie groupoids. For example it is often possible to realise a singular foliation on a manifold as the image of the anchor map of a certain Lie algebroid (not always though!). This gives a description of the foliation on the infinitesimal level, and if the algebroid arises from a Lie groupoid, then it is also possible to realise the integral submanifolds of the foliation as the orbits of this groupoid. This method finds extensive use in the process of quantization; the cotangent bundle of a given Poisson manifold is an algebroid and it realises its symplectic foliation on the infinitesimal level. If it is integrable then the orbits of the induced groupoid are exactly the symplectic leaves.

The construction of one given the other resembles the corresponding constructions for Lie algebras and Lie groups, but in the "-oid" case it is not always possible to apply Lie's third theorem. The obstructions to the integrability of a general Lie algebroid were given in [7] by Crainic and Fernandes. In this paper we focus on the transitive case; that is those algebroids whose anchor map is surjective, therefore the foliation they define on the base manifold has only one leaf. This category includes important examples, such as the algebroid associated with a symplectic manifold and the Atiyah sequence of a principal bundle. Moreover, the restriction of any Lie algebroid to a leaf of the corresponding foliation is transitive.

The transtitive case was examined thoroughly by Mackenzie in [11]. The account given there brings to the foreground the fact that such "-oids" are in fact short exact sequences. More precisely, any transitive Lie algebroid A over a manifold M can be written as an extension of Lie algebroids

$$L \rightarrowtail A \xrightarrow{\rho} \twoheadrightarrow TM \tag{1}$$

where ρ is the anchor map with kernel the Lie algebra bundle L over M. The notion of transitivity for a groupoid Ω over M is exactly that there exists an arrow in Ω over any two given points in M. This implies that Ω can be written as an exact sequence of groupoids

$$F \rightarrowtail \Omega \xrightarrow{(t,s)} \twoheadrightarrow M \times M, \tag{2}$$

where t and s are the maps associating to every arrow in Ω its target and source in M respectively. Moreover, in [11] the integrability obstruction was given. For a transitive Lie algebroid A over a simply connected case, this is a certain element of $\check{H}^2(M, Z\tilde{G})$, where

$\tilde{G}$ is the connected and simply connected Lie group integrating the fiber of the Lie algebra bundle L.

On the other hand, extensions such as the above give rise to crossed modules. This notion was introduced by Whitehead in [15] in the context of groups, and was used in the classification of extensions of Lie groups to circumvent the problem that the inner automorphism group may not be closed in the group of full automorphisms. A small reference to this is given in the introduction to section 3 of this paper. Later, Brown et al ([6], [4], [5]) used crossed modules in the context of groupoids and pointed out their relation with double structures. Last, Mackenzie in [12] used crossed modules to show that it is possible to classify principal bundles by Čech cohomology with abelian coefficients, which is computable in all cases.

The main aim of this paper is to give a different interpretation to the integrability obstruction of a transitive Lie algebroid. Namely, we show that every transitive Lie algebroid (1) gives rise to the Lie algebroid version of the notion of a crossed module which always integrates to a crossed module of Lie groupoids. Then we show that the integrability obstruction is exactly the obstruction to this crossed module to arise from an extension (2). This approach to the meaning of the integrability obstruction can be traced to the integration of Lie algebras given by Van Est, expressing them as abelian extensions of Lie algebras. On the other hand, crossed modules seem to provide a suitable framework for the handling of non-abelian problems, and this exposition of the integrability obstruction aims to enhance this point of view.

After the account on the integrability obstruction, we extend these ideas to general extensions of transitive Lie algebroids. To this end, we use the notion of a PBG structure, introduced by Mackenzie in [10]. It was shown there that such extensions correspond to a single transitive Lie algebroid plus a certain group action. This calls for an equivariant version of our previous results. In particular, it suffices to give an equivariant form of the lifting of a crossed module described in [12]. Such a generalisation is not immediate, because the methods used there involve Čech cohomology cocycles, and an equivariant version of this particular cohomology does not exist in general. We show that the *isometablic* cohomology developed in [2] is suitable for this purpose.

This paper is structured as follows: Sections 2 to 4 are concerned with the relation of the integrability obstruction with the lifting obstruction of crossed modules. In section 2 we postulate the notion of a crossed module for Lie algebroids, and show that it is a generalisation of the notion of a coupling used in [11]. In section 3 we recall crossed modules of transitive Lie groupoids and show that they differentiate to the ones we defined on the algebroid level. We also investigate the nature of the lifting obstruction for crossed modules of Lie groupoids which arise from extensions of Lie algebroids. Section 4 identifies the integrability obstruction with the lifting obstruction of a certain crossed module of Lie groupoids arising naturally from the given Lie algebroid.

The remaining sections give the generalisation of Mackenzie's results in [12] to any extension of transitive Lie groupoids. Section 5 is an account of the PBG version of the crossed modules on the algebroid level and section 6 deals with the groupoid level. The full details of the differentiation and integration procedure (when possible) for crossed modules of PBG structures is also given there. In section 7 we give a classification for PBG-groupoids which is a reformulation of the one given in [2], and can be used for the purpose of this paper. This classification clearly differentiates to the classification of transitive Lie algebroids given by

Mackenzie in [11]. Finally, sections 8 and 9 prove the corresponding results of [12] for crossed modules of PBG-groupoids.

Acknowledgments

The results in this paper are developments of ideas from the author's PhD thesis [1] at the university of Sheffield, supervised by K. Mackenzie. In particular, the idea of relating the integrability obstruction of transitive Lie algebroids to the lifting obstruction of certain crossed modules was made known to the author by private communication with K. Mackenzie. For this, as well as for the overall support and inspiration in the course of the PhD studies and beyond, the author would like to express his gratitude to K. Mackenzie.

2 Crossed modules and couplings of Lie algebroids

We begin with the notion of a crossed module on the infinitesimal level in this section. It will be proven that it generalises the notion of coupling used by Mackenzie in [11].

Definition 2.1 *A* crossed module of Lie algebroids *over the manifold M is a quadruple (K,∂,A,ρ) where $K\longrightarrow M$ is a Lie algebra bundle, $A\longrightarrow M$ is a transitive Lie algebroid, $\partial\colon\ K\longrightarrow A$ is a morphism of Lie algebroids and $\rho\colon\ A\longrightarrow \mathrm{CDO}[K]$ is a representation of A in K such that:*

(i). $\rho(\partial(V))(W)=[V,W]$ *for all* $V,W\in\Gamma K$ *and*

(ii). $\partial(\rho(X)(V))=[X,\partial(V)]$ *for all* $X\in\Gamma A,\ V\in\Gamma K$.

Since ∂ is a morphism of Lie algebroids, we have $a\circ\partial=0$, where a is the anchor of A. So $\mathrm{im}(\partial)$ lies entirely in the adjoint bundle L of A. Regarding ∂ temporarily as a morphism of Lie algebra bundles, condition (ii) is equivariance with respect to ρ and the adjoint action of A on L. It is therefore of locally constant rank (see [11, IV 1.14]), and so has a kernel Lie algebra bundle which we denote $\ker\partial$. Condition (i) now ensures that $\ker\partial$ lies in ZK. Likewise, the quotient Lie algebroid $A/\mathrm{im}(\partial)$ exists and is a Lie algebroid over M (see [11, IV§1]). This is called the *cokernel* of the crossed module, and we usually denote it $\overline{A}$. All this is described in figure 1:

Notice that ρ induces a representation of $\overline{A}$ on the vector bundle $\ker\partial$, denoted $\rho^{\ker\partial}$, by

$$\rho^{\ker\partial}(\overline{X})(V)=\rho(X)(V).$$

This is well defined because if we consider $X,Y\in\Gamma A$ such that $\overline{X}=\overline{Y}\in\overline{A}$, then there is a $W\in\Gamma K$ such that $X=Y+\partial(W)$. So

$$\rho(X)(V)=\rho(Y)(V)+\rho(\partial(W))(V)==\rho(Y)(V)+[W,V]=\rho(Y)(V)$$

since $\ker\partial\subseteq ZK$.

$$\begin{tikzcd}
\ker\partial \arrow[dd, tail] & & \\
& \rho : A \arrow[r] & \mathrm{CDO}[K] \\
K \arrow[d, two heads, "\partial"'] & & \\
\mathrm{im}(\partial) \arrow[r, tail] & A \arrow[r, two heads, "\natural"] & A/\,\mathrm{im}(\partial) = \overline{A}
\end{tikzcd}$$

Figure 1:

Throughout the rest of the paper we will be working with crossed modules with fixed cokernel, as well as fixed K and $\ker\partial$. Two such crossed modules (K, ∂, A, ρ) and $(K, \partial', A', \rho')$ are *equivalent* if there is a morphism of Lie algebroids $\theta\colon\ A \longrightarrow A'$ such that $\theta \circ \partial = \partial$, $\natural \circ \theta = \natural'$ and $\rho' \circ \theta = \rho$. The 3–lemma then shows that every such morphism is an isomorphism of Lie algebroids. From now on, every time we mention a crossed module of Lie algebroids we will refer to its equivalence class and denote it by $\langle K, \partial, A, \rho\rangle$.

There are three special types of crossed modules particularly worth noting. First of all, crossed modules of Lie algebroids with trivial kernel. These are merely extensions of Lie algebroids $K \overset{\partial}{\rightarrowtail} A \overset{\natural}{\twoheadrightarrow} A/K$ where $K \subseteq L$ is an ideal of A, and the representation ρ of A in K is the restriction to K of the adjoint representation of A on L. Namely, $\partial(\rho(X)(V)) = \mathrm{ad}_X(\partial(V))$ for all $X \in \Gamma A$ and $V \in \Gamma K$. We will also frequently need crossed modules for which the kernel is exactly the centre of K and crossed modules for which the cokernel is the tangent bundle TM.

Definition 2.2 *A crossed module of Lie algebroids* $\langle K, \partial, A, \rho\rangle$ *over the manifold* M *is called*

(i). *a* coupling crossed module *if* $\ker\partial = ZK$*;*

(ii). *a* pair crossed module *if* $\operatorname{coker}\partial = TM$*.*

If both $\ker\partial = ZK$ *and* $\operatorname{coker}\partial = TM$*, then* $\langle K, \partial, A, \rho\rangle$ *is called a* coupling pair crossed module.

We usually regard a coupling crossed module as a structure on the cokernel; if $\operatorname{coker}(\partial) = \overline{A}$ we say that $\langle K, \partial, A, \rho\rangle$ is a *coupling crossed module of* $\overline{A}$ *with* K.

The notion of coupling crossed module is equivalent to the concept of coupling of Lie algebroids, introduced in [11, IV 3.2] as the Lie algebroid form of the notion of "abstract kernel" in the sense of MacLane [9]. It will take us to the end of the section to establish this.

Consider a Lie algebra bundle K over the manifold M. The adjoint bundle of $\mathrm{CDO}[K]$ is $\mathrm{Der}(K)$, the derivations of K, and $\mathrm{ad}(K) = \mathrm{im}(\mathrm{ad}\colon K \longrightarrow \mathrm{Der}(K))$ is a Lie subalgebra bundle of $\mathrm{Der}(K)$, and an ideal of $\mathrm{CDO}[K]$. We denote the quotient Lie algebroid $\mathrm{CDO}[K]/\,\mathrm{ad}(K)$ by $\mathrm{OutDO}[K]$, and call elements of $\Gamma\,\mathrm{OutDO}[K]$ *outer covariant differential operators* on K.

Definition 2.3 *A* coupling *of the Lie algebroid* $\overline{A}$ *with the Lie algebra bundle* K *(both over the same manifold* M*) is a morphism of Lie algebroids* $\Xi\colon \overline{A} \longrightarrow \mathrm{OutDO}[K]$.

Fix a coupling Ξ of the Lie algebroid $\overline{A}$ with the Lie algebra bundle K. Since the map $\natural\colon \mathrm{CDO}[K] \longrightarrow \mathrm{OutDO}[K]$ is a surjective submersion as a map of vector bundles over M, there is a vector bundle morphism $\nabla\colon \overline{A} \longrightarrow \mathrm{CDO}[K]$, $X \mapsto \nabla_X$, such that $\natural \circ \nabla = \Xi$. We call ∇ a *Lie derivation law covering* Ξ.

Let ∇ be any such Lie derivation law. Then for $X \in \Gamma\overline{A}$ the operator $\nabla_X\colon \Gamma K \longrightarrow \Gamma K$ restricts to $\Gamma ZK \longrightarrow \Gamma ZK$, for if $Z \in \Gamma ZK$ and $V \in \Gamma K$ then

$$[V, \nabla_X(Z)] = \nabla_X([V, Z]) - [\nabla_X(V), Z] = \nabla_X(0) - 0 = 0,$$

since Z is central. Further, the restriction is independent of the choice of ∇; write ρ^{Ξ} for the restriction of ∇_X to $\Gamma ZK \longrightarrow \Gamma ZK$. Then ρ^{Ξ} defines a vector bundle map $\overline{A} \longrightarrow \mathrm{CDO}(ZK)$ which is easily seen to be a Lie algebroid morphism; that is, ρ^{Ξ} is a representation of $\overline{A}$ on ZK, called the *central representation* of Ξ.

Now we can proceed to prove the equivalence of couplings in the sense of 2.3 with coupling crossed modules of Lie algebroids.

Consider first a coupling crossed module $\langle K, \partial, A, \rho\rangle$ of the Lie algebroid $\overline{A}$ with the Lie algebra bundle K. Condition (i) of 2.1 shows that ρ sends $\mathrm{im}(\partial)$ to $\mathrm{ad}(K) \subseteq \mathrm{CDO}[K]$ and ρ therefore descends to a morphism $\Xi^{\rho}\colon \overline{A} \longrightarrow \mathrm{OutDO}[K]$ as in the diagram

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d] & \mathrm{CDO}[K] \arrow[d] \\
\overline{A} \arrow[r, "\Xi^{\rho}"'] & \mathrm{OutDO}[K]
\end{tikzcd}$$

where the two vertical maps are the natural projections. This Ξ^{ρ} is the coupling corresponding to $\langle K, \partial, A, \rho\rangle$. Note that equivalent coupling crossed modules induce the same coupling. It is also easy to see that the representation of $\overline{A}$ on ZK induced by Ξ^{ρ} is equal to the representation induced directly from ρ as in the passage following 2.1.

For the construction of the coupling crossed module corresponding to a coupling we use the construction principle of [11, IV 3.20]. Take a coupling $\Xi\colon \overline{A} \longrightarrow \mathrm{OutDO}[K]$ of the Lie algebroid $\overline{A}$ with the Lie algebra bundle K. Choose a Lie derivation law $\nabla\colon \overline{A} \longrightarrow \mathrm{CDO}[K]$ covering Ξ. This is an anchor–preserving vector bundle morphism, and so its curvature is a well defined map

$$R_{\nabla}\colon \overline{A} \oplus \overline{A} \longrightarrow \mathrm{CDO}[K], \qquad X \oplus Y \longrightarrow \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

Since $\natural \circ \nabla = \Xi$ is a morphism of Lie algebroids, it follows that $\natural \circ R_\nabla = 0$ and so R_∇ takes values in $\mathrm{ad}(K) \subseteq \mathrm{Der}(K)$.

Define a map $\overline{\nabla}\colon\ \overline{A} \longrightarrow \mathrm{CDO}[\mathrm{ad}(K)]$ by

$$\overline{\nabla}_X(\mathrm{ad}_V) = \mathrm{ad}_{\nabla_X(V)}$$

for all $X \in \Gamma\overline{A}$ and $V \in \Gamma K$. This is also an anchor preserving morphism, so its curvature is a well defined map $R_{\overline{\nabla}}\colon\ \overline{A} \oplus \overline{A} \longrightarrow \mathrm{CDO}[\mathrm{ad}(K)]$. It is easily verified that $R_{\overline{\nabla}} = \mathrm{ad} \circ R_\nabla$ and $\overline{\nabla}(R_\nabla) = 0$. Moreover, the map $\natural \circ \overline{\nabla}\colon\ \overline{A} \longrightarrow \mathrm{OutDO}[\mathrm{ad}(K)]$ has zero curvature.

Now [11, IV 3.20] shows that the formula

$$[X \oplus \mathrm{ad}_V, Y \oplus \mathrm{ad}_W] = [X,Y] \oplus \{\mathrm{ad}_{\nabla_X(W)} - \mathrm{ad}_{\nabla_Y(V)} + \mathrm{ad}_{[V,W]} - R_\nabla(X,Y)\}$$

defines a Lie bracket on $\Gamma(\overline{A} \oplus \mathrm{ad}(K))$ which makes $\overline{A} \oplus \mathrm{ad}(K)$ a Lie algebroid over M. Denote $\overline{A} \oplus \mathrm{ad}(K)$ by A. Define $\partial\colon\ K \longrightarrow A$ and $\rho\colon\ A \longrightarrow \mathrm{CDO}[K]$ by

$$\partial(V) = 0 \oplus \mathrm{ad}_V, \qquad\qquad \rho(X \oplus \mathrm{ad}_V) = \nabla_X(W) + [V,W]$$

for all $X \in \Gamma\overline{A}$ and $V, W \in \Gamma K$. These are both morphisms of Lie algebroids and the remaining steps in the following proof are straightforward.

Theorem 2.4 *The Lie algebroid A just defined, together with ∂ and ρ, constitute a coupling crossed module for $\overline{A}$, which induces the given Ξ.*

A transitive Lie algebroid A over M is in fact an extension of the tangent bundle TM by its adjoint bundle L. From this point of view, an arbitrary exptension of Lie algebroids $K \overset{\iota}{\rightarrowtail} A \overset{\pi}{\twoheadrightarrow} \overline{A}$ over a manifold M gives rise to a crossed module of Lie algebroids once we choose an ideal I of ZK, the center of the Lie algebra bundle K. The construction is described in figure 2:

$$\begin{tikzcd}
I \arrow[r, equal] \arrow[d, tail] & I \arrow[d, tail] & \\
K \arrow[r, tail, "\iota"] \arrow[d, two heads, "\partial^I"'] & A \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\natural"'] & \overline{A} \arrow[d, equal] \\
\mathrm{im}(\partial^I) \arrow[r, tail, "\iota^\natural"] & A/\iota(I) \arrow[r, two heads, "\pi^\natural"] & \overline{A}
\end{tikzcd}$$

Figure 2:

Here ∂^I is the quotient map $K \to A/\iota(I)$ and the other maps are $\iota^\natural(\partial^I(V)) = \langle\iota(V)\rangle$ for all $V \in K$ and $\pi^\natural(\langle X\rangle) = \pi(X)$ for all $X \in A$. These maps are well defined because the sequence $K \overset{\iota}{\rightarrowtail} A \overset{\pi}{\twoheadrightarrow} \overline{A}$ is exact. Now the representation $\rho^I : A/\iota(I) \to CDO[K]$ defined by

$$\iota(\rho^I(\langle X\rangle)(V)) = [X, \iota(V)]$$

is well defined because we assumed I to be an ideal of ZK and it makes the quadruple $\langle K, \partial^I, A/\iota(I), \rho\rangle$ a crossed module of Lie algebroids. The question whether every crossed module arises from an extension of Lie algebroids gives rise to the notion of an operator extension.

Definition 2.5 *Let* xm $= \langle K, \partial, A, \rho\rangle$ *be a crossed module of Lie algebroids of*$\overline{A}$ *with* K. *An* operator extension *of* xm *is a pair* $(K \overset{\iota'}{\rightarrowtail} \hat{A} \overset{\pi}{\twoheadrightarrow} \bar{A}, \mu_*)$ *of an extension of Lie algebroids together with a morphism of Lie algebroids* $\mu : \hat{A} \to A$ *which is a surjective submersion such that:*

(i). *The following diagram commutes:*

$$\begin{tikzcd}
K \arrow[r, tail, "\iota'"] \arrow[d, tail, two heads, "\partial"'] & \hat{A} \arrow[r, two heads, "\pi"] \arrow[d, tail, two heads, "\mu_*"'] & \bar{A} \arrow[d, equal] \\
Im\partial \arrow[r, tail, "\iota"'] & A \arrow[r, two heads, "\natural"'] & \bar{A}
\end{tikzcd}$$

(ii). *For all* $\hat{X} \in \Gamma A$, $V \in \Gamma K$ *we have:*

$$\iota(\rho(\mu_*(\hat{X}))(V)) = [\hat{X}, \iota(V)]$$

Definition 2.6 *The operator extensions* $(K \overset{\iota'_1}{\rightarrowtail} \hat{A}_1 \overset{\pi_1}{\twoheadrightarrow} \bar{A}, \mu_*^1)$ *and* $(K \overset{\iota'_2}{\rightarrowtail} \hat{A}_2 \overset{\pi_2}{\twoheadrightarrow} \bar{A}, \mu_*^2)$ *of the crossed module of Lie algebroids* xm $= \langle K, \partial, A, \rho\rangle$ *are* equivalent *if there is a Lie algebroid morphism* $\kappa_* : \hat{A}_1 \to \hat{A}_2$ *such that* $\mu_*^2 \circ \psi = \mu_*^1$ *and the following diagram commutes:*

$$\begin{tikzcd}
K \arrow[r, tail, "\iota'_1"] \arrow[d, equal] & \hat{A}_1 \arrow[r, two heads, "\pi_1"] \arrow[d, "\kappa_*"'] & \bar{A} \arrow[d, equal] \\
K \arrow[r, tail, "\iota'_2"] & \hat{A}_2 \arrow[r, two heads, "\pi_2"] & \bar{A}
\end{tikzcd}$$

We denote $\mathcal{O}$pext(xm) *the set of equivalence classes of operator extensions of the crossed module* xm.

We briefly recall from [11, IV§3] the construction of the obstruction class associated to a coupling $\langle K, \partial, A, \rho\rangle$ over the manifold M. Choose a connection $\gamma : TM \to A$ of the Lie algebroid $\mathrm{im}(\partial) \rightarrowtail A \twoheadrightarrow TM$. Regard its curvature $R_\gamma : TM \oplus TM \to \mathrm{im}(\partial)$ and choose a lift $\Lambda : TM \oplus TM \to K$ such that $\partial \circ \Lambda = R_\gamma$. The representation ρ induces a representation $\rho^\partial : TM \to CDO[\ker \partial]$ defined by

$$\rho^\partial(X)(V) = \rho(\gamma(X))(V)$$

for all $X \in \Gamma TM$ and $V \in \Gamma \ker \partial$. This representation is well defined and does not depend on the choice of a connection for A. Consider the map $f : TM \oplus TM \oplus TM \to K$ defined by

$$f(X, Y, Z) = \bigodot \{\rho^\partial(X)(\Lambda(Y, Z)) - \Lambda([X, Y], Z)\}$$

where $\bigodot$ denotes the cyclic sum with repsect to the permutations of X, Y and Z. Notice that

$$\begin{aligned} \partial(f(X, Y, Z)) &= \bigodot \{\partial(\rho(\gamma(X))(\Lambda(Y, Z))) - R_\gamma([X, Y], Z)\} = \\ &= \bigodot \{[\gamma(X), \partial(\Lambda(Y, Z))] - R_\gamma([X, Y], Z)\} = \bigodot \{[\gamma(X), R_\gamma(Y, Z)] - R_\gamma([X, Y], Z)\}. \end{aligned}$$

The latter is zero because of the Bianchi identity, so the map f takes values in $\ker \partial = ZK$. Moreover f is easily seen to satisfy $df = 0$ where d denotes the exterior derivative of Lie algebroid cohomology. Therefore it defines an element $[f]$ in Lie algebroid cohomology $\mathcal{H}^3(TM, \rho^\partial, ZK)$. This element does not depend on the choice of γ and Λ. Moreover $[f] = 0$ if and only if $\nabla^\gamma(\Lambda) = 0$, where $\nabla^\gamma : TM \to CDO[K]$ is defined by the same formula as ρ^∂. This is no longer a representation though, unless the Lie algebra bundle K is abelian. It is what is called a *Lie derivation law* in [11, IV§3]. It follows from the construction principle [11, IV§3] that the formula

$$[X \oplus V, Y \oplus W] = [X, Y] \oplus \{\nabla^\gamma_X(W) - \nabla^\gamma_Y(V) - \Lambda(X, Y)\}$$

defines a Lie bracket on $\Gamma(TM \oplus K)$ which makes $K \rightarrowtail TM \oplus K \twoheadrightarrow TM$ a Lie algebroid over M and an operator extension for the coupling $\langle K, \partial, A, \rho\rangle$ under consideration. Conversely, it is easily verified that if the coupling has an operator extension then $[f] = 0$. The element $[f]$ is the obstruction associated with the coupling $\langle K, \partial, A, \rho\rangle$ and we denote it by $\mathrm{Obs}\langle K, \partial, A, \rho\rangle$. If $\mathrm{Obs}\langle K, \partial, A, \rho\rangle = 0$ then the equivalence classes of operator extensions for the coupling $\langle K, \partial, A, \rho\rangle$ are classified by $\mathcal{H}^2(TM, \rho^\partial, ZK)$ (see [11, IV§3] and [1, 4.7]).

3 Crossed modules of Lie groupoids

Crossed modules of Lie groups were introduced by Whitehead [15] in the context of homotopy theory. They were used later in the classification of extensions of Lie groups to overcome the problem that the group of inner automorphisms of a Lie group may not be closed in the full automorphism group. Namely, consider an extension of groups

$$A \overset{\iota}{\rightarrowtail} H \overset{\pi}{\twoheadrightarrow} G.$$

If A is abelian, the map $\rho : G \to Aut(A)$ defined by $\rho(g) = I_h\,|_A$, where h is any element of H such that $\pi(h) = g$, is a well defined representation. Now if A is non-abelian, the automorphism ρ is no longer well defined. The usual way around this problem is to consider the map $\rho : G \to Out(A) = \frac{Aut(A)}{Inn(A)}$, given by $g \mapsto \langle I_h \mid_A \rangle$, where $\pi(h) = g$. Here, $Inn(A)$ is the group of inner automorphisms of A. This is a well defined morphism, called the *abstract kernel* of the original extension, and there is a standard classification of such extensions with a prescribed abstract kernel.

If, however one is dealing with Lie groups, the previous approach is problematic, because $Inn(N)$ need not be closed in $Aut(N)$, and the smoothness of the representation $\rho : G \to Out(A)$ has no longer a meaning. An alternative approach, circumventing this problem, was given by Mackenzie in [12], using crossed modules of Lie groups.

Crossed modules of Lie groupoids were considered by Brown and Spencer [6], Brown and Higgins [4], and by Mackenzie in [12] to classify principal bundles with prescribed gauge group bundle. We recall here the construction of a crossed module of Lie groupoids from [12].

Definition 3.1 *Let Ω be a Lie groupoid on a base manifold M and let (F, π, M) be a Lie group bundle on M. A* representation *of Ω on F is a smooth map $\rho : \Omega * F \to F$, where $\Omega * F$ is the pullback manifold $\{(\xi, f) \in \Omega \times F : \alpha(\xi) = \pi(f)\}$, such that*

(i). $\pi(\rho(\xi, f)) = \beta(\xi)$ *for* $(\xi, f) \in \Omega * F$*;*

(ii). $\rho(\eta, \rho(\xi, f)) = \rho(\eta\xi, f)$ *for all* f, η, ξ *such that* $(\xi, f) \in \Omega * F$ *and* $(\eta, \xi) \in \Omega * \Omega$*;*

(iii). $\rho(1_{\pi(f)}, f) = f$ *for all* $f \in F$*;*

(iv). $\rho(\xi) : F_{\alpha(\xi)} \to F_{\beta(\xi)},\ f \mapsto \rho(\xi, f)$ *is a Lie group isomorphism for all* $\xi \in \Omega$*.*

Representations of groupoids on fibered manifolds were introduced by Ehresmann. One can also think of a groupoid representation as a Lie groupoid morphism $\Omega \to \Phi(F)$, where $\Phi(F)$ is Lie groupoid of isomorphisms between the fibers of the Lie group bundle F, otherwise known as the *frame groupoid* of F.

Definition 3.2 *A* crossed module *of Lie groupoids is a quadruple $xm = (F, \partial, \Omega, \rho)$, where $\Omega \rightrightarrows M$ is a Lie groupoid over M, F is a Lie group bundle on the same base, $\partial : F \to \Omega$ is a morphism of Lie groupoids over M, and where ρ is a representation of Ω on F, all such that*

(i). $\partial(\rho(\xi, f)) = \xi\partial(f)\xi^{-1}$ *for all* $(\xi, f) \in \Omega * F$*;*

(ii). $\rho(\partial(f), f') = ff'f^{-1}$ *for all* $f, f' \in F$ *with* $\pi(f) = \pi(f')$*;*

(iii). $Im(\partial)$ *is a closed embedded submanifold of* Ω*.*

The conditions of this definition show that $\operatorname{im}\partial$ lies entirely in $I\Omega$ and is normal in Ω. The normalcy of $\operatorname{im}\partial$ then ensures that it is a Lie group bundle. The quotient $\Omega/\operatorname{im}\partial$ therefore exists and is a Lie groupoid over M. This is called the *cokernel* of the crossed module and we usually denote it by $\overline{\Omega}$. On the other hand, condition (ii) ensures that $\ker\partial$ lies in ZF. All this is described in figure 3.

$$\begin{tikzcd}
\ker\partial \arrow[dd, tail] & & \\
 & \rho : \Omega * F \arrow[r] & F \\
F \arrow[d, two heads, "\partial"'] & & \\
Im(\partial) \arrow[r, tail] & \Omega \arrow[r, two heads, "\natural"] & \Omega/Im(\partial) = \overline{\Omega}
\end{tikzcd}$$

Figure 3:

Notice that ρ here also induces a representation of $\overline{\Omega}$ on $\ker\partial$, denoted by $\rho^{\ker\partial}$, by

$$\rho^{\ker\partial}(\overline{\xi}, f) = \rho(\xi, f).$$

This is well defined because if we consider $\xi, \eta \in \Omega$ such that $\overline{\xi} = \overline{\eta}$ then there is a $f' \in F$ such that $\eta = \xi \cdot \partial(f')$. So

$$\rho(\eta, f) = \rho(\xi, \rho(\partial(f'), f)) == \rho(\xi, f'f(f')^{-1}) = \rho(\xi, f)$$

since $\ker\partial \subseteq ZF$.

The definition of *equivalence* of crossed modules of Lie groupoids is identical in form to that of Section 1, and we denote an equivalence class of Lie groupoid crossed modules by $\langle F, \partial, \Omega, \rho\rangle$. Again, we regard a crossed module of Lie groupoids as a structure on the cokernel; if $\mathrm{coker}(\partial) = \overline{\Omega}$ we say that $\langle F, \partial, \Omega, \rho\rangle$ is a *crossed module of* $\overline{\Omega}$ *with* F.

There are three special types of crossed modules worth noting on the groupoid level also. First, crossed modules of Lie groupoids with trivial kernel. These are merely extensions of Lie groupoids $F \overset{\partial}{\rightarrowtail} \Omega \overset{\natural}{\twoheadrightarrow} \Omega/F$ where $F \subseteq I\Omega$ is a normal subbundle of $I\Omega$ and the representation ρ of Ω to F is the restriction of the inner representation of Ω on $I\Omega$. namely, $\partial(\rho(\xi, f)) = I_\xi(f)$ for all $(\xi, f) \in \Omega * F$. The other two types are crossed modules for which the center is exactly the centre of F and crossed modules for which the cokernel is the pair groupoid $M \times M$.

Definition 3.3 *A crossed module of Lie groupoids* $\langle F, \partial, \Omega, \rho\rangle$ *over the manifold* M *is called*

(i). *a* coupling crossed module *if* $\ker\partial = ZK$*;*

(ii). *a* pair crossed module *if* $\mathrm{coker}\,\partial = M \times M$.

If both $\ker\partial = ZK$ *and* $\mathrm{coker}\,\partial = M \times M$*, then* $\langle F, \partial, \Omega, \rho\rangle$ *is called a* coupling pair *crossed module.*

We describe briefly now the differentiation of Lie groupoid crossed modules to crossed modules of Lie algebroids. Given a crossed module of Lie groupoids $\langle F, \partial, \Omega, \rho\rangle$ over M it is well known that the Lie group bundle F differentiates to a Lie algebra bundle F_* over M, the Lie groupoid Ω to a Lie algebroid $A\Omega$ over M and the morphism ∂ to a morphism of Lie algebra bundles $\partial_* : F_* \to L\Omega \subseteq A\Omega$. The part that needs some attention is the differentiation of the representation ρ. For every $\xi \in \Omega$ the map $\rho(\xi) : F_{\alpha(\xi)} \to F_{\beta(\xi)}$ is a Lie group isomorphism. The Lie functor then shows that it differentiates to an isomorphism of Lie algebras $(\rho(\xi))_* : (F_{\alpha(\xi)})_* \to (F_{\beta(\xi)})_*$. Denoting $\Phi(F_*) \rightrightarrows M$ the Lie groupoid of isomorphisms between the fibers of the Lie algebra bundle F_* (otherwise known as the frame groupoid of F_*), we get a well defined morphism of Lie groupoids

$$\widetilde{\rho} : \Omega \to \Phi(F_*), \xi \mapsto (\rho(\xi))_*$$

Now apply the Lie functor to $\widetilde{\rho}$ to get the morphism of Lie algebroids $\rho_* : A\Omega \to CDO[F_*]$. This is the representation ρ differentiates to. It remains to show that it satisfies

(i). $\rho_*(\partial_*(V)) = \mathrm{ad}_V$ for all $V \in F_*$ and

(ii). $\partial_* \circ \rho_*(X) = \mathrm{ad}_X \circ \partial_*$ for all $X \in A\Omega$.

Using the definitions of ρ_* and ∂_* we have:

$$\rho_*(\partial_*(V)) = \rho_*(T_{e_x}\partial(V)) = (T_{1_x}\widetilde{\rho} \circ T_{e_x}\partial)(V) = T_{e_x}(\widetilde{\rho} \circ \partial)(V)$$

for any $V \in (F_*)_x$ and $x \in M$. On the other hand, for all $f \in F$ we have

$$(\widetilde{\rho} \circ \partial)(f) = T_{e_{\pi(f)}}(\rho(\partial(f))) = T_{e_{\pi(f)}}(I_f) = Ad_f$$

So,

$$\rho_*(\partial_*(V)) = T_{e_x}(\widetilde{\rho} \circ \partial)(V) = T_{e_x}Ad(V) = ad_V.$$

For (ii) we know that $\partial \circ \rho(\xi) = I_\xi \circ \partial$ for all $\xi \in \Omega$. Therefore,

$$T_{e_{\alpha(\xi)}}(\partial \circ \rho(\xi)) = T_{e_{\alpha(\xi)}}(I_\xi \circ \partial) = Ad_\xi \circ T_{e_{\alpha(\xi)}}\partial \Rightarrow T_{e_{\beta\xi}}\partial \circ \widetilde{\rho}(\xi) = Ad_\xi \circ T_{e_{\alpha(\xi)}}\partial.$$

By differentiating the last equality and using the fact that $T_{e_x}\partial$ is linear, therefore it is its own derivative, we get $\partial_* \circ \rho_*(X) = ad_X \circ \partial_*$. We have therefore proven the following proposition

Proposition 3.4 *If* $\mathrm{xm} = \langle F, \partial, \Omega, \rho\rangle$ *is a crossed module of Lie groupoids then* $\mathrm{xm}_* = \langle F_*, \partial_*, A\Omega, \rho_*\rangle$ *is a crossed module of Lie algebroids.*

Extensions of Lie groupoids give rise to crossed modules of Lie groupoids in a manner analogous to the one described in Section 1. Namely, given an extension of Lie groupoids $F \overset{\iota}{\rightarrowtail} \Omega \overset{\pi}{\twoheadrightarrow} \overline{\Omega}$ then the choice of a normal subbundle N of F which lies entirely in ZF allows the construction of the crossed module $\langle F, \partial, \Omega/N, \rho\rangle$ of $\overline{\Omega}$ with F, where ∂ is the projection $F \to F/N$ and $\rho : \Omega/N * F \to F$ is the representation defined by

$$\rho(\langle\xi\rangle, f) = \xi \cdot \iota(f) \cdot \xi^{-1}$$

Again, the question whether every crossed module of Lie groupoids arises in this manner from an extension leads to the notion of an operator extension.

Definition 3.5 *An* operator extension *of the crossed module of Lie groupoids* $(F, \partial, \Omega, \rho)$ *over the manifold* M *with cokernel* $\overline{\Omega}$ *is a pair* $(\widehat{\Omega}, \mu)$ *where* $\widehat{\Omega} \Longrightarrow M$ *is a Lie groupoid extension of* $\overline{\Omega}$ *by* F *and* $\mu : \widehat{\Omega} \to \Omega$ *is a morphism of Lie groupoids over* M *such that:*

(i). *The following diagram commutes:*

$$\begin{tikzcd}
F \arrow[r, tail, "\iota"] \arrow[d, two heads, "\partial"'] & \widehat{\Omega} \arrow[r, two heads] \arrow[d, two heads, "\mu"'] & \overline{\Omega} \arrow[d, equal] \\
\operatorname{im}\partial \arrow[r, tail] & \Omega \arrow[r, two heads] & \overline{\Omega}
\end{tikzcd}$$

(ii). $\iota(\rho(\mu(\widehat{\xi}), f)) = \widehat{\xi} \cdot \iota(f) \cdot \widehat{\xi}^{-1}$ *for all* $(\widehat{\xi}, f) \in \widehat{\Omega} * F$.

Definition 3.6 *Let* $(F, \partial, \Omega, \rho)$ *be a pair crossed module of Lie groupoids over the manifold* M. *Two operator extensions* $(\widehat{\Omega}, \mu)$ *and* $(\widehat{\Omega}', \mu')$ *are called* equivalent *if there is an isomorphism of Lie groupoids* $\kappa : \widehat{\Omega} \to \widehat{\Omega}'$ *such that* $\mu' \circ \kappa = \mu$.

Notice that if an operator extension exists (for an arbitrary crossed module of Lie groupoids) then it differentiates to an operator extension for the corresponding crossed module of Lie algebroids. The obstruction associated with a pair crossed module of Lie groupoids was given by the second author in [12]. We will show in Section 4 that in order to understand the obstruction associated with a general crossed module of Lie groupoids it suffices to understand the obstruction in the particular case of pair crossed modules. Let us recall now the construction of the obstruction from [12].

A pair crossed module of Lie groupoids $\langle F, \partial, \Omega, \rho\rangle$ is, as we discussed earlier, a crossed module of $M \times M$ with F. This means that the gauge group bundle of the Lie groupoid Ω is the image of ∂. The groupoid Ω can therefore be written as an extension of groupoids in the form

$$\operatorname{im}\partial \rightarrowtail \Omega \xrightarrow{(\beta,\alpha)} \!\!\!\!\rightarrow M \times M.$$

Fix an element $x_0 \in M$ and denote the Lie group F_{x_0} by H. Choose an open simple cover $\{U_i\}_{i \in I}$ of M. We write U_{ij} for the intersection of two open sets U_i and U_j, also U_{ijk} for the intersection of three open sets, etc. Let $\{s_{ij} : U_{ij} \to \partial(H)\}_{i,j \in I}$ be a cocycle of transition functions for the Lie groupoid $\Omega \Longrightarrow M$ and $\{\widehat{s}_{ij} : U_{ij} \to H\}_{i,j\in I}$ be smooth lifts of the transition functions to H, such that $s_{ij} = \partial \circ \widehat{s}_{ij}$. Now consider the failure of these lifts to form a cocycle

$$e_{ijk} : U_{ijk} \to H,\ e_{ijk} = \widehat{s}_{jk} \cdot \widehat{s}_{ik}^{-1} \cdot \widehat{s}_{ij}.$$

It follows from the fact that the s_{ij}'s form a cocycle that this function takes values in ZH. Therefore it defines a class $[e] \in \check{H}^2(M, ZH)$. This class depends neither on the choice of cocycle for Ω nor from the choice of lifts for this cocycle. Of course it is zero if and only if the lifts $\widehat{s}_{ij}$ form a cocycle and in this case they define a Lie groupoid $\widehat{\Omega} \Longrightarrow M$. It is proven in [12] that $\widehat{\Omega}$ is an operator extension for the crossed module. This element is called the *obstruction* of the crossed module and we denote it by $\operatorname{Obs}\langle F, \partial, \Omega, \rho\rangle$. In [12] it was also shown that if $\mathrm{Obs}\langle F, \partial, \Omega, \rho\rangle = 0$ then the equivalence classes of operator extensions of the crossed module $\langle F, \partial, \Omega, \rho\rangle$ are classified by $\check{H}^1(M, ZH)$.

Proposition 3.7 *If a pair crossed module of Lie groupoids differentiates to a coupling crossed module of Lie algebroids for which the obstruction class vanishes, then the obstruction class for the crossed module of Lie groupoids takes values in Čech cohomology with coefficients in constant functions.*

Proof. Consider a pair crossed module of Lie groupoids $\langle F, \partial, \Omega, \rho\rangle$. If the obstruction class of the coupling $\langle F_*, \partial_*, A\Omega, \rho_*\rangle$ vanishes then it has an operator extension, i.e. there exists a (transitive) Lie algebroid $\widehat{A}$ over M and a morphism of Lie algebroids $\mu_* : \widehat{A} \to A\Omega$ which is a surjective submersion such that the diagram in figure 4 commutes:

$$\begin{tikzcd}
F_* \arrow[r, tail, "\iota'"] \arrow[d, two heads, "\partial_*"'] & \widehat{A} \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\mu_*"'] & TM \arrow[d, equal] \\
\operatorname{im}\partial_* \arrow[r, tail, "\iota"'] & A\Omega \arrow[r, two heads, "\natural"'] & TM.
\end{tikzcd}$$

Figure 4:

Let H denote the fiber of the Lie group bundle $F \to M$ and $\mathfrak{h}$ its Lie algebra. Then the vertex groups of the Lie groupoid $\Omega \rightrightarrows M$ are isomorphic to $\partial(H)$. Take a simple open cover $\{U_i\}_{i\in I}$ of M. Choose a section atlas $\{s_{ij} : U_{ij} \to \partial(H)\}_{i,j\in I}$ for the Lie groupoid $\Omega \rightrightarrows M$ over this cover. Consider the family of Maurer-Cartan forms $\chi_{ij} : TU_{ij} \to U_{ij} \times \partial_*(\mathfrak{h})$ defined as $\chi_{ij} = \Delta(s_{ij})$ and the cocycle $\alpha_{ij} = \mathrm{Ad}_{s_{ij}}$ with values in $\mathrm{Aut}(H)$. Here Δ denotes the Darboux derivative, otherwise known as the right-derivative of functions with values in a Lie group. These define a system of transition data (χ, α) in the sense of [11, III§5]. This means that this pair is compatible in the sense

$$\Delta(\alpha_{ij}) = \mathrm{ad} \circ \chi_{ij} \tag{3}$$

and it satisfies the cocycle condition

$$\chi_{ik} = \chi_{ij} + \alpha_{ij}(\chi_{jk}) \tag{4}$$

for all $i, j, k \in I$. In fact, it is proven in [11, III§5] that systems of transition data classify transitive Lie algebroids. This and the commutativity of the diagram in figure 4 show that there exists a system of transition data $(\widehat{\chi}, \widehat{\alpha})$ with values in $\mathfrak{h}$ for the Lie algebroid $\widehat{A}$ such that $\partial_* \circ \widehat{\chi_{ij}} = \chi_{ij}$ and $\partial_* \circ \widehat{\alpha}_{ij} = \alpha_{ij}$. The $\widehat{\chi}_{ij}$'s are Maurer-Cartan forms, so they integrate uniquely to smooth functions $\widehat{s}_{ij} : U_{ij} \to H$ such that $\widehat{\chi}_{ij} = \Delta(\widehat{s}_{ij})$. Following the same steps as in [11, V§1] it is proven that the compatibility condition 3 for the system of transition data $(\widehat{\chi}, \widehat{\alpha})$ gives

$$\widehat{\alpha}_{ij} = \mathrm{Ad}_{\widehat{s}_{ij}}.$$

The system $(\widehat{\chi}, \widehat{\alpha})$ satisfies the cocycle condition 4. This gives

$$\Delta(\widehat{s}_{ik}) = \Delta(\widehat{s}_{ij}) + \mathrm{Ad}_{\widehat{s}_{ij}}(\Delta(\widehat{s}_{jk})) \Rightarrow \Delta(\widehat{s}_{ik}) = \Delta(\widehat{s}_{ij} \cdot \widehat{s}_{jk}).$$

A uniqueness argument now shows that there is a constant $c_{ijk} \in H$ such that $e_{ijk} = \widehat{s}_{jk} \cdot \widehat{s_{ik}}^{-1} \cdot \widehat{s}_{ij} = c_{ijk}$. In fact, c_{ijk} lies in the center of the Lie group H because e_{ijk} takes values exactly there. So, the obstruction class of the pair crossed module of Lie groupoids $\langle F, \partial, \Omega, \rho \rangle$ takes values in Čech cohomology with constant coefficients. ■

Regarding the integration of crossed modules of Lie algebroids to crossed modules of Lie groupoids, the following is true: A crossed module of Lie algebroids $\langle K, \partial, A, \rho \rangle$ integrates to a crossed module of Lie groupoids if the Lie algebroid A is integrable. A detailed account of this is given in Section 5.

Even when a crossed module integrates though, it does not follow that its operator extensions (if they exist) also integrate. It may well be the case that an integrable crossed module of Lie algebroids has an operator extension while the crossed module of Lie groupoids it integrates to does not. Examples of such a situation are the non-integrable transitive Lie algebroids. Every such algebroid A over a manifold M is a Lie algebroid extension $L \rightarrowtail A \twoheadrightarrow TM$. The crossed module induced by the choice of an ideal $I \subseteq ZL$ may be integrable, but the crossed module of Lie groupoids it would integrate to can not have an operator extension. If an operator extension on the groupoid level existed, then it would have to differentiate to $L \rightarrowtail A \twoheadrightarrow TM$ and in this case the Lie algebroid A would be integrable, which is a contradiction. We will study a specific example of this type in the next Section.

4 The integrability of (transitive) Lie algebroids via crossed modules

In this section we recall the integrability obstruction for a transitive Lie algebroid, originally given in [11]. We present it here in a quite different way, using crossed modules, which clarifies the nature of the obstruction. The approach presented here was sketched very briefly in [12].

Consider a transitive Lie algebroid $L \overset{\iota}{\rightarrowtail} A \overset{q_A}{\twoheadrightarrow} TM$. The method we described in Section 1 shows that the choice of an ideal I of L such that $I \subseteq ZL$ gives rise to a pair crossed module of Lie algebroids. In particular, if we choose $I = ZL$ then we obtain the coupling crossed module $\langle L, \natural, A/ZL, \rho_* \rangle$, where A/ZL is a Lie algebroid over M with Lie bracket

$$[X + ZL, Y + ZL]_{A/ZL} = [X, Y]_A + ZL$$

for all $X, Y \in \Gamma A$. Its anchor map is $q_{A/ZL}(X + ZL) = q_A(X)$, therefore it can be written as an extension of Lie algebroids in the form

$$L/ZL \rightarrowtail A/ZL \overset{q_{A/ZL}}{\twoheadrightarrow} TM.$$

The map $\natural : L \to A/ZL$ is of course the quotient projection and its image is the adjoint bundleL/ZL. Notice that the Lie algebroid A/ZL can be identified canonically with the

$$\begin{tikzcd}
ZL \arrow[r, equal] \arrow[d, tail] & ZL \arrow[d, tail] & \\
L \arrow[r, tail, "\iota"] \arrow[d, two heads, "\mathrm{ad}"'] & A \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\mathrm{ad}"'] & TM \arrow[d, equal] \\
\mathrm{ad}(L) \arrow[r, tail, "\iota^\natural"] & \mathrm{ad}(A) \arrow[r, two heads, "\pi^\natural"] & TM
\end{tikzcd}$$

$$\rho_* : \mathrm{ad}(A) \to \mathrm{CDO}[L]$$

Figure 5:

Lie subalgebroid $\mathrm{ad}(A)$ of $\mathrm{CDO}[L]$. This is the image of the adjoint representation ad : $A \to CDO[L]$. Namely, every $X + ZL \in \Gamma A/ZL$ defines the operator $\mathrm{ad}_X : \Gamma L \to \Gamma L$ by $\mathrm{ad}_X(V) = [X, \iota(V)]_A$ for every $V \in \Gamma L$. This element is well defined because if $Y \in \Gamma A$ is another representative of the class $X + ZL$ then there exists an element $W \in \Gamma ZL$ such that $Y = X + \iota(W)$. Then

$$\mathrm{ad}_Y(V) = [Y, \iota(V)]_A = [X, \iota(V)]_A + [\iota(W), \iota(V)]_A = [X, \iota(V)]_A = \mathrm{ad}_X(V),$$

since $W \in \Gamma ZL$. On the other hand, every element $\mathrm{ad}_X \in \mathrm{ad}(A)$ can be canonically identified with $X + ZL \in A/ZL$. A similar argument shows that L/ZL can be identified canonically with $\mathrm{ad}(L)$, the bundle of inner automorphisms of the fibers of L.

Now that we have established this identification, it is easier to regard the representation $\rho_* : A/ZL \to \mathrm{CDO}[L]$ as the natural inclusion of algebroids $\mathrm{ad}(A) \rightarrowtail \mathrm{CDO}[L]$ and the quotient map $\natural$ as $\mathrm{ad} : L \to \mathrm{ad}(L)$. All this is described in figure 5:

The coupling induced by A can now be written as $\langle L, \mathrm{ad}, \mathrm{ad}(A), \rho_* \rangle$. The Lie algebroid $\mathrm{CDO}[L]$ integrates to the frame groupoid $\Phi[L] \rightrightarrows M$ (see [11, III§4]). Therefore, $\mathrm{ad}(A)$ also integrates as a Lie subalgebroid of $\mathrm{CDO}[L]$. The Lie groupoid it integrates to is denoted by $\mathrm{Int}(A) \rightrightarrows M$. It is a Lie subgroupoid of the frame groupoid $\Phi(L)$ and it is called the *groupoid of inner automorphisms* of L. Let $F \to M$ be the Lie group bundle L integrates to. Then the Lie algebroid coupling crossed module $\langle L, \mathrm{ad}, \mathrm{ad}(A), \rho_* \rangle$ integrates to a coupling crossed module of Lie groupoids, namely $\langle F, I, \mathrm{Int}(A), \rho \rangle$. Here $I : F \to \mathrm{Int}(A)$ maps an element f which belongs to the fiber F_x to the inner automorphism I_f of the fiber F_x. The image of this map is the Lie group bundle $\mathrm{Inn}(F)$ of inner automorphisms of the fibers of F and it is immediate that it differentiates to $\mathrm{ad}(L)$. Finally, the representation $\rho : \mathrm{Int}(A) * F \to F$ is $\rho(\varphi, f) = \varphi(f)$ for all $(\varphi, f) \in \mathrm{Int}(A) * F$.

Proposition 4.1 *The obstruction to the integrability of the Lie algebroid A is* $\mathrm{Obs}\langle F, I, \mathrm{Int}(A), \rho \rangle$.

Proof. If $\mathrm{Obs}\langle F, I, \mathrm{Int}(A), \rho \rangle = 0$, then the coupling crossed module of Lie groupoids has operator extensions which differentiate to operator extensions of the coupling crossed module of Lie algebroids $\langle L, \mathrm{ad}, \mathrm{ad}(A), \rho_* \rangle$. Therefore there exists a Lie groupoid which integrates the Lie algebroid A. ■

The following result is an immediate consequence of the previous proposition and 3.7.

Corollary 4.2 *The integrability obstruction of a transitive Lie algebroid takes values in Čech cohomology with constant coefficients.*

This result clarifies the nature of the integrability obstruction for transitive Lie algebroids.

5 PBG–structures

In order to extend these results to general extensions, we need the concept of a PBG–algebroid.

Definition 5.1 *Let $P(M,G)$ be a principal bundle. A transitive Lie algebroid A over P is called a* PBG-algebroid *if the Lie group G acts on the manifold A so that for every $g \in G$, the diagram*

$$\begin{tikzcd}
A \arrow[r, "R_g"] \arrow[d] & A \arrow[d] \\
P \arrow[r, "R_g"] & P
\end{tikzcd}$$

is an automorphism of Lie algebroids.

A PBG-algebroid A over the principal bundle $P(M,G)$ is denoted by $A \Rightarrow P(M,G)$. If A and A' are PBG-algebroids over the same principal bundle $P(M,G)$, a morphism of PBG-algebroids is a map $\psi : A \to A'$ which is a morphism of Lie algebroids and satisfies $\psi(Xg) = \psi(X)g$ for all $X \in A$ and $g \in G$. A Lie algebra bundle $K \to P(M,G)$ is a PBG-Lie algebra bundle if, as a totally intransitive Lie algebroid, it is a PBG-algebroid. Given a PBG-Lie algebra bundle $K \to P(M,G)$, it is straightforward that $\mathrm{CDO}[K]$ is itself a PBG-algebroid. A *representation of PBG-algebroids* is a morphism of PBG-algebroids $\rho : A \to \mathrm{CDO}[K]$.

In [10] it is shown that PBG-algebroids correspond to extensions of integrable Lie algebroids by Lie algebra bundles. To give an outline of this correspondence, let us start by considering a PBG-algebroid A over the principal bundle $P(M,G)$. This can be written as an extension of Lie algebroids as $K \rightarrowtail A \twoheadrightarrow TP$, where K is a PBG-Lie algebra bundle over $P(M,G)$. It is shown in [2] that the quotient space $\frac{A}{G}$ is always a manifold, therefore the above extension gives rise to an extension of Lie algebroids $\frac{K}{G} \rightarrowtail \frac{A}{G} \twoheadrightarrow \frac{TP}{G}$ over M.

On the other hand, consider a transitive Lie groupoid $\Omega \rightrightarrows M$, a Lie algebra bundle K over M, and an extension of Lie algebroids $K \rightarrowtail A \twoheadrightarrow A\Omega$. Choose a basepoint in M and let $P(M,G)$ denote the principal bundle corresponding to the groupoid Ω. Then the pull-back of A over the bundle projection $P \to M$ is a PBG-algebroid over $P(M,G)$.

Note that the right-splittings of the extension $K \rightarrowtail A \twoheadrightarrow A\Omega$ correspond to those splittings of the pullback PBG-algebroid which are equivariant with respect to the group action.

Definition 5.2 *Let $A \Rightarrow P(M,G)$ be a PBG-algebroid. An* isometablic *connection of A is a vector bundle morphism $\gamma : TP \to A$ such that*

(i). *$q \circ \gamma = id_{TP}$, where q is the anchor of A;*

(ii). *$\gamma(Xg) = \gamma(X)g$ for all $X \in TP$ and $g \in G$.*

The respective notion for PBG-Lie algebra bundles is the following:

Definition 5.3 *Let $K \to P(M,G)$ be a PBG-Lie algebra bundle. An* isometablic *Koszul connection of K is a vector bundle morphism $\nabla : TP \to \mathrm{CDO}(K)$ such that*

$$\nabla_{Xg}(Vg) = [\nabla_X(V)]g$$

for all $X \in TP$, $V \in K$ and $g \in G$.

The purpose of this paper is to extend the crossed module approach described in the previous section, to the integrability problem of more general extensions of integrable (transitive) Lie algebroids by Lie algebra bundles. Since such extensions are equivalent with PBG-algebroids, we need to postulate a notion of crossed module which is compatible with the PBG structure.

Definition 5.4 *A* crossed module of PBG-algebroids *over the principal bundle $P(M,G)$ is a crossed module of Lie algebroids $\langle K, \partial, A, \rho \rangle$, where $K \to P(M,G)$ is a PBG-Lie algebra bundle, $A \Rightarrow P(M,G)$ is a PBG-algebroid, $\partial : K \to A$ is a morphism of PBG-algebroids and $\rho : A \to CDO[K]$ is a representation of PBG-algebroids.*

We will now give a brief outline of the correspondence of pair crossed modules of PBG-algebroids with general crossed modules of integrable Lie algebroids. To start with this, consider a pair crossed module of PBG-algebroids $\langle K, \partial, A, \rho \rangle$ over the principal bundle $P(M,G,p)$. Let $L \to P(M,G,p)$ be the kernel of the anchor of A, itself a PBG-Lie algebra bundle. It was shown in [2] that, because both K, L and A are PBG as Lie algebroids, the quotient manifolds $\frac{K}{G}$, $\frac{L}{G}$ and $\frac{A}{G}$ exist. Also, since the map $\partial : K \to L$ is equivariant, it quotients to a morphism of vector bundles $\partial^{/G} : \frac{K}{G} \to \frac{L}{G}$. This is a surjective morphism of Lie algebra bundles because ∂ itself is surjective morphism of Lie algebra bundles. The representation $\rho : A \to \mathrm{CDO}[K]$ induces a representation $\rho^{/G} : \frac{A}{G} \to \mathrm{CDO}[\frac{K}{G}]$, defined by

$$\rho^{/G}(\langle X \rangle)(\langle \mu \rangle) = \langle \rho(X)(\mu) \rangle$$

for all $X \in A$ and $\mu \in K$. The fact that $\partial^{/G}$ and $\rho^{/G}$ satisfy the properties of definition 2.1 follows from the fact that ∂ and ρ satisfy the respective properties for a pair crossed module of PBG-algebroids. Therefore we get the following crossed module of Lie algebroids:

$$\begin{tikzcd}
\ker\partial^{/G} \arrow[d, tail] & \rho^{/G}:\dfrac{A}{G}\longrightarrow \mathrm{CDO}[\dfrac{K}{G}] & \\
\dfrac{K}{G} \arrow[d, two heads, "\partial^{/G}"'] & & \\
\mathrm{im}(\partial^{/G}) \arrow[r, tail] & \dfrac{A}{G} \arrow[r, two heads, "\natural"] & \dfrac{TP}{G}
\end{tikzcd}$$

Conversely, suppose $\Omega \Longrightarrow M$ is a transitive Lie groupoid and $\langle K, \partial, A, \rho\rangle$ is a crossed module of $A\Omega$. Choose a basepoint in M and denote $P(M,G,p)$ the principal bundle corresponding to Ω. We will show that this crossed module induces a pair crossed module of PBG-algebroids over $P(M,G)$. First, take the extension of Lie algebroids $Im(\partial) \rightarrowtail A \overset{\pi}{\twoheadrightarrow} A\Omega$. It was proven in [10] that this extension induces a PBG-algebroid $A \lessdot p \Rightarrow P(M,G)$, where $A \lessdot p$ is the pullback vector bundle of A over $p : P \to M$. That is $A \lessdot p = \{(u,X) \in P \times A : X \in A_{p(u)}\}$. This Lie algebroid can be realised as an extension in the following way:

$$(Im(\partial)) \lessdot p \rightarrowtail A \lessdot p \overset{q^!}{\twoheadrightarrow} TP.$$

Recall that $\Gamma(A \lessdot p) \equiv C^\infty(P) \otimes \Gamma A$, where $f \otimes X$ corresponds to $f \cdot (X \circ p)$. The PBG-algebroid structure can be given both immediately and on the module of sections. Thus the anchor is $q^!(u,X) = TR_u(\pi \circ X)$ and on the sections, $q^!(f \otimes X)_u = f(u) \cdot TR_u(\pi(X_{p(u)}))$. The Lie bracket is

$$[f \otimes X, h \otimes Y] = (f \cdot h) \otimes [X,Y] + (f \cdot \pi(\vec{X})(h)) \otimes Y - (h \cdot \pi(\vec{Y})(f)) \otimes X,$$

where $\pi(\vec{X})_u = TR_u(\pi(X)_{1_{\beta(u)}})$ is the right-invariant vector field on P corresponding to $\pi(X) \in \Gamma A\Omega$. We denote the action of G on $A \lessdot p$ by $R^!_g$ for every $g \in G$. This is defined as $R^!_g(u,X) = (u \cdot g, X)$ or, on the sections by $R^!_g(f \otimes X) = (f \circ R_{g^{-1}}) \otimes X$.

The PBG-Lie algebra bundle $K \lessdot p$ is the pullback vector bundle of K over $p : P \to M$. Namely, $K \lessdot p = \{(u,V) \in P \times K : V \in K_{p(u)}\}$. Again, from the standard result for vector bundles we have $\Gamma(K \lessdot p) = C^\infty(P) \otimes_{C^\infty(M)} \Gamma K$ where $f \otimes V$ corresponds to $f \cdot (V \circ p)$.

The PBG-Lie algebra bundle structure of $K \lessdot p$ can also be given both immediately and on the sections. To show that $K \lessdot p$ is indeed a Lie algebra bundle, we have the following theorem:

Theorem 5.5 *Suppose $K \to M$ is a vector bundle with a Lie bracket $[,]^K : M \to Alt^2(K;K)$, P a manifold and $p : P \to K$ a smooth map. Also, suppose $\nabla^K : TM \to CDO[K]$ is a connection of K such that*

$$\nabla^K_X[V,W] = [\nabla^K_X(V), W]^K + [V, \nabla^K_X(W)]^K.$$

If $[,]^!P \to Alt^2(K \sphericalangle p, K \sphericalangle p)$ *is the Lie bracket on* $K \sphericalangle p$ *defined as*

$$[(u,V),(u,W)]^! = (u,[V,W]^K_{p(u)}),$$

then the map $\nabla^! : TP \to CDO[K \sphericalangle p]$ *defined by* $\nabla^!_Y(u,V) = (u, \nabla^K_{Tp(Y)}(V))$ *is a connection in* $K \sphericalangle p$ *and it satisfies*

$$\nabla^!_Y([(u,V),(u,W)]^!) = [\nabla^!_Y(u,V),(u,W)]^! + [(u,V),\nabla^!_Y(u,W)]^!$$

PROOF. It suffices to prove that $\nabla^!$ satisfies the last equality. This follows immediately from the respective equality for ∇^K. ■

Remark. The connection $\nabla^!$ of the previous theorem on the section-level is given by the formula:

$$\nabla^!_Y(f \otimes V) = f \otimes \nabla^K_{Tp(Y)}(V) + Tp(Y)(f) \otimes V.$$

On the level of sections, the expression for the Lie bracket of $K \sphericalangle p$ defined above is given by $[f \otimes V, h \otimes W]^! = f \cdot h \otimes [V,W]$.

We now give an isometablic version of [11, III§7, Thm 7.12]. It gives a criterion for the existence of a PBG-Lie algebra bundle structure on a vector bundle.

Theorem 5.6 *Let* K *be a vector bundle over* $P(M,G)$ *on which* G *acts by isomorphisms and* $[\ ,\ ]$ *a section of the vector bundle* $Alt^2(K;K)$. *Then the following three conditions are equivalent:*

(i). *The fibers of* K *are pairwise isomorphic as Lie algebras.*

(ii). K *admits an isometablic connection* ∇ *such that*

$$\nabla_X[V,W] = [\nabla_X(V),W] + [V,\nabla_X(W)]$$

for all $X \in \Gamma TP$ *and* $V,W \in \Gamma K$.

(iii). K *is a PBG-Lie algebra bundle.*

Corollary 5.7 *If* $K \to M$ *is a Lie algebra bundle and* $P(M,G,p)$ *a principal bundle then* $K \sphericalangle p$ *is a PBG-Lie algebra bundle.*

PROOF. Define the action of G on $K \sphericalangle p$ to be $R^!_g(u,V) = (u \cdot g, V)$. On the section-level, it will be $R^!_g(f \otimes V) = (f \circ R_{g^{-1}}) \otimes V$. We showed in 5.6 that a vector bundle is a PBG-Lie algebra bundle if and only if it has an isometablic Lie connection. If $K \to M$ has a Lie connection ∇^K then the connection $\nabla^!$ constructed in the previous theorem is also a Lie connection. It is moreover isometablic because:

$$\nabla^!_{TR_g(Y)}(u \cdot g, V) = (u \cdot g, \nabla^K_{Tp \circ TR_g(Y)}(V)) = (u \cdot g, \nabla^K_{T(p \circ R_g)}(V)) = (u \cdot g, \nabla^K_{Tp(Y)(V)}).$$

■

The next step is to define the morphism of PBG-algebroids $\partial^! : K \lessdot p \to A \lessdot p$. This is defined by $\partial^!(u, V) = (u, \partial(V))$, or, on the section-level by $\partial^!(f \otimes V) = f \otimes \partial(V)$. It is a straightforward calculation to show that it is a morphism of PBG-algebroids.

Finally we need to define a representation $\rho^! : A \lessdot p \to CDO[K \lessdot p]$ and show that it satisfies the properties of definition 4.1.1. This is defined as $\rho^!(u, X)(u, V) = (u, \rho(X)(V))$, or, on the section-level as

$$\rho^!(f \otimes X)(h \otimes V) = (f \cdot h) \otimes \rho(X)(V) + (f \cdot \pi(\vec{X})(h)) \otimes V.$$

Again, the proof that $\partial^!$ and $\rho^!$ satisfy the necessary properties which make $\langle K \lessdot p, \partial^!, A \lessdot p, \rho^! \rangle$ a pair crossed module of PBG-algebroids is a straightforward calculation. These considerations can be formulated to the following result:

Theorem 5.8 *Pair crossed modules of PBG-algebroids are equivalent to crossed modules of integrable Lie algebroids.*

Therefore, it suffices to work with pair crossed modules of PBG-algebroids. As far as their operator extensions are concerned, these are pairs (A, μ) where $A \Rightarrow P(M, G)$ is a PBG-algebroid and μ is a morphism of PBG-algebroids. Following the same process as the one described in section 1, and working with isometablic connections instead, the obstruction to the existence of an operator extensions is an element of equivariant Lie algebroid cohomology $\mathcal{H}^3_G$ and if it vanishes the operator extensions are classified by $\mathcal{H}^2_G$.

6 Crossed modules of PBG-groupoids

Every extension $K \rightarrowtail A \twoheadrightarrow A\Omega$ of transitive Lie algebroids over M (where $A\Omega$ integrates to the Lie groupoid $\Omega \rightrightarrows M$) gives rise to a Lie algebroid crossed module of $A\Omega$, and in the previous section we showed that such crossed modules correspond to pair crossed modules of PBG-algebroids. Therefore, bearing in mind the ideas explained in section 3, the obstruction to the integrability of a general extension $K \rightarrowtail A \twoheadrightarrow A\Omega$ must coincide with the obstruction associated with a certain crossed module on the groupoid level, which involves a PBG structure as well.

In this section we give the definition of this particular crossed module and its operator extensions, and show that such crossed modules correspond to pair crossed modules of PBG-algebroids via the processes of differentiation and integration. Let us start with a brief account on the prerequisites of PBG structures on the groupoid level.

Definition 6.1 *A* PBG-groupoid *is a Lie groupoid $\Upsilon \rightrightarrows P$ whose base is the total space of a principal bundle $P(M, G)$ together with a right action of G on the manifold Υ such that for all $(\xi, \eta) \in \Upsilon \times \Upsilon$ such that $s\xi = t\eta$ and $g \in G$ we have:*

(i). $t(\xi \cdot g) = t(\xi) \cdot g$ *and* $s(\xi \cdot g) = s(\xi) \cdot g$

(ii). $1_{u \cdot g} = 1_u \cdot g$

(iii). $(\xi\eta)\cdot g = (\xi\cdot g)(\eta\cdot g)$

(iv). $(\xi\cdot g)^{-1} = \xi^{-1}\cdot g$

The properties of a PBG-groupoid imply that the right translation on Υ is a Lie groupoid automorphism over the right translation of the principal bundle. A morphism φ of Lie groupoids between two PBG-groupoids Υ and Υ' over the same principal bundle is called a morphism of PBG-groupoids if it preserves the group actions. Namely, if $\varphi\circ\widetilde{R}_g = \widetilde{R}'_g\circ\varphi$ for all $g\in G$. In the same fashion, a PBG-Lie group bundle (PBG-LGB) is a Lie group bundle F over the total space P of a principal bundle $P(M,G)$ such that the group G acts on F by Lie group bundle automorphisms. We denote a PBG-LGB by $F\to P(M,G)$. It is easy to see that the gauge group bundle $I\Upsilon$ of a PBG-groupoid $\Upsilon \Longrightarrow P(M,G)$ is a PBG-LGB. It is straightforward that PBG-groupoids differentiate to PBG-algebroids.

The class of transitive PBG-groupoids is of interest here, and that is because these groupoids are equivalent to extensions of transitive Lie groupoids. Namely, given a transitive PBG-groupoid $\Upsilon \Longrightarrow P(M,G)$, its corresponding extension $I\Upsilon \rightarrowtail \Upsilon \twoheadrightarrow P\times P$ can be quotiened by G (see [10]) to give rise to the extension of transitive Lie groupoids over M.

$$\frac{I\Upsilon}{G} \rightarrowtail \frac{\Upsilon}{G} \twoheadrightarrow \frac{P\times P}{G}.$$

On the other hand, given an extension of transitive Lie groupoids $F \rightarrowtail \Omega \twoheadrightarrow \Phi$ over M, choose a basepoint and consider the corresponding extension of principal bundles $N \rightarrowtail Q(M,H) \twoheadrightarrow P(M,G)$. This gives rise to the principal bundle $Q(P,N)$, and in turn this forms the Lie groupoid $\Upsilon = \frac{Q\times Q}{N} \Longrightarrow P$. Now the Lie group G acts on Υ by

$$\langle q_2, q_1\rangle g = \langle q_2 h, q_1 h\rangle,$$

where h is any element of H which projects to g. A detailed account of these constructions can be found in [10], as well as the proof that they are mutually inverse.

Definition 6.2 *A* crossed module of PBG-groupoids *is a quadruple* (F,∂,Ω,ρ)*, where* $\Omega \Longrightarrow P(M,G)$ *is a PBG-groupoid,* $\pi : F\to P(M,G)$ *is a PBG-Lie group bundle,* $\partial : F\to\Omega$ *is a morphism of PBG-groupoids over* $P(M,G)$ *and* ρ *is a representation of* Ω *on* F*, all such that*

(i). $\rho(\xi g, fg) = \rho(\xi,f)g$ *for all* $(\xi,f)\in\Omega * F$ *and* $g\in G$*;*

(ii). $\partial(\rho(\xi,f)) = \xi\partial(f)\xi^{-1}$ *for all* $(\xi,f)\in\Omega * F$*;*

(iii). $\rho(\partial(f),f') = ff'f^{-1}$ *for all* $f,f'\in F$ *with* $\pi(f)=\pi(f')$*;*

(iv). $Im(\partial)$ *is a closed embedded submanifold of* Ω*.*

In the same fashion, $Im(\partial)$ is a PBG-Lie group bundle which lies entirely in $I\Omega$ and is normal in Ω, and the cokernel $\frac{\Omega}{Im(\partial)}$ is a PBG-groupoid over $P(M,G)$. If the cokernel of a crossed module of PBG-groupoids is the pair groupoid $P\times P$, then the crossed module is called *pair*. If, moreover, $\ker\partial = ZF$, then it is called a coupling. In the remaining of this paper we will be concerned only with pair crossed modules of PBG-groupoids.

Definition 6.3 *An* operator extension *of a* pair *crossed module of PBG-groupoids* $\langle F, \partial, \Omega, \rho\rangle$ *over the principal bundle* $P(M,G)$ *is a pair* (Φ, μ) *such that* Φ *is a PBG-groupoid over* $P(M,G)$, $\mu : \Phi \to \Omega$ *is a morphism of PBG-groupoids, and the pair is an operator extension in the sense of 3.5.*

Differentiation

Now consider a pair crossed module of PBG-groupoids $pxm = (F, \partial, \Omega, \rho)$ over the principal bundle $P(M,G)$. From definition 6.2 we then have:

(i). $\rho(\xi g, fg) = \rho(\xi, f)g$ for all $(\xi, f) \in \Omega * F$ and $g \in G$;

(ii). $\partial(\rho(\xi, f)) = \xi \cdot \partial(f) \cdot \xi^{-1}$ for all $(\xi, f) \in \Omega * F$;

(iii). $\rho(\partial(f), f') = ff'f^{-1}$ for all $f, f' \in F$ with $\pi(f) = \pi(f')$.

In order to differentiate pxm to a pair crossed module of PBG-algebroids, consider the PBG-Lie algebra bundle $F_* \to P(M,G)$, the PBG-algebroid $A\Omega \Rightarrow P(M,G)$ and the morphism of PBG-Lie algebra bundles $\partial_* : F_* \to L\Omega$.

First of all we construct a representation $\rho_* : A\Omega \to CDO[F_*]$ which preserves the $G-$actions. Since ρ is an equivariant representation we have that $\rho(\xi) : F_{\alpha(\xi)} \to F_{\beta(\xi)}$ is a Lie group isomorphism for all $\xi \in \Omega$ such that for every $g \in G$ the isomorphism $\rho(\xi g) : F_{\alpha(\xi g)} \to F_{\beta(\xi g)}$ is equal to $\rho(\xi)g$. Applying the Lie functor we have that $(\rho(\xi g))_* = (\rho(\xi))_* g$ for all $g \in G$. Thus, we get a well defined morphism of PBG-groupoids

$$\tilde{\rho} : \Omega \to \Pi[F_*],\ \xi \mapsto (\rho(\xi))_*.$$

Denote $\rho_* : A\Omega \to CDO[F_*]$ the morphism of PBG-algebroids $\widetilde{\rho}$ differentiates to. This is the representation we are looking for.

Proposition 6.4 *Let* $pxm = (F, \partial, \Omega, \rho)$ *be a pair crossed module of PBG-groupoids over* $P(M,G)$. *Then,*

(i). $\rho_*(\partial_*(V))(W) = [V, W]$ *for all* $V, W \in F_*$ *and*

(ii). $\partial_*(\rho_*(X)(V)) = [X, \partial_*(V)]$ *for all* $X \in A\Omega$ *and* $V \in F_*$.

Proof. Using the definitions of ρ_* and ∂_* we have:

$$\rho_*(\partial_*(V)) = \rho_*(T_{e_u}\partial(V)) = (T_{1_u}\widetilde{\rho} \circ T_{e_u}\partial)(V) = T_{e_u}(\widetilde{\rho} \circ \partial)(V)$$

for any $V \in (F_*)_u$ and $u \in P$. However,

$$(\widetilde{\rho} \circ \partial)(f) = T_{e_{\pi(f)}}(\rho(\partial(f))) = T_{e_{\pi(f)}}(I_f) = Ad_f$$

for all $f \in F$. So,

$$\rho_*(\partial_*(V)) = T_{e_u}(\widetilde{\rho} \circ \partial)(V) = T_{e_u}Ad(V) = ad_V.$$

For the second equality, we know that $\partial \circ \rho(\xi) = I_\xi \circ \partial$ for all $\xi \in \Omega$. Therefore,

$$T_{e_{\alpha(\xi)}}(\partial \circ \rho(\xi)) = T_{e_{\alpha(\xi)}}(I_\xi \circ \partial) = Ad_\xi \circ T_{e_{\alpha(\xi)}}\partial \Rightarrow T_{e_{\beta\xi}}\partial \circ \widetilde{\rho}(\xi) = Ad_\xi \circ T_{e_{\alpha(\xi)}}\partial.$$

By differentiating the last equality and using the fact that $T_{e_u}\partial$ is linear, therefore it is its own derivative, we get the desired equation, i.e. $\partial_* \circ \rho_*(X) = ad_X \circ \partial_*$. ■

We have therefore proven the following theorem:

Theorem 6.5 *If $pxm = (F, \partial, \Omega, \rho)$ is a pair crossed module of PBG-groupoids over the principal bundle $P(M,G)$ then $pxm_* = (F_*, \partial_*, A\Omega, \rho_*)$ is a pair crossed module of PBG-algebroids over the same principal bundle.*

Next, suppose that pxm has an operator extension $(F \overset{\iota}{\rightarrowtail} \Phi \overset{(\beta,\alpha)}{\twoheadrightarrow} P \times P, \mu)$. That is to say that $\mu : \Phi \to \Omega$ is a morphism of PBG-groupoids such that the diagram

$$\begin{tikzcd}
F \arrow[r, tail, "\iota"] \arrow[d, two heads, "\partial"'] & \Phi \arrow[r, two heads, "{(\beta,\alpha)}"] \arrow[d, two heads, "\mu"] & P \times P \arrow[d, equal] \\
I\Omega \arrow[r, tail] & \Omega \arrow[r, two heads] & P \times P
\end{tikzcd}$$

commutes and $(\iota \circ \rho \circ \mu)(\omega) = I_\omega \circ \iota$ for all $\omega \in \Phi$. It is immediate that the diagram

$$\begin{tikzcd}
F_* \arrow[r, tail, "\iota_*"] \arrow[d, two heads, "\partial_*"'] & A\Phi \arrow[r, two heads] \arrow[d, two heads, "\mu_*"] & TP \arrow[d, equal] \\
L\Omega \arrow[r, tail] & A\Omega \arrow[r, two heads] & TP
\end{tikzcd}$$

commutes, and, in the same fashion as with $\partial_*(\rho_*(X)(V)) = [X, \partial_*(V)]$, one can prove that

$$(\iota_* \circ \rho_* \circ \mu_*)(X')(V) = [X', \iota(V)]$$

for all $X' \in A\Phi$ and $V \in F_*$.

Integration

Suppose given a pair crossed module of PBG-algebroids $pxm_* = (K, \partial_*, A, \rho_*)$ over the principal bundle $P(M,G)$. The general theory ([11], [10]) induces that the PBG-Lie algebra bundle $K \Rightarrow P(M,G)$ integrates to a PBG-Lie group bundle $F \to P(M,G)$ with connected and simply connected fibers. This section proves that if the PBG-algebroid A integrates to a PBG-groupoid $\Omega \Longrightarrow P(M,G)$ which is α-connected and α-simply connected, then the pair crossed module pxm_* integrates to a pair crossed module of PBG-groupoids $pxm = (F, \partial, \Omega, \rho)$ over the principal bundle $P(M,G)$.

Since the PBG-Lie algebra bundle $F \to P(M,G)$ has connected and simply connected fibers, [13] shows that $\partial_* : F_* \to L\Omega$ integrates uniquely to a morphism of PBG-groupoids $\partial : F \to I\Omega$ which is onto. Thus, all we need to show in order to prove the integrability of pxm_* is that ρ_* integrates uniquely to an equivariant representation $\rho : \Omega \to \Pi(F)$ such that:

(i). $\rho(\partial(f)) = I_f$ for all $f \in F$ and

(ii). $(\partial \circ \rho)(\xi) = I_\xi \circ \partial$ for all $\xi \in \Omega$.

Let us start with the integration of a representation of PBG-algebroids. Consider a PBG-groupoid $\Omega \rightrightarrows P(M,G)$ which is $\alpha-$connected and $\alpha-$simply connected, a PBG-Lie group bundle $F \to P(M,G)$ with simply connected fibers and an equivariant representation $\rho_* : A\Omega \to CDO[F_*]$ of the PBG-algebroid $A\Omega$ on the PBG-Lie algebra bundle F_*. Since Ω is supposed to be $\alpha-$connected and $\alpha-$simply connected, [13] shows that this integrates uniquely to a morphism of PBG-groupoids $\overline{\rho} : \Omega \to \Pi(F_*)$ such that $\overline{\rho}_* = \rho_*$. For every $\xi \in \Omega$ we then have an isomorphism of Lie algebras

$$\overline{\rho}(\xi) : (F_*)_{\alpha(\xi)} \to (F_*)_{\beta(\xi)}.$$

Moreover, for all $g \in G$ and $\xi \in \Omega$ we have $\overline{\rho}(\xi g) = \overline{\rho}(\xi)g$ because ρ_* is equivariant. Therefore, from the general Lie theory for every $\xi \in \Omega$ there is a Lie group isomorphism $\rho(\xi) : F_{\alpha(\xi)} \to F_{\beta(\xi)}$ such that $(\rho(\xi))_* = \overline{\rho}(\xi)$. So we get a map $\rho : \Omega \to \Pi(F)$ sich that $\rho(\xi g) = \rho(\xi)g$ for all $g \in G$.

Proposition 6.6 *The map ρ is $C^\infty-$differentiable.*

PROOF. We work locally to prove this. Let $\{P_i\}_{i\in I}$ be an open cover of P. Then $\Omega_{P_i}^{P_i} \cong P_i \times \partial(H)P_i$ and $\Pi(F)_{P_i} \cong P_i \times Aut(H) \times P_i$, where H is the fiber type of F. Then ρ over P_i is the map

$$\rho(u, \partial(h), v) = (u, \theta(u) \circ f(\partial(h)) \circ \theta(v)^{-1}, v)$$

where $\theta : P_i \to Aut(H)$ is a map (not C^∞) and $f : H \to Aut(H)$ is a Lie group morphism. Also, $\Pi(F_*)_{P_i} \cong P_i \times Aut(\mathfrak{h}) \times P_i$ anf $\overline{\rho}$ over P_i becomes

$$\overline{\rho}(u, h, v) = (u, \overline{\theta}(u) \circ \overline{f}(h) \circ \overline{\theta}v^{-1}, v)$$

where $\overline{\theta} : P_i \to Aut(\mathfrak{h})$ is a $C^\infty-$map and $\overline{f} : H \to Aut(\mathfrak{h})$ is a Lie group morphism. We know that $(\rho(u,h,v))_* = \overline{\rho}(u,h,v)$, therefore the map $P_i \times H \times P_i \to Aut(\mathfrak{h})$ defined by $(u,h,v) \mapsto (\theta(u))_* \circ (f(h))_* \circ (\theta(v))_*^{-1}$ is smooth. It follows that the map $P_i \times H \times P_i \to Aut(H)$ defined by $(u,h,v) \mapsto \theta(u) \circ f(h) \circ \theta(v)^{-1}$ is smooth. That is because of a more general result which says that a linear first order system of equations, whose right-hand sides depend smoothly on auxiliary parameters, has solutions which depend smoothly on these parameters, providing that the initial conditions vary smoothly. Therefore, ρ is smooth. ■

Proposition 6.7 *The map ρ is a morphism of Lie groupoids.*

Proof. All we need to prove is $\rho(\eta\xi) = \rho(\eta) \circ \rho(\xi)$ for all $(\eta, \xi) \in \Omega * \Omega$. To this end, take $\eta, \xi \in \Omega$ such that $\alpha(\xi) = u$, $\beta(\xi) = v = \alpha(\eta)$ and $\beta(\eta) = w$. Now consider the Lie group automorphism

$$f_{\eta\xi} = \rho(\eta\xi) \circ (\rho(\xi))^{-1} \circ (\rho(\eta))^{-1} : F_w \to F_w.$$

We will show that $f_{\eta\xi} = id_{F_w}$. Indeed, if e_w is the identity element in F_w we have:

$$\begin{aligned} T_{e_w} F_{\eta\xi} = T_{e_w}(\rho(\eta\xi) \circ (\rho(\xi))^{-1} \circ (\rho(\eta))^{-1}) = \\ = T_{((\rho(\xi))^{-1} \circ (\rho(\eta))^{-1})(e_w)} \rho(\eta\xi) \circ T_{e_w}((\rho(\xi))^{-1} \circ (\rho(\eta))^{-1}) = \\ = T_{e_w}(\rho(\eta\xi)) \circ T_{(\rho(\eta))^{-1}(e_w)}(\rho(\xi))^{-1} \circ T_{e_w}(\rho(\eta))^{-1} = \\ = T_{e_w}(\rho(\eta\xi)) \circ [T_{e_u}(\rho(\xi))]^{-1} \circ [T_{e_v}(\rho(\eta))]^{-1} = \\ = \overline{\rho}(\eta\xi) \circ (\overline{\rho}(\xi))^{-1} \circ (\overline{\rho}(\eta))^{-1} = id_{(F_*)_w}. \end{aligned}$$

Since F has connected and simply connected fibers we get $f_{\eta\xi} = id_{F_w}$, thus ρ is indeed a morphism of Lie groupoids. ■

Now we can proceed to the integration of the pair crossed module. We need to prove that ρ and θ satisfy the identities mentioned in the beginning. To this end, we need to establish the PBG-Lie group bundle morphisms I and Ad and the PBG-Lie algebra bundle morphism ad.

Consider the PBG-Lie group bundle $F \to P(M, G)$ with fiber type H and let $\{\psi_i : P_i \times H \to F_{P_i}\}_{i \in I}$ be a section atlas of it. It is easily verified that the Lie group bundle $Aut(F) \to P(M, G)$ is a PBG-Lie group bundle and the family of maps $\{\psi_i^{Aut} : P_i \times Aut(H) \to Aut(F)_{P_i}\}_{i \in I}$ defined by

$$\psi_i^{Aut}(u, \varphi \in Aut(H)) = \psi_{i,u} \circ \varphi \circ \psi_{i,u}^{-1}$$

is a section atlas for this bundle.

Proposition 6.8 *The map* $I : F \to Aut(F)$ *defined by* $I_f(f') = f f' f^{-1}$ *for all* $f, f' \in F$ *such that* $\pi(f) = \pi(f')$ *is a PBG-Lie group bundle morphism. Locally it is of the form* $I^i : F_{P_i} \to (Aut(F))_{P_i}$ *where*

$$I^i(u, h) = (u, I_h^H)$$

for all $(u, h) \in P_i \times H \overset{\psi_i}{\cong} F_{P_i}$. *Here* I^H *is the inner automorphism of* H.

Proof. Immediate. ■

Next we consider the PBG-Lie group bundle $Aut(F_*) \to P(M, G)$. The section atlas of this bundle is $\{(\psi_i^{Aut})_* : P_i \times Aut(\mathfrak{h}) \to Aut(F_*)_{P_i}\}_{i \in I}$ defined by

$$(\psi_i^{Aut})_*(u, \varphi_* \in Aut(\mathfrak{h})) = (\psi_{i,u})_* \circ \varphi_* \circ (\psi_{i,x}^{-1})_* = T_e(\psi_i^{Aut}(u, \varphi))$$

for all $i \in I$, $u \in P_i$ and $\varphi \in Aut(H)$.

Proposition 6.9 *The map* $Ad : F \to Aut(F_*)$ *defined by* $Ad_f = T_{e_u} I_f$ *for all* $f \in F_u$, $u \in P$ *is a PBG-Lie group bundle morphism. Locally it is of the form* $Ad^i : F_{P_i} \to Aut(F_*)_{P_i}$ *where*

$$Ad^i(u, h) = (u, Ad_h^H)$$

for all $(u,h) \in P_i \times H \overset{\psi_i}{\cong} F_{P_i}$. *Here* Ad^H *is the adjoint representation on* H.

Proof. Immediate ■

The representation Ad differentiates to the PBG-Lie algebra bundle morphism $ad : F_* \to Der(F_*)$ defined by

$$ad_V(W) = [V,W] = (T_{e_u}Ad(V))(W)$$

for all $V, W \in (F_*)_u,\ u \in P$.

Now we can proceed to the proof of the first identity. For all $V, W \in F_*$ we have $\rho_*(\partial_*(V))(W) = [V,W]$, or $(\overline{\rho} \circ \partial)_* = Ad_*$. Since F has connected and simply connected fibers we have $\overline{\rho}(\partial(f)) = Ad_f$ for all $f \in F$. Therefore,

$$\rho \circ \partial = I.$$

For the second identity, take an $X \in \Gamma A\Omega$. Then X induces a vector field $\vec{X} \in \mathfrak{X}(\Omega)$ which is defined as $\vec{X}_\xi = T_{1_{\beta(\xi)}}R_\xi(X_{\beta(\xi)})$ for all $\xi \in \Omega$. This is an $\alpha-$vertical ($\vec{X}_\xi \in T_\xi\Omega_{\alpha(\xi)}$) and right-invariant ($\vec{X} \circ R_\xi = TR_\xi \circ \vec{X}$) vector field on Ω. Let $\varphi : (-\epsilon,\epsilon) \times \mathcal{U}_0 \to \mathcal{V}_0$ be the flow of $\vec{X}$, where $\mathcal{U}_0, \mathcal{V}_0 \subseteq \Omega$. Then, it is immediate that every $\varphi_t : \mathcal{U}_0 \to \mathcal{V}_0$ has the properties $\alpha \circ \varphi_t = \alpha$ and $\varphi_t \circ R_\xi = R_\xi \circ \varphi_t$ for all $\xi \in \Omega$ and $t \in (-\epsilon,\epsilon)$.

Denote $\mathcal{U} = \beta(\mathcal{U}_0)$ and $\mathcal{V} = \beta(\mathcal{V}_0)$ and let $\Psi : (-\epsilon,\epsilon) \times \mathcal{U} \to \mathcal{V}$ be the map $\psi_t(u) = \beta(\varphi_t(\eta))$ for all $\eta \in \mathcal{U}_0^u$. This is well defined because if we consider an $\eta' \in \mathcal{U}_0^u$ then there is a $\xi \in \mathcal{U}_0$ such that $\eta' = \eta \cdot \xi$. Consequently,

$$\psi_t(u) = \beta(\varphi_t(\eta\xi)) = \beta(\varphi_t(\eta) \cdot \xi) = \beta(\varphi_t(\eta)).$$

Finally, for all $t \in (-\epsilon,\epsilon)$ we have $\alpha \circ \varphi_t = \alpha$, $\beta \circ \varphi_t = \psi_t \circ \beta$ and $\varphi_t(\xi\eta) = \varphi_t(\xi) \cdot \eta$. Therefore, proposition 5.8 in chapter 3 of [11] shows that φ_t is the restriction to $\mathcal{U}_0$ of a unique local left-translation $L_{\sigma_t} : \Omega^{\mathcal{U}} \to \Omega^{\mathcal{V}}$, where

$$\sigma_t(u) = \varphi_t(\xi) \cdot \xi^{-1}$$

for all $\xi \in \mathcal{U}_0^u$. We define the exponential map $Exp : (-\epsilon,\epsilon) \times \Gamma A\Omega \to \Gamma_{\mathcal{U}}\Omega$ by

$$ExptX = \sigma_t.$$

Now take an $X \in \Gamma A\Omega$ and a $V \in \Gamma F_*$. From the properties of the exponential, for all $u \in P$ we have:

$$\begin{aligned}
\partial_*(\rho(X)(V_u)) &= -\frac{d}{dt}\partial_*(\overline{\rho}(ExptX(u))(V_u)) \mid_0 = \\
&= -\frac{d}{dt}\partial_*(\overline{\rho}(\varphi_t(\xi) \cdot \xi^{-1})(V_u)) \mid_0 = -\frac{d}{dt}\partial_*(\overline{\rho}(\varphi_t(\xi))[\overline{\rho}(\xi^{-1})(V_u)]) \mid_0 = \\
&= -\frac{d}{dt}(\partial \circ \rho(\varphi_t(\xi)))_*[\overline{\rho}(\xi)^{-1}(V_u)] \mid_0
\end{aligned}$$

and

$$\begin{aligned} ad_X(\partial_*(V_u)) = -\frac{d}{dt}Ad_{(ExptX(u))}(\partial_*(V_u)) \mid_0 &= \\ = -\frac{d}{dt}(I_{\sigma_t(u)})_*(\partial_*(V_u)) \mid_0 = -\frac{d}{dt}(I_{\varphi_t(\xi)\xi^{-1}})_*(\partial_*(V_u)) \mid_0 &= \\ = -\frac{d}{dt}(I_{\varphi_t(\xi)})_*[(I_{\xi^{-1}} \circ \partial)_*(V_u)] \mid_0 . \end{aligned}$$

Consider the curves $\delta_u, \gamma_u : (-\epsilon, \epsilon) \to F_*$ defined by

$$\gamma_u(t) = (\partial \circ \rho(\varphi_t(\xi)))_*[\overline{\rho}(\xi)^{-1}(V_u)]$$

and

$$\delta_u(t) = (I_{\varphi_t(\xi)})_*[(I_{\xi^{-1}} \circ \partial)_*(V_u)].$$

Obviously, $\gamma_u(0) = \delta_u(0) = \partial_*(V_u)$. Since $\partial_*(\rho(X)(V)) = [X, \partial_*(V)]$ we have

$$\frac{d}{dt}\gamma_u(t) \mid_0 = \frac{d}{dt}\delta_u(t) \mid_0 .$$

Lemma 6.10 *There is a* $\delta < \epsilon$ *such that* $\frac{d}{dt}\gamma_u(t) \mid_{t_0} = \frac{d}{dt}\delta_u(t) \mid_{t_0}$ *for all* $|t_0| < \delta$.

PROOF. For all $t \in (-\epsilon, \epsilon)$ we have:

$$\gamma_u(t) = (\partial \circ \rho(\varphi_{(t-t_0)+t_0}(\xi)))_*[\overline{\rho}(\xi)^{-1}(V_u)] = (\partial \circ \rho(\varphi_{t-t_0}(\xi)))_*[\overline{\rho}(\xi)^{-1}(V_u)].$$

Therefore,

$$\begin{aligned} \gamma_u(t) - \gamma_u(t_0) = (\partial \circ \rho(\varphi_{t-t_0}(\xi)))_*[\overline{\rho}(\xi)^{-1}(V_u)] - (\partial \circ \rho(\varphi_{t_0}(\xi)))_*[\overline{\rho}(\xi)^{-1}(V_u)] &= \\ = (\partial \circ \rho(\varphi_{t-t_0}(\xi)))_*[\overline{\rho}(\xi)^{-1}(V_u)] = \gamma_u(t-t_0). \end{aligned}$$

And of course, the same is true for δ_u. Define $\widetilde{\gamma_u}(t) = \gamma_u(t-t_0)$ and $\widetilde{\delta_u}(t) = \delta_u(t-t_0)$. Then,

$$\begin{aligned} \frac{d}{dt}\gamma_u(t) \mid_{t_0} = lim_{t \to t_0}\frac{\gamma_u(t) - \gamma_u(t_0)}{t - t_0} = lim_{t \to 0}\frac{\widetilde{\gamma_u}(t)}{t} &= \\ = \frac{d}{dt}\widetilde{\gamma_u}(t) \mid_0 = \frac{d}{dt}\tilde{\delta_u}(t) \mid_0 = \ldots = \frac{d}{dt}\delta_u(t) \mid_{t_0} . \end{aligned}$$

■

So, for all $|t| < \delta < \epsilon$ we have $(\partial \circ \rho(ExptX))_* = (I_{ExptX} \circ \partial)_*$ and since Ω is $\alpha-$connected and $\alpha-$simply connected we finally get

$$\partial \circ \rho(ExptX) = I_{ExptX} \circ \partial$$

for all $|t| < \delta$. Hence the desired equality. Combining this result with Proposition 4.4 of [10] we get the following result.

Theorem 6.11 *Suppose given a PBG-Lie group bundle* F *and a PBG-groupoid* Ω, *both over the same principal bundle* $P(M,G)$. *Then any pair crossed module of PBG-algebroids* $(F_*, \partial_*, A\Omega, \rho_*)$ *integrates to a pair crossed module of PBG-groupoids* $(F, \partial, \Omega, \rho)$.

7 Classification of PBG-groupoids

In order to characterize cohomologically the obstruction associated with a pair crossed module of PBG-groupoids, it is necessary to give a classification of such groupoids. Such a classification was given in [2], and it will be used in Section 8, for the enumeration of operator extensions. In this section we give a different cohomological classification of PBG-groupoids, which is consistent with the classification of transitive Lie algebroids given in [11, IV§4].

It was shown there that a transitive Lie algebroid $L \rightarrowtail A \twoheadrightarrow TM$ is locally described by the following data: If $\mathfrak{h}$ denotes the fibre type of L, then for a simple open cover $\{U_i\}_{i\in I}$ of M, there exists a family of differential-2-forms $\chi = \{\chi_{ij}\colon TU_{ij}\times TU_{ij} \to U_{ij}\times\mathfrak{h}\}_{i,j\in I}$ and a cocycle $\alpha = \{\alpha_{ij}\colon U_{ij}\to \mathrm{Aut}(\mathfrak{h})\}_{i,j\in I}$ such that

(i). $\delta\chi_{ij} + [\chi_{ij},\chi_{ij}] = 0$, whenever $U_{ij}\neq\emptyset$, namely the χ_{ij}s satisfy the Maurer-Cartan equation,

(ii). $\chi_{ik} = \chi_{ij} + \alpha_{ij}(\chi_{jk})$, whenever $U_{ijk}\neq\emptyset$,

(iii). $\Delta(\alpha_{ij}) = \mathrm{ad}\circ\chi_{ij}$, whenever $U_{ij}\neq\emptyset$.

The α_{ij}s here are the transition functions of the Lie algebra bundle L. The notation Δ stands for the Darboux derivative. More than that, it is shown that this data classifies transitive Lie algebroids.

Transitive Lie groupoids differentiate to transitive Lie algebroids, however the classification on the groupoid level is by the transition functions of the respective principal bundle, and it is not clear how these differentiate to the above data. An attempt to reformulate the classification of PBG-groupoids in such a form which clearly differentiates to the above data was made in [2]. In this section we briefly recall the account given there and proceed to show that indeed this gives a classification of PBG-groupoids. In the following sections of this paper it will be shown that this is the appropriate classification for the formulation of the integrability obstruction of PBG-algebroids.

Let $\Omega \rightrightarrows P(M,G)$ be a PBG-groupoid and $\{P_i \equiv U_i\times G\}_{i\in I}$ an atlas of its base principal bundle. It was shown in [2] that for every $i\in I$ there exists a flat isometablic connection $\gamma_i : TP_i \to A\Omega_{P_i}$. More than that, it was shown that as a morphism of PBG-algebroids, every γ_i integrates to a morphism of PBG-groupoids $\theta_i : P_i\times P_i \to \Omega_{P_i}^{P_i}$. Now fix a $u_0\in P$ and denote $H = \Omega_{u_0}^{u_0}$. For every $i\in I$ choose a $u_i\in P_i$ and an arrow $\xi_i\in\Omega_{u_0}^{u_i}$. Now define the maps

$$\sigma_i : P_i \to \Omega_{u_0},\ \sigma_i(u) = \theta_i(u,u_i)\cdot\xi_i.$$

These are sections of Ω and they respect the G-action in the following sense:

$$\sigma_i(ug) = [\sigma_i(u)g]\cdot(\xi_i^{-1}g)\cdot\sigma_i(u_ig).$$

These sections give rise to a family of representations $\{\varphi_i : G\to Aut(H)\}_{i\in I}$ of G on H, namely

$$\varphi_i(g)(h) = \sigma_i(u_ig)^{-1}\cdot(\xi_ig)\cdot(hg)\cdot(\xi_ig)^{-1}\cdot\sigma_i(u_ig).$$

It was shown in [2] that these representations are local expressions of the automorphism action of G on the Lie group bundle $I\Omega$.

If we begin with a different local family $\{\gamma'_i\}_{i\in I}$ of flat isometablic connections, there exist 1-forms $\ell^*_i : TP_i \to P_i \times \mathfrak{h}_i$ such that $\gamma'_i = \gamma_i + \ell^*_i$. Here $\mathfrak{h}_i$ is the Lie algebra of the Lie group $\Omega^{u_i}_{u_i}$. Therefore the ℓ^*_is integrate to maps $\ell_i : P_i \times P_i \to \Omega^{u_i}_{u_i}$ such that $\theta'_i = \theta_i + \ell_i$. Define

$$r_i : P_i \to H,\ r_i(u) = \xi_i^{-1} \cdot \ell_i(u, u_i) \cdot \xi_i.$$

Now the respective sections are related by $\sigma'_i = \sigma_i \cdot r_i$, and with respect to the G-action the r_is satisfy

$$r_i(ug) = \varphi_i(g)(r_i(u)) \cdot r_i(u_i g).$$

Last, the representations arising from σ'_i and σ_i are related by

$$\varphi'_i(g)(h) = r_i(u_i g)^{-1} \cdot \varphi_i(g)(h) \cdot r_i(u_i g).$$

Now, instead of classifying Ω by the transition functions $s_{ij} : P_{ij} \to H$ associated with the sections σ_i, let us consider the following maps:

$$\chi_{ij} : P_{ij} \times P_{ij} \to H,\ \chi_{ij}(u, v) = s_{ij}(u) \cdot s_{ji}(v)$$

and

$$\alpha_{ij} : P_{ij} \to Aut(H),\ \alpha_{ij}(u)(h) = s_{ij}(u) \cdot h \cdot s_{ji}(u).$$

The α_{ij}s are the transition functions of the PBG-Lie group bundle $I\Omega$. Together with the χ_{ij}s they satisfy:

(i). $\chi_{ik}(u, v) = \chi_{ij}(u, v) \cdot \alpha_{ij}(v)(\chi_{jk}(u, v))$.

(ii). For a choice of $u_{ij} \in P_{ij}$, $\alpha_{ij}(u) = I_{\chi_{ij}(u,u_{ij})} \circ I_{s_{ij}(u_{ij})}$.

(iii). $\chi_{ij}(ug, vg) = \varphi_i(g)(\chi_{ij}(u, v))$.

(iv). $\alpha_{ij}(ug)(\varphi_j(g)(h)) = \varphi_i(g)(\alpha_{ij}(u)(h))$.

Definition 7.1 *A pair (χ, α) satisfying (i)–(iv) is called an φ-*isometablic pair of transition data.

The relation between two isometablic systems of transition data given in the following proposition is proven in [2].

Proposition 7.2 *Two ρ-isometablic and ρ'-isometablic systems of transition data (χ, α) and (χ', α') respectively are related by*

$$\chi'_{ij}(u, v) = r_i(u)^{-1}[\chi_{ij}(u, v) \cdot \alpha_{ij}(v)(r_i(u) \cdot r_j(v)^{-1})] \cdot r_i(v) \tag{5}$$

and

$$\alpha'_{ij}(u) = I_{r_i(u)^{-1}} \circ \alpha_{ij}(u) \circ I_{r_j(u)} \tag{6}$$

It is straightforward that the relation between isometablic systems of transition data is an equivalence relation, therefore it is legitimate to give the following definition.

Definition 7.3 *Two isometablic systems of transition data which satisfy (5) and (6) are called* equivalent*.*

Now we can proceed to show that isometablic systems of transition data classify PBG-groupoids.

Theorem 7.4 *Suppose $P(M,G)$ be a principal bundle and $\{U_i\}_{i\in I}$ is a simple open cover of M, whereas $\{P_i \equiv U_i \times G\}_{i\in I}$ is an atlas over this cover. Let $\varphi = \{\varphi_i : G \to Aut(H)\}_{i\in I}$ be a family of representations of G on a Lie group H and (χ, α) a family of φ-isometablic transition data. Then there exists a PBG-groupoid over $P(M,G)$ and a local family of flat isometablic connections which give rise to this data.*

PROOF. For each $i \in I$, let $\Upsilon^i = P_i \times H \times P_i$ and on the disjoint sum of the Υ^is define an equivalence relation $\cong$ by

$$(i,u,h,v) \cong (j,u',h',v') \Leftrightarrow u = u',\ v = v',\ h' = \chi_{ji}(u,v) \cdot \alpha_{ji}(v)(h).$$

Denote the quotient set by Υ and the equivalence classes by $\langle i,(u,h,v)\rangle$. Define maps $\alpha, \beta : \Upsilon \to P$ by $\langle i,(u,h,v)\rangle \mapsto v$ and $\langle i,(u,h,v)\rangle \mapsto u$ respectively. The object inclusion map is $P_i \ni u \mapsto \langle i,(u,e_H,u)\rangle$. it is easy to see that the map

$$\overline{\Psi}_i : P_i \times H \times P_i \to (\beta,\alpha)^{-1}(P_i),\ (u,h,v) \mapsto \langle i,(u,h,v)\rangle$$

is a bijection. Give Υ the smooth structure induced from the manifolds $P_i \times H \times P_i$ via $\overline{\Psi}_i$.

Now we define a multiplication in Υ. For $\xi, \eta \in \Upsilon$ such that $\alpha(\xi = \beta(\eta)) = u$, choose a P_i containing u and write $\xi = \langle i,(v,h,u)\rangle$, $\eta = \langle i,(u,h',w)\rangle$. Define

$$\xi \cdot \eta = \langle i,(v,hh',w)\rangle.$$

Finally, G acts on Υ by

$$\langle i,(u,h,v)\rangle g = \langle i,(ug,\varphi_i(g)(h),vg)\rangle.$$

It is left to the reader to verify that Υ is a well defined PBG-groupoid over $P(M,G)$. The PBG-algebroid it differentiates to is the one given in [11, III, 5.15], and the connections associated with the transition data we began with are the ones given there. ∎

Remark. Note that for the previous construction the only property that we use is the cocycle condition that (χ,α) satisfy, namely $\chi_{ij}(u,v) = \chi_{ik}(u,v) \cdot \alpha_{ik}(v)(\chi_{kj}(u,v))$, and that is to show that the relation $\cong$ is indeed an equivalence relation. On the other hand, the compatibility condition is not used here.

The following proposition shows that the PBG-groupoid arising from isometablic transition data is well defined up to equivalence. Its proof is a straightforward calculation.

Proposition 7.5 *Let $P(M,G)$ be a principal bundle, $\{P_i\}_{i\in I}$ an open cover of P by principal bundle charts, H a Lie group and φ', φ be two families of representations of G on H by which are equivalent under a family of maps $r = \{r_i : P_i \to H\}_{i\in I}$ such that $r_i(ug) = \varphi_i(g)(r_i(u))\cdot r_i(u_i g)$ for all $u\in P_i, g\in G$ and $i\in I$. Let (χ', α') and (χ, α) be ρ'-isometablic and ρ-isometablic systems of transition data with values in H respectively which are equivalent under the family of maps r. Let Ω' and Ω be the associated PBG-groupoids respectively. Then the map $\Xi : \Omega' \to \Omega$ defined by*

$$\langle i, (u,h,v)\rangle \mapsto \langle i, (u, r_i(u)^{-1}\cdot h\cdot r_j(v), v)\rangle$$

is an isomorphism of PBG-groupoids over $P(M,G)$.

8 The obstruction of a pair crossed module of PBG-groupoids

In this section we give the cohomological obstruction to the existence of an operator extension for a pair crossed module of PBG-groupoids. Let us start with such a crossed module $\langle F, \partial, \Omega, \rho\rangle$ over the principal bundle $P(M,G)$. Then the PBG-groupoid $\Omega \Longrightarrow P(M,G)$ is the extension of PBG-groupoids

$$\begin{tikzcd} Im(\partial) \arrow[r, tail] & \Omega \arrow[r, two heads, "{(\beta,\alpha)}"] & P\times P. \end{tikzcd}$$

Choose a simple open cover $\{U_i\}_{i\in I}$ of M and an atlas $\{P_i \equiv U_i\times G\}_{i\in I}$ of the principal bundle, and consider a φ-isometablic system of transition data (χ, α). Note that in this context we denote H the fiber type of F, therefore every φ_i is a representation of G on $\partial(H)$, namely $\varphi_i : G \to Aut(\partial(H))$. Now the following proposition shows that there are canonical lifts of the representations φ_i. Its proof is a straightforward calculation.

Proposition 8.1 *For every $i \in Im$ the map $\widehat{\varphi}_i : G \to Aut(H)$ defined by*

$$\widehat{\varphi}_i(g)(h) = \rho(\sigma_i(u_i g)^{-1}\cdot(\xi_i g), hg)$$

for all $g\in G$ and $h\in H$ is a representation of G on H and $\partial\circ\widehat{\varphi}_i = \varphi_i$.

The next two results show that there also exist canonical lifts of the transition functions α_{ij} of $Im(\partial)$, to transition functions of F.

Proposition 8.2 *Let $(F, \partial, \Omega, \rho)$ be a pair crossed module of PBG-groupoids. With the previous notation, the maps $\psi_i : P_i\times H \to F_{P_i}$ defined by $\psi_i(u,h) = \rho(\sigma_i(u), h)$ are charts of the Lie group bundle F and they are isometablic in the sense*

$$\psi_i(ug, \widehat{\varphi}_i(g^{-1})(h)) = \psi_i(u,h)\cdot g$$

for all $g\in G$, $u\in P_i$ and $h\in H$.

Proof. First of all, to prove that the ψ_is are well defined, we need to ensure that the restriction of ρ on $\Omega_{u_0}^{P_i} * H$ takes values in $\pi^{-1}(P_i)$. Indeed, if $\xi\in\Omega_{u_0}^{P_i}$ and $f\in H$ is such that $\pi(f) = u_0$ then $\pi(\rho(\xi, f)) = \beta(\xi) \in P_i$. The ψ_is are injective because for all $u, u' \in P_i$ and $f, f' \in H$ we have:

$$\begin{aligned} &\rho(\sigma_i(u), f) = \rho(\sigma_i(u'), f') \Rightarrow \pi(\rho(\sigma_i(u), f)) = \pi(\rho(\sigma_i(u'), f')) \Rightarrow \\ &\qquad\qquad\qquad\qquad \Rightarrow \beta(\sigma_i(u)) = \beta(\sigma_i(u')) \Rightarrow u = u'. \end{aligned}$$

Since $\rho(\sigma_i(u))$ is an isomorphism on the fibers of F, we also have $f = f'$.

For the surjectivity of the ψ_is, consider an $f \in F_u \subseteq F_i$ for some $u \in P_i$. Then, because $\rho(\sigma_i(\pi(f)))$ is an isomorphism $H \to F_u$, there is an $f' \in H$ such that $f = \rho(\sigma_i(\pi(f)), f') = \psi_i(\pi(f), f')$. Last, the following diagram commutes

$$\begin{tikzcd} P_i \times N \arrow[rr, "\psi_i"] \arrow[dr, "pr_1"'] & & \pi^{-1}(P_i) \arrow[dl, "\pi"] \\ & P_i & \end{tikzcd}$$

because $\pi(\psi_i(u, f)) = \pi(\rho(\sigma_i(u), f)) = \beta(\sigma_i(u)) = u$. For the isometablicity of the ψ_i's we have:

$$\begin{aligned} &\psi_i(ug, \widehat{\rho}_i(g^{-1})(h)) = \rho(\sigma_i(ug), \widehat{\rho}_i(g^{-1})(h)) = \\ &= \rho(\sigma_i(ug), \rho(\sigma_i(u_ig)^{-1} \cdot (\xi_i g), hg)) = \rho([\sigma_i(u)g] \cdot (\xi_i^{-1} g) \cdot \sigma_i(u_i g) \cdot \sigma_i(u_i g)^{-1} \cdot (\xi_i g), hg) = \\ &\qquad\qquad\qquad = \rho(\sigma_i(u)g, hg) = \rho(\sigma_i(u), h) \cdot g = \psi_i(u, h) \cdot g. \end{aligned}$$

■

Now let us look at the transition functions of the Lie group bundle charts defined in the previous theorem.

Proposition 8.3 *The transition functions of the charts* $\{\psi_i\}_{i\in I}$ *are lifts of the transition functions* $\{\widehat{\alpha}_{ij}\}_{i,j\in I}$*, form a Čech-1-cocycle and are isometablic with respect to the representations* $\{\widehat{\varphi}_i\}_{i\in I}$.

PROOF. For all $u \in P_{ij}$ and $h \in H$, we have:

$$\psi_{ij}(u)(h) \;=\; \psi_{i,u}^{-1}(\psi_{j,u}(h)) \;=\; \psi_{i,u}^{-1}(\rho(\sigma_j(u), h)) \;=\; \rho(\sigma_i(u)^{-1} \cdot \sigma_j(u), h) \;=\; \rho(s_{ij}(u), h).$$

Therefore, $\partial(\psi_{ij}(u)(h)) = I_{s_{ij}(u)}(h) = \alpha_{ij}(u)(h)$, so the ψ_{ij}'s are lifts of the α_{ij}'s. They form a Čech-1-cocycle because:

$$\begin{aligned} &[\psi_{jk}(u) \circ \psi_{ik}(u)^{-1} \circ \psi_{ij}(u)](h) = \rho(s_{jk}(u), \rho(s_{ik}(u)^{-1}, \rho(s_{ij}(u), h))) = \\ &\qquad\qquad\qquad = \rho(s_{jk}(u) \cdot s_{ik}(u)^{-1} \cdot s_{ij}(u), h) = \rho(1_u, h) = h \end{aligned}$$

Moreover, they are isometablic with respect to the lifts $\{\widehat{\rho}_i\}_{i\in I}$ of the representations $\{\overline{\rho}_i\}_{i\in I}$ because:

$$\begin{aligned}\psi_{ij}(ug)(\widehat{\varphi}_j(g^{-1})(h)) = \rho(s_{ij}(ug), \rho(\sigma_j(u_jg)^{-1}\cdot(\xi_jg), hg)) = \\ = \rho(\sigma_i(u_ig)^{-1}\cdot(\xi_ig)\cdot(s_{ij}(u)g)\cdot(\xi_j^{-1}g)\cdot\sigma_j(u_jg)\cdot\sigma_j(u_jg)^{-1}\cdot(\xi_jg), hg) = \\ = \widehat{\varphi}_i(g^{-1})(\psi_{ij}(u)(h)).\end{aligned}$$

■

Now let us show that there also exist canonical $\widehat{\varphi}$-isometablic lifts of the χ_{ij}s. Consider the quotient maps $\chi_{ij}^{/G} : \frac{P_{ij}\times P_{ij}}{G} \to \frac{\partial(H)}{G}$ and $\partial^{/G} : \frac{F}{G} \to \frac{Im(\partial)}{G}$. The restriction of $\partial^{/G}$ to $H = F_{u_0}$ is a a map $\partial^{/G}|_H \colon \frac{H}{G} \to \frac{\partial(H)}{G}$, where the action of G on $\partial(H)$ implied is φ_i, and the action of G on H is $\widehat{\varphi_i}$. This happens because the φ_is are local expressions of the G-action on $Im(F)$ as was shown in [2]. Bearing in mind that $P_{ij} \equiv U_{ij} \times G$, the quotient $\frac{P_{ij}\times P_{ij}}{G}$ is just $U_{ij} \times U_{ij}$, therefore we have the following diagram:

$$\begin{tikzcd} & \frac{H}{G} \arrow[d, "\partial^{/G}"] \\ U_{ij}\times U_{ij} \arrow[r, "\chi_{ij}^{/G}"'] & \frac{\partial(H)}{G} \end{tikzcd}$$

Note that $\frac{H}{G}(\frac{\partial(H)}{G}, \ker(\partial), \partial^{/G})$ is a principal bundle in a trivial way. Since the U_{ij}s are simply connected, it follows from [8] that there exists a differentiable map $\widehat{\chi}_{ij}^{/G} : U_{ij}\times U_{ij} \to \frac{H}{G}$ such that the above diagram commutes.

Denote $\sharp : P_{ij}\times P_{ij} \to U_{ij}\times U_{ij}$ and $\sharp^H : H \to \frac{H}{G}$ the natural projections. Since $\sharp^H$ is a pullback over the projection $\partial^{/G}$ of the principal bundle $\frac{H}{G}(\frac{\partial(H)}{G}, \ker(\partial), \partial^{/G})$, there is a unique map $\widehat{\chi}_{ij} : P_{ij}\times P_{ij} \to H$ such that

$$\sharp^H \circ \widehat{\chi}_{ij} = \widehat{\chi}_{ij}^{/G} \circ \sharp.$$

Due to the G-invariance of $\sharp$ and $\sharp^H$, the map $\widehat{\varphi}_i(g)^{-1}\circ\widehat{\chi}_{ij}\circ(R_g\times R_g)$ also satisfies the previous equation for every $g\in G$, therefore it follows from the uniqueness argument that $\widehat{\chi}_{ij}$ is φ_i-isometablic. these considerations consist the proof of the following result.

Theorem 8.4 *Let $\langle F, \partial, \Omega, \rho\rangle$ be a pair crossed module of PBG-groupoids over a principal bundle $P(M,G)$ and $\varphi = \{\varphi_i : G \to Aut(\partial(H))\}_{i\in I}$ a family of representations of G on the image by ∂ of the fiber type H of the PBG-Lie group bundle F. Then there exists a canonical family of representations $\widehat{\varphi} = \{\widehat{\varphi}_i : G \to Aut(H)\}_{i\in I}$ such that*

(i). $\partial\circ\widehat{\varphi}_i = \varphi_i$ *for all* $i\in I$;

(ii). *For every* φ*-isometablic system of transition data* (χ, α) *of* Ω*, there exists a canonical pair* $(\widehat{\chi}, \widehat{\alpha})$ *with values in* H*, such that* $\widehat{\alpha}$ *is an isometablic cocycle of transition functions for the PBG-Lie group bundle* F*, and* $\widehat{\chi}$ *is a* $\widehat{\varphi}$*-isometablic family of differential maps* $\{\widehat{\chi}_{ij} : P_{ij} \times P_{ij} \to H\}_{i,j \in I}$ *such that* $\partial(\widehat{\chi}, \widehat{\alpha}) = (\chi, \alpha)$.

If this lift of the transition data of Ω is a $\widehat{\varphi}$-isometablic system of transition data itself, then it gives rise to a PBG-groupoid, with adjoint bundle F, and this would play the role of an operator extension for the given pair crossed module. We saw that this lift is indeed $\widehat{\varphi}$-isometablic. As we remarked in 7.4, the only the only thing that is required is for the pair $(\widehat{\chi}, \widehat{\alpha})$ to satisfy the cocycle condition. This can be reformulated to

$$\psi_{i,v}(\widehat{\chi}_{ij}(u,v)) = \psi_{i,v}(\widehat{\chi}_{ik}(u,v)) \cdot \psi_{k.v}(\widehat{\chi}_{kj}(u,v)).$$

Thus the failure of the pair $(\widehat{\chi}, \widehat{\alpha})$ to satisfy the cocycle condition is the map $e_{ijk} : P_{ijk} \times P_{ijk} \to \ker \partial \leq ZF$, defined by

$$e_{ijk}(u,v) = \psi_{i,v}(\widehat{\chi}_{ij}(u,v)) \cdot [\psi_{k.v}(\widehat{\chi}_{kj}(u,v))]^{-1} \cdot \psi_{i,v}(\widehat{\chi}_{ik}(u,v)).$$

The fact that it takes values in $\ker \partial$ follows from the fact that the original system of transition data (χ, α) does satisfy the cocycle condition. A routine calculation shows that $e_{ijk}(ug, vg) = e_{ijk}(u,v)g$ for all $g \in G$, and for $P_{ijkl} \neq \emptyset$

$$e_{jkl} - e_{ikl} - e_{ijl} - e_{ijk} = 0 \in ZF$$

and so e is a 2-cocycle in $\check{H}^2_G(P \times P, ZF)$, the G-isometablic Čech cohomology of $P \times P$ with respect to the atlas $\{P_i \cong U_i \times G\}_{i \in I}$ of the principal bundle $P(M,G)$, and with coefficients in the sheaf of germs of local isometablic maps from $P \times P$ to ZF.

It is trivial to see that if a second family of lifts $P_{ij} \times P_{ij} \to H$ of the χ_{ij}s is chosen then the resulting cocycle is cohomologous to e. More generally, if (χ', α') is a second φ'-isometablic system of transition data for Ω, over the same atlas $\{P_i \equiv U_i \times G\}_{i \in I}$ of the principal bundle $P(M,G)$, then it follows from the relations we gave in 7.2 and 8.2 that $e'_{ijk} = e_{ijk}$.

Theorem 8.5 *Continuing the above notation, there exists an operator extension* (Υ, μ) *for the pair crossed module of PBG-groupoids* $\langle F, \partial, \Omega, \rho \rangle$ *iff* $e = 0 \in \check{H}^2_G(P \times P, ZF)$.

Proof. Assume that $e = 0$ and consider the PBG-groupoid $\Upsilon \rightrightarrows P(M,G)$ constructed directly from the pair $(\widehat{\chi}, \widehat{\alpha})$ as in 7.4. Recall that the representation ρ induces an atlas of PBG-Lie group bundle charts $\rho(\sigma_i(u), h)$ for $F \to P(M,G)$. Thus every element of F, say $\lambda \in F_u$, can be represented as $\rho(\sigma_i(u), h)$ for any $i \in I$ with $u \in P_i$. Define $\iota : F \to \Upsilon$ by mapping $\rho(\sigma_i(u), h) \in F_u$ to $\langle i, (u, h, u) \rangle$. It is trivial to check that ι is well defined, and an isomorphism of PBG-Lie group bundles over $P(M,G)$ onto $I\Upsilon$. Thus we have the extension of PBG-groupoids

$$\begin{tikzcd} F \arrow[r, tail, "\iota"] & \Upsilon \arrow[r, two heads, "(\beta{,}\alpha)"] & P \times P. \end{tikzcd}$$

Define $\mu : \Upsilon \to \Omega$ by $\langle i, (u, h, v) \rangle \mapsto \sigma_i(u)\partial(h)\sigma_i(v)^{-1}$. Again one checks that μ is well defined, a surjective submersion and a morphism of PBG-groupoids over $P(M,G)$.

To see that the diagram

$$\begin{tikzcd}
F \arrow[r, "\iota"] \arrow[d, two heads, "\partial"'] & \Upsilon \arrow[d, two heads, "\mu"'] \\
Im\partial \arrow[r] & \Omega
\end{tikzcd}$$

commutes, recall that $\partial(\rho(\xi,\lambda)) = \xi\partial(\lambda)\xi^{-1}$ for $\xi \in \Omega, \lambda \in F_{\alpha\xi}$. Taking $\xi = \sigma_i(u)$ and $\lambda = h \in H = F_{u_0}$, this gives

$$\partial(\rho(\sigma_i(u),h)) = (\mu\circ\iota)(\rho(\sigma_i(u),h)),$$

as required.

It remains to verify that the action of Ω on F induced by the diagram

$$\begin{tikzcd}
\ker\partial \arrow[r, equal] \arrow[d, tail] & \ker\partial \arrow[d, tail] & \\
F \arrow[r, tail, "\iota"] \arrow[d, two heads, "\partial"'] & \Upsilon \arrow[r, two heads, "{(\beta,\alpha)}"] \arrow[d, two heads, "\mu"'] & P\times P \arrow[d, equal] \\
Im\partial \arrow[r, tail] & \Omega \arrow[r, two heads, "{(\beta,\alpha)}"] & P\times P
\end{tikzcd}$$

coincides with the given ρ. Take $\omega \in \Upsilon$, say $\omega = \langle j,(u,h,v)\rangle$, and $\lambda \in F_{\alpha\omega}$, say $\lambda = \rho(\sigma_{i'}(u),h')$; it is no loss of generality to assume that $j = i'$. Now $\omega\iota(\lambda)\omega^{-1} = \langle j,(u,hh'h^{-1},v)\rangle$, by the definition of ι and the multiplication in Υ. On the other hand, $\mu(\omega)$ is equal to $\sigma_j(u)\partial(h)\sigma_j(v)^{-1}$ and so

$$\rho(\mu(\omega),\lambda) = \rho(\sigma_j(u)\partial(h),h') = \rho(\sigma_j(u),hh'h^{-1}).$$

So $\omega\iota(\lambda)\omega^{-1} = \iota(\rho(\mu(\omega),\lambda))$, as required. This completes the proof that (Υ,μ) is an operator extension of the pair crossed module of PBG-groupoids $\langle F,\partial,\Omega,\rho\rangle$. The converse is a trivial verification. ■

The element $e \in \check{H}^2_G(P\times P, ZF)$ is the *obstruction* associated with the pair crossed module of PBG-groupoids $\langle F,\partial,\Omega,\rho\rangle$. Following the notation of [12], we denote it by $\mathrm{Obs}\langle F,\partial,\Omega,\rho\rangle$. The following theorem is an immediate consequence of the previous considerations.

Theorem 8.6 *Let $\Omega \Longrightarrow M$ be a transitive Lie groupoid and*

$$K \rightarrowtail A \twoheadrightarrow A\Omega \tag{7}$$

be an extension of Lie algebroids over the manifold M. Choose a baspoint in M and let $P(M,G,p)$ be the principal bundle corresponding to Ω. If $F \to P(M,G)$ is the PBG-Lie group bundle integrating the PBG-Lie algebra bundle $K \gtrdot p$, then the integrability obstruction of 7 is the obstruction $e \in \check{H}^2_G(P\times P, ZF)$ associated with the pair crossed module of PBG-groupoids corresponding to the extension 7.

9 Classification of operator extensions for coupling pair crossed modules of PBG-groupoids

Suppose given a coupling pair crossed module of PBG-groupoids $\langle F, \partial, \Omega, \rho\rangle$ over the principal bundle $P(M,G)$. Recall that *coupling* means $\ker\partial = ZF$ whereas *pair* means that the cokernel $\frac{\Omega}{\ker\partial} = P \times P$. In this section we show that if its obstruction cocycle vanishes then its operator extensions are classified by $\check{H}^1_G(P, ZH)$, where H is the fiber type of the Lie group bundle F. This cohomology, defined on P instead of $P \times P$ is the isometablic cohomology given in [2], and we start with a brief recollection of it.

Consider a PBG-groupoid $\Xi \Longrightarrow P(M,G)$. Choose an atlas $\{P_i \equiv U_i \times G\}_{i\in I}$ for the principal bundle $P(M,G)$, where $\{U_i\}_{i\in I}$ is a simple open cover of M, and a family of local flat isometablic connections $TP_i \to A\Xi_{P_i}$. As we discussed in the beginning of section 6, this data gives rise to sections $\sigma_i : P_i \to \Xi_{u_0}$ of the PBG-groupoid, which are isometablic in the sense

$$\sigma_i(ug) = [\sigma_i(u)g] \cdot (\xi_i^{-1}g) \cdot \sigma_i(u_i g).$$

An alternative classification of PBG-groupoids, given in [2], is by the transition functions $\{s_{ij} : P_{ij} \to \Xi^{u_0}_{u_0}\}_{i,j\in I}$ of these sections. The isometablicity of these functions is expressed by

$$s_{ij}(ug) = \varphi_{ij}(g)(s_{ij}(u)).$$

Here, the φ_{ij}s are the actions of G on $\Xi^{u_0}_{u_0}$ defined by

$$\varphi_{ij}(g)(h) = \sigma_i(u_i g)^{-1} \cdot (\xi_i g) \cdot (hg) \cdot (\xi_j g)^{-1} \cdot \sigma_j(u_j g).$$

Note that the φ_{ij}s are just actions, *not* representations of G on $\Xi^{u_0}_{u_0}$. That is because for every $g \in G$, the map $\varphi_{ij}(g) : \Xi^{u_0}_{u_0} \to \Xi^{u_0}_{u_0}$ does not preserve the multiplication on $\Xi^{u_0}_{u_0}$. Instead, it is a straightforward calculation that for $P_{ijk} \neq \emptyset$ they satisfy

$$\varphi_{ij}(g)(h_1 h_2) = \varphi_{ik}(g)(h_1)\varphi_{kj}(g)(h_2).$$

This property was called a *cocycle morphism* in [2], and it was shown that such data (that is to say cocycle morphisms $\varphi = \{\varphi_{ij} : G \times H \to H\}_{i,j\in I}$, together with a φ-isometablic cocycle $\{s_{ij} : P_{ij} \to H\}_{i,j\in I}$, where H is a Lie group) classifies PBG-groupoids. Let us recall briefly the construction of a PBG-groupoid from such data.

Take X to be the disjoint union of the manifolds $P_i \times H \times P_j$ and define on X the equivalence relation

$$(j, u, h, v, i) \cong (j'.u', h', v', i') \Leftrightarrow u = u',\ v = v' \ and\ h' = s_{j'j}(u) h s_{ii'}(v).$$

Let Ξ be $X/_{\cong}$ and denote elements of Ξ by $\langle j, u, h, v, i\rangle$. the groupoid structure on Ξ is

$$\alpha(\langle j, u, h, v, i\rangle) = v,\ \ \beta(\langle j, u, h, v, i\rangle) = u,$$

$$1_u = \langle i, u, 1, u, i\rangle\ for\ any\ i\ with\ u \in P_i,$$

and multiplication

$$\langle k, w, h_2, u, j_2\rangle\langle j_1, u, h_1, v, i\rangle = \langle k, w, h_2 s_{j_2 j_1}(u) h_1, v, i\rangle.$$

The group G acts on Ξ by

$$\langle j,u,h,v,i\rangle g = \langle j,ug,\varphi_{ji}(g)(h),vg,i\rangle.$$

Finally, place a manifold structure on Ξ using the charts $P_i \times H \times P_j \to \Xi_{P_i}^{P_j}$, $(u,h,v) \mapsto \langle j,u,h,v,i\rangle$. It is then straightforward to verify that Ξ is a PBG-groupoid over the principal bundle $P(M,G)$. Note that the cocycle morphism property is necessary to show that the G-action respects the groupoid multiplication.

Now the definition of isometablic Čech cohomology with respect to the family of actions φ_{ij} was given in [2, §VII], and it was shown that $\check{H}^1_G(P,H)$ classifies those PBG-groupoids over the principal bundle $P(M,G)$ such that the fiber of the adjoint bundle is the Lie group H.

Let us make a fresh start now, considering a coupling pair crossed modules of PBG-groupoids $\langle F,\partial,\Omega,\rho\rangle$ over the principal bundle $P(M,G)$. Let H denote the fiber type of the PBG-Lie group bundle F, and suppose that $\mathrm{Obs}\langle F,\partial,\Omega,\rho\rangle = 0$. Let $\mathcal{O}\mathrm{pext}\langle F,\partial,\Omega,\rho\rangle$ denote the set of equivalence classes of operator extensions. We define an action of $\check{H}^1_G(P,ZH)$ on $\mathcal{O}\mathrm{pext}\langle F,\partial,\Omega,\rho\rangle$ in the following way:

Consider an operator PBG-groupoid $(F \overset{\iota}{\rightarrowtail} \Upsilon \overset{(\beta,\alpha)}{\twoheadrightarrow} P\times P,\mu)$ for $\langle F,\partial,\Omega,\rho\rangle$, and an element $f \in H^1_G(P\times P,ZH)$. Note that $f : P_{ij}\times P_{ij} \to ZH$ is isometablic in the sense

$$f_{ij}(ug,vg) = \varphi_{ij}(g)(f_{ij}(u,v))$$

for all $g \in G$. Therefore, if $\widehat{s}_{ij}$ are the transition functions of the PBG-groupoid Υ, arising from an isometablic section-atlas $\widehat{\sigma}_i : P_i \to \Upsilon_{u_0}$, the maps $\widehat{s}_{ij}f_{ij} : P_{ij} \to H$ satisfy the cocycle equation and $[s_{ij}f_{ij}](ug) = \varphi_{ij}(g)(s_{ij}f_{ij}(u))$. Moreover, $\partial\circ(\widehat{s}_{ij}f_{ij}) = \partial\circ\widehat{s}_{ij}$.

Proposition 9.1 *The PBG-groupoid $\Upsilon^f \Longrightarrow P(M,G)$ constructed from the $\widehat{s}_{ij}f_{ij}$ s, is an operator PBG-groupoid for $(F \overset{\iota}{\rightarrowtail} \Upsilon \overset{(\beta,\alpha)}{\twoheadrightarrow} P\times P,\mu)$.*

PROOF. Same as [12, 3.4]. ∎

The proof that this action is well defined is exactly the same as in [12, §3], taking into account the isometablicity considerations of section 6 in the present paper. Moreover, applying these considerations to the proof of [12, 3.5], we get the following classification of operator extensions for a coupling pair crossed module of PBG-groupoids:

Theorem 9.2 *The above action of $\check{H}^1_G(P,ZH)$ on the set of operator extensions of a coupling pair crossed module of PBG-groupoids $\langle F,\partial,\Omega,\rho\rangle$ is free and transitive.*

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