

# FUNCTORIALITY IN INTERSECTION THEORY AND A CONJECTURE OF COX, KATZ, AND LEE

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ABSTRACT. A functoriality property of the virtual fundamental class on the moduli of stable maps is proven. The property is used to supply a proof of a conjecture of Cox, Katz, and Lee.

## 1. INTRODUCTION

A recent paper by Cox et al. [3] states the following conjecture on virtual moduli cycles. Let  $X$  be a nonsingular complex projective variety; a vector bundle  $V$  on  $X$  is *convex* if, for every genus zero stable map  $\varphi: C \rightarrow X$ , we have  $H^1(C, f^*V) = 0$ . Over the moduli stack of genus zero stable maps  $\bar{M}_{0,n}(X, \beta)$ , we have the universal curve  $\pi_{n+1}: \bar{M}_{0,n+1}(X, \beta) \rightarrow \bar{M}_{0,n}(X, \beta)$  and evaluation morphism  $e_{n+1}: \bar{M}_{0,n+1}(X, \beta) \rightarrow X$ . If  $V$  is convex, then

$$\mathcal{V}_{\beta,n} := (\pi_{n+1})_* e_{n+1}^* V$$

is a vector bundle on  $\bar{M}_{0,n}(X, \beta)$ .

**Conjecture** (Cox et al. [3]). *Fix  $X$  and  $n$ , and let  $V$  be a convex vector bundle on  $X$ . Denote by  $i: Y \rightarrow X$  the inclusion defined by the zero locus of a regular section of  $V$ , and for  $\gamma \in H_2(Y, \mathbb{Z})$ , denote by  $j_\gamma$  the natural inclusion  $\bar{M}_{0,n}(Y, \gamma) \rightarrow \bar{M}_{0,n}(X, i_*\gamma)$ . Then, for any  $\beta \in H_2(X, \mathbb{Z})$ , we have*

$$\sum_{i_*\gamma=\beta} (j_\gamma)_* [\bar{M}_{0,n}(Y, \gamma)]^{\text{virt}} = c_{\text{top}}(\mathcal{V}_{\beta,n}) \cap [\bar{M}_{0,n}(X, \beta)]^{\text{virt}}, \quad (1)$$

where  $[\ ]^{\text{virt}}$  denotes the virtual fundamental class of Behrend–Fantechi [2].

The given section of  $V$  determines a section  $s: \bar{M}_{0,n}(X, \beta) \rightarrow \mathcal{V}_{\beta,n}$  whose zero locus is precisely the disjoint union of the  $\bar{M}_{0,n}(Y, \gamma)$  in (1). Denote by  $0_{\beta,n}$  the zero section of  $\mathcal{V}_{\beta,n}$ . Since the Conjecture is implied by the statement

$$0_{\beta,n}^! [\bar{M}_{0,n}(X, \beta)]^{\text{virt}} = \sum_{i_*\gamma=\beta} [\bar{M}_{0,n}(Y, \gamma)]^{\text{virt}}, \quad (2)$$

we can regard the Conjecture as an instance of functoriality for the virtual fundamental class.

The purpose of this note is twofold. First, we prove (Section 2) a general functoriality result for the virtual fundamental class of Behrend and Fantechi, strengthening the result appearing in their paper [2, Proposition 7.5]. As a corollary, we obtain a proof, entirely within the framework of the Behrend–Fantechi construction and using essentially the same techniques (perfect obstruction theories, deformation to the normal cone, etc.), of the Conjecture.

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The second purpose is to point out (Section 3) that the Conjecture has actually already been proved—using the virtual class of construction of Li and Tian [10] in place of Behrend and Fantechi! Indeed, the flexibility of the construction of Li and Tian (which involves formal neighborhoods of points and subvarieties of moduli spaces) allows them to obtain a functoriality result, Proposition 3.9 of [10], which is free of the restrictive hypotheses of the functoriality result of Behrend and Fantechi. Hence one obtains a proof of the Conjecture by combining the functoriality result of Li and Tian with the observation that the virtual fundamental class constructed by Behrend and Fantechi reproduces the one constructed by Li and Tian. This last statement, while well-known in some circles, appears to have no proof in the literature, so we also include an argument for its validity in Section 3.

## 2. FUNCTORIALITY OF THE BEHREND–FANTECHI CLASS

We fix a target stack  $\mathfrak{M}$ , algebraic, locally of finite type, and pure-dimensional over a given base field. Stacks  $X$ ,  $Y$ , etc. will always be algebraic and of finite type over the base field. Let  $X$  and  $Y$  be stacks. Let  $X \rightarrow \mathfrak{M}$  and  $Y \rightarrow \mathfrak{M}$  be morphisms of relative Deligne–Mumford type. Let  $u: X \rightarrow Y$  be a morphism which fits into a 2-commutative triangle with the given morphisms to  $\mathfrak{M}$ . Let  $E$  be a perfect relative obstruction theory for  $X$  over  $\mathfrak{M}$  and let  $F$  be a perfect relative obstruction theory for  $Y$  over  $\mathfrak{M}$ . Let  $Z$  and  $W$  be stacks, and let  $v: Z \rightarrow W$  be a local complete intersection morphism of relative Deligne–Mumford type, such that there is a 2-cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ p \downarrow & & \downarrow q \\ Z & \xrightarrow{v} & W \end{array}$$

Recall [2] we say  $E$  and  $F$  are *compatible over  $v$*  if we are supplied with a triple  $(\varphi, \psi, \chi)$  of morphisms giving rise to a morphism of distinguished triangles

$$\begin{array}{ccccccc} u^*F & \xrightarrow{\varphi} & E & \xrightarrow{\psi} & p^*L_{Z/W} & \xrightarrow{\chi} & u^*F[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u^*L_{Y/\mathfrak{M}} & \longrightarrow & L_{X/\mathfrak{M}} & \longrightarrow & L_{X/Y} & \longrightarrow & u^*L_{Y/\mathfrak{M}}[1] \end{array} \quad (3)$$

in the derived category of sheaves over  $X$ .

**Theorem 1.** *If  $E$  and  $F$  are compatible over  $v$ , then*

$$v^! [Y, F] = [X, E],$$

where  $[Y, F]$  and  $[X, E]$  denote the virtual fundamental class of [2, 8].

One needs first a preliminary result on normal cone stacks, and then this result follows in a straightforward manner, following the outline of the proof of functoriality of the Gysin map in standard intersection theory [4, Section 6.5].

**Proposition 1.** *Let  $X$ ,  $Y$ , and  $Z$  be stacks, and let  $i: X \rightarrow Y$  and  $j: Y \rightarrow Z$  be morphisms of relative Deligne–Mumford type. Then there is a natural identification*

$$N_{X \times \mathbb{P}^1 / M_{Y/Z}^{\circ}} \simeq h^1 / h^0(c(f)^{\vee}) \quad (4)$$

where  $c(f)$  is the mapping cone to the morphism  $f := (T \cdot \text{id}, U \cdot \text{can})$  of cotangent complexes on  $X \times \mathbb{P}^1$  (here  $\text{can}$  is the canonical morphism and  $T$  and  $U$  are homogeneous coordinates on  $\mathbb{P}^1$ ):

$$i^* L_{Y/Z} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{f} i^* L_{Y/Z} \oplus L_{X/Z}.$$

*Proof.* We use notation  $N_{X/Y}$  for normal sheaf and  $M_{X/Y}^\circ$  for deformation stack; these are defined for a morphism  $X \rightarrow Y$  of relative Deligne–Mumford type. Recall [4, 8] that  $M_{X/Y}^\circ$  is a stack over  $\mathbb{P}^1$  whose general fiber is isomorphic to  $Y$  and whose fiber over a chosen point, which by convention we take to be  $\{0\}$ , is the normal cone stack  $C_{X/Y}$  (but note  $M_{X/Y}^\circ$  may not be an algebraic stack; it does, however, have representable finite-type locally separated diagonal, and it admits a smooth cover by a scheme). The abelian hull of  $C_{X/Y}$  is the abelian cone stack  $N_{X/Y}$ . Always,  $L_{X/Y}$  denotes the cotangent complex, defined for a general morphism of algebraic stacks [9]. We prove the result by treating successively more general cases.

*Case 1.* The morphisms  $X \rightarrow Y$  and  $Y \rightarrow Z$  are closed immersions. The left- and right-hand sides of (4) are abelian cones, and  $h^1/h^0(c(f)^\vee) = \text{Spec Sym coker}(h^{-1}(f))$ . Denote the ideal sheaves to  $Y$  in  $Z$ , resp. to  $X$  in  $Z$ , by  $\mathcal{J}$ , resp.  $\mathcal{K}$ . The stack  $M_{Y/Z}^\circ$  is affine over  $Z \times \mathbb{P}^1$ , being the identity over  $Z \times (\mathbb{P}^1 \setminus \{0\})$  and given over  $Z \times \mathbb{A}^1 = \text{Spec } \mathcal{O}_Z[T]$  as

$$\text{Spec}(\cdots \oplus \mathcal{J}^2 T^{-2} \oplus \mathcal{J} T^{-1} \oplus \mathcal{O}_Z \oplus \mathcal{O}_Z T \oplus \mathcal{O}_Z T^2 \oplus \cdots).$$

The ideal sheaf of the morphism  $X \times \mathbb{P}^1 \rightarrow M_{Y/Z}^\circ$ , restricted to  $X \times \mathbb{A}^1$ , is

$$\tilde{\mathcal{K}} := \cdots \oplus \mathcal{J}^2 T^{-2} \oplus \mathcal{J} T^{-1} \oplus \mathcal{K} \oplus \mathcal{K} T \oplus \mathcal{K} T^2 \oplus \cdots,$$

and hence

$$\tilde{\mathcal{K}}/\tilde{\mathcal{K}}^2 = (\mathcal{J}/\mathcal{J}\mathcal{K})T^{-1} \oplus (\mathcal{K}/\mathcal{K}^2) \oplus (\mathcal{K}/\mathcal{K}^2)T \oplus \cdots.$$

So there is an epimorphism  $(\mathcal{J}/\mathcal{J}\mathcal{K} \oplus \mathcal{K}/\mathcal{K}^2) \otimes \mathcal{O}_X[T] \rightarrow \tilde{\mathcal{K}}/\tilde{\mathcal{K}}^2$  on  $X \times \mathbb{A}^1$ , and  $\mathcal{J}/\mathcal{J}\mathcal{K} \otimes \mathcal{O}_X[T]$  maps onto the kernel. This epimorphism extends to one defined on all of  $X \times \mathbb{P}^1$ , where now  $\mathcal{J}/\mathcal{J}\mathcal{K} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$  maps by the indicated map  $f$  onto the kernel.

*Case 2.* The morphisms  $X \rightarrow Y$  and  $Y \rightarrow Z$  are representable and unramified (i.e., local embeddings). As in Case 1, the result is an assertion that two sheaves are isomorphic. So it is enough to reason locally, i.e., we may assume  $Z$  is a scheme, and now replacing  $X$ ,  $Y$ , and  $Z$  by étale covers, we are reduced to Case 1.

*Case 3.* There is a morphism from  $X$  to a smooth scheme  $V$  such that the induced  $X \rightarrow Y \times V$  is a local embedding, and  $j$  factors (up to 2-isomorphism) as  $Y \rightarrow Z' \rightarrow Z$ , with  $Z' \rightarrow Z$  smooth and representable, and  $Y \rightarrow Z'$  a local embedding. Let  $Z'' = Z' \times_Z Z'$ . We have  $M_{Y/Z'}^\circ \rightarrow M_{Y/Z}^\circ$  smooth and representable, and the fiber product of  $M_{Y/Z'}^\circ$  with itself over  $M_{Y/Z}^\circ$  is identified with  $M_{Y/Z''}^\circ$ . In the factorization  $X \times \mathbb{P}^1 \rightarrow M_{Y \times V/Z' \times V}^\circ \rightarrow M_{Y/Z}^\circ$ , the first map is a local embedding, and hence

$$N_{X \times \mathbb{P}^1/M_{Y/Z}^\circ} = [N_{X \times \mathbb{P}^1/M_{Y \times V/Z' \times V}^\circ} \rightrightarrows N_{X \times \mathbb{P}^1/M_{Y \times V/Z' \times V}^\circ}]. \quad (5)$$

By Case 2, each normal sheaf on the right-hand side of (5) is known, and now by standard exact sequences of conormal sheaves, we have

$$N_{X \times \mathbb{P}^1/M_{Y/Z}^\circ} = h^1/h^0([\mathcal{C} \rightarrow \Omega_{Z'/Z}(1) \oplus \Omega_V]^\vee),$$

where  $\mathcal{C} = \text{coker}(i^*\mathcal{N}_{Y/Z'}^\vee(-1) \rightarrow i^*\mathcal{N}_{Y/Z'}^\vee \oplus \mathcal{N}_{X/Z' \times V}^\vee)$ . The exact sequence of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & i^*\mathcal{N}_{Y/Z'}^\vee(-1) & \longrightarrow & i^*\mathcal{N}_{Y/Z'}^\vee \oplus \mathcal{N}_{X/Z' \times V}^\vee & \longrightarrow & \mathcal{C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{Z'/Z}(-1) & \longrightarrow & \Omega_{Z'/Z} \oplus \Omega_{Z'/Z} \oplus \Omega_V & \longrightarrow & \Omega_{Z'/Z}(1) \oplus \Omega_V \longrightarrow 0 \end{array}$$

identifies this cone stack with  $h^1/h^0(c(f)^\vee)$ .

*Case 4.* The general case. Choose a smooth atlas  $Z_0$  for  $Z$ . Let  $Y_0$  be an affine scheme which is an étale atlas for  $Y \times_Z Z_0$ , and factor  $Y_0 \rightarrow Z_0$  as a local embedding  $Y_0 \rightarrow Z'_0$  followed by a smooth representable morphism  $Z'_0 \rightarrow Z_0$  (e.g., by taking  $Z'_0 = Z_0 \times \mathbb{A}^n$  for some  $n$ ). Choose an affine scheme  $X_0$  which is an étale atlas for  $X \times_Y Y_0$ , and choose a smooth scheme  $V$  into which  $X_0$  embeds. Denote  $X_0 \times_X X_0$  by  $X_1$  and similarly for  $Y_1$  and  $Z_1$ ; let  $Z'_1 = Z'_0 \times_Z Z_0$ . Factoring  $X \times \mathbb{P}^1 \rightarrow M_{Y/Z}^\circ$ , we have

$$N_{X \times \mathbb{P}^1 / M_{Y/Z}^\circ} = [N_{X_1 \times \mathbb{P}^1 / M_{Y_1 \times V \times V / Z'_1 \times V \times V}^\circ} \rightrightarrows N_{X_0 \times \mathbb{P}^1 / M_{Y_0 \times V / Z'_0 \times V}^\circ}].$$

By Case 2, the right-hand side is identified with

$$[\text{Spec Sym } \mathcal{D} \rightrightarrows \text{Spec Sym } \mathcal{C}], \quad (6)$$

where  $\mathcal{C}$  is as in Case 3 (but with  $X, Y, Z'$  replaced by  $X_0, Y_0, Z'_0$  respectively) and where  $\mathcal{D} = \text{coker}(i^*\mathcal{N}_{Y_1/Z'_1}^\vee(-1) \rightarrow i^*\mathcal{N}_{Y_1/Z'_1}^\vee \oplus \mathcal{N}_{X_1/Z'_1 \times V \times V}^\vee)$ . Now  $\text{Spec Sym } \mathcal{C}$  serves as a smooth atlas for  $h^1/h^0(c(f)^\vee)$ , and with this atlas, (6) is the groupoid presentation for the cone stack  $h^1/h^0(c(f)^\vee)$ . One verifies, further, the compatibility of the remaining morphisms in the groupoid presentation, and the proof is complete.  $\square$

*Proof of Theorem 1.* Let  $N = g^*N_{Z/W}$ . Consider the vector bundle stacks

$$\begin{aligned} \rho: & \quad h^1/h^0(E^\vee) \rightarrow X, \\ \sigma: & \quad N \oplus u^*(h^1/h^0(F^\vee)) \rightarrow X, \\ \pi: & \quad h^1/h^0(F^\vee) \rightarrow Y. \end{aligned}$$

By homotopy invariance for vector bundle stacks, flat pullbacks by  $\rho, \sigma$ , and  $\pi$  induce isomorphisms on Chow groups [8]. We have, by Behrend and Fantechi [2] and Kresch [8],

$$[Y, F] = (\pi^*)^{-1}([C_{Y/\mathfrak{M}}]) \quad \text{and} \quad [X, E] = (\rho^*)^{-1}([C_{X/\mathfrak{M}}]).$$

Let  $C_0 = C_{Y/\mathfrak{M}}$ . Then the normal cone stack  $C_{X/C_0}$  naturally embeds in  $N \oplus u^*(h^1/h^0(F^\vee))$ . We see that  $v^1([Y, F])$  is represented by the cycle  $[C_{X/C_0}]$  in the vector bundle stack  $N \oplus u^*(h^1/h^0(F^\vee))$  by essentially the argument of [4, Section 6.5]: replace  $[C_0]$  by  $\pi^*$  of a cycle on  $Y$  representing the class  $[Y, F]$ , argue by local analysis that the corresponding cone in  $N \oplus u^*(h^1/h^0(F^\vee))$  represents  $v^1([Y, F])$ , and now use the fact that the Fulton–MacPherson construction respects equivalence in Chow groups. So we are reduced to showing

$$(\sigma^*)^{-1}([C_{X/C_0}]) = (\rho^*)^{-1}([C_{X/\mathfrak{M}}]) \quad (7)$$

in  $A_*X$ .

Introduce the double deformation space  $M_{X \times \mathbb{P}^1 / M_{Y/\mathfrak{M}}}^{\circ} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ , with fiber  $C_{X \times \mathbb{P}^1 / M_{Y/\mathfrak{M}}}^{\circ}$  over  $\{0\} \times \mathbb{P}^1$ . Consider the rational equivalence on the double deformation space coming from  $\mathbb{P}^1 \times \{0\} \sim \mathbb{P}^1 \times \{1\}$ ; this rational equivalence, intersected [7] with the divisor  $C_{X \times \mathbb{P}^1 / M_{Y/\mathfrak{M}}}^{\circ}$ , is a rational equivalence

$$[C_{X/C_0}] \sim [C_{X/\mathfrak{M}}] \quad \text{on } C_{X \times \mathbb{P}^1 / M_{Y/\mathfrak{M}}}^{\circ}. \quad (8)$$

The normal cone stack  $C_{X \times \mathbb{P}^1 / M_{Y/\mathfrak{M}}}^{\circ}$  is a substack of its abelian hull, and the abelian hull  $N_{X \times \mathbb{P}^1 / M_{Y/\mathfrak{M}}}^{\circ}$  is in turn identified with the abelian cone stack  $h^1/h^0(c(f)^{\vee})$  of Proposition 1. Letting  $g = (T \cdot \text{id}, U \cdot \varphi)$ , now, from (3) we get a morphism of distinguished triangles

$$\begin{array}{ccccccc} u^*F(-1) & \xrightarrow{g} & u^*F \oplus E & \longrightarrow & c(g) & \longrightarrow & u^*F(-1)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u^*L_{Y/\mathfrak{M}}(-1) & \xrightarrow{f} & u^*L_{Y/\mathfrak{M}} \oplus L_{X/\mathfrak{M}} & \longrightarrow & c(f) & \longrightarrow & u^*L_{Y/\mathfrak{M}}(-1)[1] \end{array}$$

over  $X \times \mathbb{P}^1$ . So, by functoriality of the  $h^1/h^0$  construction, the rational equivalence (8) pushes forward to a vector bundle stack over  $X \times \mathbb{P}^1$  whose fiber over  $\{0\} \in \mathbb{P}^1$  is  $\sigma$  and whose fiber over  $\{1\} \in \mathbb{P}^1$  is  $\rho$ , and (7) is established.  $\square$

We return to the Conjecture:  $X$  is a nonsingular projective variety with convex vector bundle  $V$  and section of  $V$  whose zero locus is nonsingular  $Y \subset X$ , and there is the inclusion  $r: MY \rightarrow MX$ , where  $MX$  denotes  $\bar{M}_{0,n}(X, \beta)$  and  $MY$  denotes  $\coprod_{\gamma} \bar{M}_{0,n}(Y, \gamma)$ , the disjoint union over all  $\gamma \in H_2(Y, \mathbb{Z})$  whose image in  $H_2(X, \mathbb{Z})$  equals  $\beta$ . We have a compatibility of distinguished triangles

$$\begin{array}{ccccccc} r^*(R\pi'_*e'^*T_X)^{\vee} & \longrightarrow & (R\pi_*e^*T_Y)^{\vee} & \longrightarrow & r^*\mathcal{V}_{\beta,n}^{\vee}[1] & \longrightarrow & r^*(R\pi'_*e'^*T_X)^{\vee}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ r^*L_{MX/\mathfrak{M}} & \longrightarrow & L_{MY/\mathfrak{M}} & \longrightarrow & L_{MY/MX} & \longrightarrow & r^*L_{MX/\mathfrak{M}}[1] \end{array}$$

where  $\pi'$  and  $e'$  denote the projection and evaluation morphisms from the Introduction, with  $\pi$  and  $e$  their restrictions over  $MY$ , and where  $\mathfrak{M} = \mathfrak{M}_{0,n}$ . Applying Theorem 1 with  $v = 0_{\beta,n}$  establishes the Conjecture.

### 3. COMPARING VIRTUAL FUNDAMENTAL CLASSES

In this section, the base field  $k$  is assumed to be algebraically closed. Let  $M$  be a Deligne–Mumford stack of finite type over  $k$ ; “points” of  $M$  always refer to isomorphism classes of objects of  $M$  over  $\text{Spec } k$ . We work with the parallel notions of perfect obstruction theory on  $M$  from [2] and perfect tangent-obstruction complex from [10]. Here we record a proof of the (already known) observation that the virtual fundamental cycle classes constructed in these two papers coincide.

Let us start with the framework of Behrend and Fantechi [2]. Let  $E \in D(\mathcal{O}_M)$ , with  $\varphi: E \rightarrow L_M$ , be a perfect obstruction theory, with virtual fundamental class  $[M, E] \in A_*M$ . We have to assume  $E$  is globally presented by a two-term locally free complex to draw a parallel with the construction of [10]. So, let us assume there exist locally free coherent sheaves  $\mathcal{G}_1$  and  $\mathcal{G}_2$  such that

$$E^{\vee} \cong [\mathcal{G}_1 \rightarrow \mathcal{G}_2] \quad (9)$$

(the indexing of the sheaves  $\mathcal{G}_i$  is meant to be consistent with [10]). At a given point  $p$  of  $M$ , then, the tangent space is

$$T_1 = (T_1)_p = \ker((\mathcal{G}_1)_p \rightarrow (\mathcal{G}_2)_p)$$

and the obstruction space is

$$T_2 = (T_2)_p = \operatorname{coker}((\mathcal{G}_1)_p \rightarrow (\mathcal{G}_2)_p).$$

The intrinsic obstruction space of  $M$  at the point,  $\operatorname{Ext}^1(L_M|_p, \mathcal{O}_p)$ , is then naturally a subspace of  $T_2$ . With  $C_M = C_{M/\operatorname{Spec} k}$ , the intrinsic normal cone, the class  $[M, E]$  is the intersection of  $[C_M] \in A_* h^1/h^0(E^\vee)$  with the zero section of  $h^1/h^0(E^\vee) \rightarrow M$ . By (9),  $h^1/h^0(E^\vee)$  can be identified with the stack quotient  $[\operatorname{Vect}(\mathcal{G}_2)/\mathcal{G}_1]$ . Let us denote by

$$\pi: \operatorname{Vect}(\mathcal{G}_2) \rightarrow [\operatorname{Vect}(\mathcal{G}_2)/\mathcal{G}_1] \quad (10)$$

the morphism to the stack quotient. Then, we have

$$[M, E] = 0_{\operatorname{Vect}(\mathcal{G}_2)}^*(\pi^*[C_M]),$$

where  $0_{\operatorname{Vect}(\mathcal{G}_2)}^*$  denotes the intersection with the zero section of the vector bundle whose sheaf of sections is  $\mathcal{G}_2$ .

Now let us pass to the setting of the virtual fundamental class construction of Li and Tian [10]. Given the perfect obstruction theory  $\varphi: E \rightarrow L_M$  and the global presentation (9), it follows that  $h^*(\mathcal{G}^*)$  is a perfect tangent-obstruction complex as in [10]. Now Li and Tian use relative Kuranishi families to produce, canonically, a cycle  $[C^{\mathcal{G}^*}]$  on  $\operatorname{Vect}(\mathcal{G}_2)$ . The virtual fundamental class of Li and Tian is  $0_{\operatorname{Vect}(\mathcal{G}_2)}^*([C^{\mathcal{G}^*}])$ .

**Proposition 2.** *Let  $\mathcal{G}^* = [\mathcal{G}_1 \rightarrow \mathcal{G}_2]$  be a two-term complex of locally free coherent sheaves on a Deligne–Mumford stack  $M$ , and let  $\varphi: [\mathcal{G}^*]^\vee \rightarrow L_M$  be a perfect obstruction theory. Let  $[C^{\mathcal{G}^*}]$  be the virtual normal cone of Li and Tian for the perfect tangent-obstruction complex  $h^*(\mathcal{G}^*)$ . Then we have the equality*

$$[C^{\mathcal{G}^*}] = \pi^*[C_M]$$

of cycles on  $\operatorname{Vect}(\mathcal{G}_2)$ , where  $C_M$  is the intrinsic normal cone of  $M$  and  $\pi$  is the morphism to the stack quotient (10).

**Corollary 1.** *Given a perfect obstruction theory on a Deligne–Mumford stack which admits a global presentation, the virtual fundamental class constructed by Behrend and Fantechi [2] is equal to the virtual cycle class of Li and Tian for the corresponding tangent-obstruction complex [10].*

We remark that Li and Tian [10] assume  $\operatorname{char} k = 0$ , but this is essential only so that the moduli stacks  $\bar{M}_{g,n}(X, \beta)$  will be of Deligne–Mumford type. The construction is valid for Deligne–Mumford stacks over an algebraically closed field of arbitrary characteristic.

*Proof of Proposition 2.* Let  $p$  be a point of  $M$ . Let  $T_1$  and  $T_2$  be the tangent space and obstruction space, respectively, at  $p$ . Denote by  $\hat{p}$  the formal neighborhood of  $p$  in (any étale atlas for)  $M$ , so we have  $u: \hat{p} \rightarrow M$ , a formally étale morphism. There exists a closed immersion

$$\hat{p} \rightarrow \operatorname{Spec} \operatorname{Sym}(T_1^\vee)$$

defined by some ideal  $I$ , so then,

$$\tau_{\geq -1} L_{\hat{p}} = [I/I^2 \rightarrow T_1^\vee \otimes \mathcal{O}_{\hat{p}}].$$

We have  $u^*E = [u^*\mathcal{G}_2^\vee \rightarrow u^*\mathcal{G}_1^\vee]$ , and we replace this with a quasi-isomorphic complex as follows. Because  $\varphi: E \rightarrow L_M$  is a perfect obstruction theory, there is an epimorphism  $u^*\mathcal{G}_1^\vee \rightarrow \Omega_{\hat{p}}$ , and hence the natural map  $T_1^\vee \otimes \mathcal{O}_{\hat{p}} \rightarrow \Omega_{\hat{p}}$  can be lifted to a morphism  $T_1^\vee \otimes \mathcal{O}_{\hat{p}} \rightarrow u^*\mathcal{G}_1^\vee$ . Now, in the standard way, we obtain a two-term complex with second term  $T_1^\vee \otimes \mathcal{O}_{\hat{p}}$ , quasi-isomorphic to  $u^*E$ ; moreover, we can identify the first term with  $T_2^\vee \otimes \mathcal{O}_{\hat{p}}$ . Since the terms are free, the morphism  $u^*E \rightarrow u^*L_M = L_{\hat{p}}$  in the derived category is realized by a morphism of complexes. In summary, we have

$$\begin{array}{ccc}
 u^*\mathcal{G}_2^\vee & \longrightarrow & u^*\mathcal{G}_1^\vee \\
 \uparrow & & \uparrow \\
 T_2^\vee \otimes \mathcal{O}_{\hat{p}} & \longrightarrow & T_1^\vee \otimes \mathcal{O}_{\hat{p}} \\
 \downarrow & & \parallel \\
 I/I^2 & \longrightarrow & T_1^\vee \otimes \mathcal{O}_{\hat{p}}
 \end{array} \tag{11}$$

where the map from the middle row to the top row is a quasi-isomorphism and the map to the bottom row is given by the perfect obstruction theory, restricted to  $\hat{p}$ .

Let  $S = \text{Spec } k$  and  $Z = \hat{p}$ . Now, if  $f$  denotes any lift  $T_2^\vee \rightarrow I \subset \text{Sym}(T_1^\vee)$  of the map  $T_2^\vee \rightarrow I/I^2$  coming from the diagram (11), then the ideal  $(f)$  generated by the image of  $f$  is equal to  $I$ ; it is straightforward, now, to verify that the pair consisting of the map  $f$  and the resulting isomorphism  $\text{Spec Sym}(T_1^\vee)/(f) \rightarrow Z$  is a relative Kuranishi family for  $Z/S$  [10, Definition 2.3]. Hence the cone produced by Li and Tian from this Kuranishi family is the normal cone

$$C^f = C_{\hat{p}/\text{Spec Sym } T_1^\vee},$$

with closed immersion  $C^f \rightarrow \text{Vect}(T_2) \times \hat{p}$  determined by the map  $T_2^\vee \otimes \mathcal{O}_{\hat{p}} \rightarrow I/I^2$ . As in [2], the cone  $C^f$  is  $T_1$ -equivariant, and the stack quotient can be identified with  $C_M \times_M \hat{p}$ .

Let  $j$  denote the vector bundle surjection  $\text{Vect}(\mathcal{G}_2) \times_M \hat{p} \rightarrow \text{Vect}(T_2) \times \hat{p}$  (coming from the morphism in the diagram (11)). Then,

$$j^*([C^f]) = [\pi^{-1}(C_M) \times_M \hat{p}]. \tag{12}$$

But according to Li and Tian ([10], Remark after Corollary 3.5), the virtual normal cone  $[C^{\mathcal{G}^*}]$  is characterized as follows: For every point  $p$ , there is a vector bundle surjection  $j$  extending  $(\mathcal{G}_2)_p \rightarrow T_2$ , such that  $j^*([C^f]) = r^*[C^{\mathcal{G}^*}]$ , where  $r: \text{Vect}(u^*\mathcal{G}_2) \rightarrow \text{Vect}(\mathcal{G}_2)$  is the induced morphism. So, from (12), we have  $[C^{\mathcal{G}^*}] = \pi^*[C_M]$ .  $\square$

Recall, under the Behrend–Fantechi approach,  $[\bar{M}_{g,n}(X, \beta)]^{\text{virt}}$  is defined using a *relative* perfect obstruction theory over  $\mathfrak{M}_{g,n}$ , the (Artin) stack of prestable  $n$ -pointed genus  $g$  curves [1]. The issue of relative versus absolute perfect obstruction theory was dealt with in [6]; we summarize as follows.

**Proposition 3.** *Let  $M$  be a finite-type Deligne–Mumford stack and  $\tau: M \rightarrow \mathfrak{M}$  a morphism to a smooth Artin stack  $\mathfrak{M}$  which is locally of finite type and of pure dimension. Let  $\varphi: E \rightarrow L_{M/\mathfrak{M}}$  be a relative perfect obstruction theory. If  $h$  denotes the composite  $E \rightarrow L_{M/\mathfrak{M}} \rightarrow \tau^*L_{\mathfrak{M}}[1]$  and we set  $F = c(h)[-1]$  (the shifted mapping cone), then the induced  $\psi: F \rightarrow L_M$  is a perfect obstruction theory. Moreover,  $\varphi$  and  $\psi$  determine the same virtual class in  $A_*M$ .*

*Proof.* As in the proof of [2, Proposition 3.14], we have a short exact sequence of cone stacks

$$h^1/h^0(\tau^*L_{\mathfrak{M}}^\vee[-1]) \rightarrow C_{M/\mathfrak{M}} \rightarrow C_M,$$

as well as similar exact sequence relating  $h^1/h^0(E^\vee)$  with  $h^1/h^0(F^\vee)$ . Now  $[C_{M/\mathfrak{M}}]$  is the pullback of  $[C_M]$  along the (smooth) projection  $h^1/h^0(E^\vee) \rightarrow h^1/h^0(F^\vee)$ , hence the virtual classes agree.  $\square$

A perfect obstruction theory gives rise to an obstruction theory in the usual sense, i.e., obstruction classes for square-zero extensions. For  $M = \bar{M}_{g,n}(X, \beta)$  and  $\mathfrak{M} = \mathfrak{M}_{g,n}$ , it is routine to check that the resulting obstruction theory coincides with the standard one (described, e.g., in [10, Proposition 1.5]). Now, Corollary 1 coupled with Proposition 3 gives:

**Corollary 2.** *Let  $X$  be a nonsingular projective variety over an algebraically closed field  $k$  of characteristic zero, and let  $\beta$  be an element in the group of one-dimensional cycles on  $X$  modulo algebraic equivalence. Then the virtual moduli cycle class  $LT_{g,n}(X, \beta)$  defined by Li and Tian [10] is equal to the virtual fundamental class  $[\bar{M}_{g,n}(X, \beta)]^{\text{virt}}$  of Behrend and Fantechi [1, 2].*

The functoriality result in the Li–Tian setting, Proposition 3.9 of [10], now implies the form of the Conjecture where the left- and right-hand sides of (1) are interpreted as virtual fundamental classes in the sense of [10]. Since these are equal to the virtual classes of Behrend and Fantechi, the Conjecture is proved in its original form. To apply Li–Tian’s functoriality result, it remains only to note that their technical hypothesis is satisfied.

This is routine, but we provide details which, also, serve to make the machinery of Section 2 more concrete. We start with a general fact about moduli spaces of rational maps in genus zero. Let  $Y$  be any nonsingular complex projective variety, and choose an embedding  $j: Y \rightarrow P$  into a convex variety  $P$  (e.g., a projective space). Fix a class  $\beta \in H_2(Y, \mathbb{Z})$  and an integer  $n$ . Denote by  $N_Y$  the normal bundle to  $Y$  in  $P$ . Convexity of  $P$  implies  $\pi_*e^*N_Y$  is locally free, where  $\pi$  and  $e$  are the projection and evaluation morphisms, respectively, from the universal curve over  $MY := \bar{M}_{0,n}(Y, \beta)$ . Denote the universal sections of  $\pi$  by  $s_i$ ,  $i = 1, \dots, n$ . Then the relative, resp. absolute, obstruction theory on  $MY$  is represented by

$$[(\pi_*e^*N_Y)^\vee \rightarrow (\pi_*e^*j^*T_P)^\vee],$$

respectively

$$[(\pi_*e^*N_Y)^\vee \rightarrow \text{Ext}^1([e^*j^*\Omega_P \rightarrow \Omega_\pi(\sum s_i)], \mathcal{O})^\vee] =: [\mathcal{E}_2^\vee \rightarrow \mathcal{E}_1^\vee],$$

where in both cases the morphism to the cotangent complex involves a natural map  $(\pi_*e^*N_Y)^\vee \rightarrow \mathcal{I}/\mathcal{I}^2$ , with  $\mathcal{I}$  the ideal sheaf to  $MY$  in  $MP := \bar{M}_{0,n}(P, j_*\beta)$ .

The technical hypothesis is that the compatibility of tangent-obstruction complexes is given by a surjective map of two-term complexes with kernel  $[0 \rightarrow r^*\mathcal{V}_{\beta,n}]$ , with  $r: MY \rightarrow MX$  the inclusion, as before. In the setting of the Conjecture, such a map of complexes is  $\mathcal{E}^* \rightarrow [\mathcal{E}_1 \rightarrow \pi_*e^*i^*N_X]$ .

Let us return briefly to the approach taken in Section 2. We have, now,

$$[MY]^{\text{virt}} = 0_{\pi_*e^*N_Y}^*[C_{MY/MP}],$$

and similarly for  $[MX]^{\text{virt}}$ . So, the Conjecture can be deduced from intersection theory on Deligne–Mumford stacks [5, 11] coupled with Proposition 1 applied to the



sequence of morphisms  $MY \rightarrow MX \rightarrow MP$ . Note that the statement and the proof of Proposition 1 in the case of a sequence of closed immersions make no reference to cone stacks. Now one who so wishes can reprove the Conjecture using only normal cones, coherent sheaves, and vector bundles on Deligne–Mumford stacks; it is a routine matter to fill in details.

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