ARITHMETIC PROPERTIES OF EQUIVARIANT BIRATIONAL TYPES

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Abstract. We study arithmetic properties of equivariant birational types introduced by Kontsevich, Pestun, and the second author.

1. Introduction

Let G be a finite abelian group and k an algebraically closed field of characteristic zero. Investigations of obstructions to G-equivariant birationality over k led to the definition, in [2], of new invariants of actions of G on algebraic varieties X defined over k. These invariants were further developed in [4], where specialization maps were defined, generalizing the ones from the non-equivariant setting [3].

The invariants from [2] are computed on a suitable smooth projective model X, where G acts regularly. To such an action one associates a formal sum

$$[X \circlearrowleft G] := \sum_{\alpha} \beta_\alpha, \tag{1.1}$$

where the sum is over components of the fixed point locus $F_\alpha \subset X^G$, and β_α are the characters of G appearing in the tangent bundle to a point $x_\alpha \in F_\alpha$. Equivariant birational maps can be factored into sequences of blowups (and blowdowns) of smooth G-stable subvarieties, thanks to equivariant weak factorization. To obtain an equivariant birational invariant, one imposes relations on the formal sums in (1.1), of the type

$$[\tilde{X} \circlearrowleft G] - [X \circlearrowleft G] = 0, \tag{1.2}$$

for every equivariant blowup $\tilde{X} \to X$.

This construction motivated the introduction of two closely related quotients of the free abelian group $\mathcal{S}_n(G)$, generated by symbols

$$\beta = [a_1, \dots, a_n] = [a_{\sigma(1)}, \dots, a_{\sigma(n)}], \quad \forall \sigma \in \mathfrak{S}_n, \tag{1.3}$$

where β is an n-dimensional *faithful* representation of G over k, i.e., a collection of characters $a_1, \dots, a_n$ of G, up to permutation, spanning the character group of G. This group receives the formal sums as in (1.1).

Date: December 7, 2020.

Consider the following relations on elements of $\mathcal{S}_n(G)$:

(**B**) **Blow-up:** for all $[a_1, a_2, b_1, \dots, b_{n-2}] \in \mathcal{S}_n(G)$ one has

$$[a_1, a_2, b_1, \dots, b_{n-2}] = \begin{cases} [a_1, a_2 - a_1, b_1, \dots, b_{n-2}] + [a_1 - a_2, a_2, b_1, \dots, b_{n-2}], & a_1 \neq a_2, \\ [0, a_1, b_1, \dots, b_{n-2}], & a_1 = a_2. \end{cases} \tag{1.4}$$

Let $\mathcal{B}_n(G)$ be the quotient by these relations, and

$$\mathsf{b} : \mathcal{S}_n(G) \to \mathcal{B}_n(G)$$

the corresponding projection homomorphism. One of the main results in [2] is the following

Theorem 1.1. *Let X be a smooth projective algebraic variety of dimension n over k, with a regular action of G. The class*

$$[X \circlearrowleft G] \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

In other words, *all* relations from (1.2) are implied by relations (**B**), which explains the terminology *blow-up relations*.

Numerical experiments revealed an interesting structure of *another* quotient

$$\mathfrak{m} : \mathcal{S}_n(G) \to \mathcal{M}_n(G),$$

by similar relations:

(**M**) **Modular blow-up:** for all $[a_1, a_2, b_1, \dots, b_{n-2}] \in \mathcal{S}_n(G)$ one has

$$\begin{aligned} &[a_1, a_2, b_1, \dots, b_{n-2}] = \\ &\qquad [a_1, a_2 - a_1, b_1, \dots, b_{n-2}] + [a_1 - a_2, a_2, b_1, \dots, b_{n-2}]. \end{aligned} \tag{1.5}$$

To distinguish, we write

$$[a_1, \dots, a_n], \qquad \text{respectively}, \qquad \langle a_1, \dots, a_n \rangle,$$

for the image of a generator in $\mathcal{B}_n(G)$, respectively, the image of a generator in $\mathcal{M}_n(G)$.

When $a_1 \neq a_2$, the relations are *identical*; the only difference is

$$\begin{aligned} [a_1, a_1, \dots, a_n] &= [a_1, 0, \dots, a_n] \quad \in \mathcal{B}_n(G) \\ \langle a_1, a_1, \dots, a_n \rangle &= 2\langle a_1, 0, \dots, a_n \rangle \in \mathcal{M}_n(G). \end{aligned}$$

There is a homomorphism

$$\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G), \quad n \geq 2, \tag{1.6}$$

defined on symbols by:

$$\mu([a_1, \dots, a_n]) := \begin{cases} \langle a_1, \dots, a_n \rangle & \text{if all } a_i \neq 0, \\ 2\langle a_1, \dots, a_n \rangle & \text{if exactly one } a_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [2] it was shown that this map on symbols is compatible with relations. Thus, we have a diagram

$$\begin{array}{ccc} \mathcal{S}_n(G) & \xrightarrow{\mathsf{b}} & \mathcal{B}_n(G) \\ & & \downarrow \mu \\ \mathcal{S}_n(G) & \xrightarrow{\mathsf{m}} & \mathcal{M}_n(G) \end{array} \tag{1.7}$$

Geometric and structural considerations motivated the introduction of an additional relation:

Antisymmetry.

$$[-a_1, \dots, a_n] = -[a_1, \dots, a_n],$$

for all $[a_1, \dots, a_n] \in \mathcal{S}_n(G)$. This is defined only for nontrivial G.

This yields a diagram of homomorphisms

$$\begin{array}{ccc} \mathcal{B}_n(G) & \twoheadrightarrow & \mathcal{B}_n^-(G) \\ \mu \downarrow & & \downarrow \mu^- \\ \mathcal{M}_n(G) & \twoheadrightarrow & \mathcal{M}_n^-(G) \end{array}$$

where the horizontal maps are projections to the corresponding quotients by this additional relation. On symbols, the map μ^- is the same as μ; its compatibility with defining relations is obvious.

In this note, we prove a comparison, left open in [2, Conjecture 8]:

Theorem 1.2. *Both homomorphisms μ and μ^- are isomorphisms, after tensoring with $\mathbb{Q}$.*

This implies that the main constructions connected with $\mathcal{M}_n(G)$, from Sections 4,5,6, and 9 of [2], also apply to $\mathcal{B}_n(G) \otimes \mathbb{Q}$. We briefly sketch these structures:

- **Lattices and cones:** elements $\langle a_1, \dots, a_n \rangle \in \mathcal{M}_n(G)$ can be identified with isomorphism classes of triples
$$(\mathbf{L}, \chi, \Lambda), \tag{1.8}$$
where $\mathbf{L} = \mathbb{Z}^n$ is a lattice, $\chi \in \mathbf{L} \otimes A$, and $\Lambda \subset \mathbf{L} \otimes \mathbb{R}$ is a basic simplicial cone. Here A denotes the character group of G, and by a basic simplicial cone we mean one that is spanned by a basis of $\mathbf{L}$. Concretely, choosing a basis $e_1, \dots, e_n$ of lattice vectors spanning Λ, one can write
$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \quad a_i \in A,$$
and put
$$(\mathbf{L}, \chi, \Lambda) \mapsto \langle a_1, \dots, a_n \rangle.$$
Changing the basis spanning Λ permutes the entries $a_1, \dots, a_n$, and relation (**M**) arises from decompositions of a simplicial cone into simplicial subcones. We will discuss this in more detail in Section 3.
- **Operations:** Given an exact sequence of groups
$$0 \to G' \to G \to G'' \to 0$$
there is a $\mathbb{Z}$-bilinear *multiplication* homomorphism
$$\nabla : \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}(G'') \to \mathcal{M}_{n'+n''}(G), \quad n', n'' \geq 1,$$
which descends to the antisymmetric versions, as well as a *co-multiplication* homomorphism
$$\Delta : \mathcal{M}_{n'+n''}(G) \to \mathcal{M}_{n'}(G') \otimes \mathcal{M}^-_{n''}(G''),$$
(the minus on the second factor is not an error), which also comes with an antisymmetric version
$$\Delta^- : \mathcal{M}^-_{n'+n''}(G) \to \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G'').$$
These homomorphisms allow to decompose $\mathcal{M}_n(G)$ into *primitive* pieces, and reveal a rich internal structure.
- **Hecke operators:** The lattice-theoretic interpretation of $\mathcal{M}_n(G)$ leads to the definition of commuting operators
$$T_{\ell,r} : \mathcal{M}_n(G) \otimes \mathbb{Q} \to \mathcal{M}_n(G)$$
for all $1 \leq r \leq n-1$ and primes ℓ not dividing the order of G. By Theorem 1.2, the groups $\mathcal{B}_n(G) \otimes \mathbb{Q}$ also carry Hecke operators.

- **Cohomology of arithmetic groups:** Let
$$\Gamma(G,n)\subset \mathrm{GL}_n(\mathbb{Z})$$
be the stabilizer of χ in (1.8). Let
  - $\mathcal{F}_n$ be the $\mathbb{Q}$-vector space generated by characteristic functions of convex finitely generated rational polyhedral cones $\Lambda\subset\mathbb{R}^n$, modulo those of dimension $\le n-1$,
  - St_n be the *Steinberg*-module, and
  - or_n be the *sign of the determinant* module.

  By [2, Prop. 22] and Theorem 1.2, we have a commutative diagram
$$\begin{array}{ccc}
\mathcal{M}_n(G)\otimes\mathbb{Q} & \twoheadrightarrow & \mathcal{M}_n^-(G)\otimes\mathbb{Q}\\
\simeq\downarrow & & \downarrow\simeq\\
H_0(\Gamma(G,n),\mathcal{F}_n) & \twoheadrightarrow & H_0(\Gamma(G,n),\mathrm{St}_n\otimes\mathrm{or}_n)\\
\simeq\uparrow & & \uparrow\simeq\\
\mathcal{B}_n(G)\otimes\mathbb{Q} & \twoheadrightarrow & \mathcal{B}_n^-(G)\otimes\mathbb{Q},
\end{array}$$

The connection between the groups $\mathcal{B}_n(G)\otimes\mathbb{Q}$, encoding invariants of abelian actions on algebraic varieties, and the theory of automorphic forms, via cohomology of congruence subgroups, seems intriguing to us. However, given the link between $\mathcal{B}_n(G)$ and $\mathcal{M}_n(G)$ it is natural to seek a lattice theoretic interpretation of $\mathcal{B}_n(G)$ as well. This is done in Section 3. One of the byproducts is the definition of Hecke operators
$$T_{\ell,r}:\mathcal{B}_n(G)\to\mathcal{B}_n(G),$$
where ℓ is a prime not dividing the order of G and $1\le r\le n-1$, over the integers.

Acknowledgments: We are very grateful to Brendan Hassett for his interest and help on this and related projects. The first author was partially supported by the Swiss National Science Foundation. The second author was partially supported by NSF grant 2000099.

2. Comparison

We continue to assume that G is a finite abelian group. This section is closely related to [2, Sections 3, 5, and 11]. In particular, we settle Conjecture 8 from *ibid*, asserting that
$$\mathcal{B}_n(G)\otimes\mathbb{Q}\simeq\mathcal{M}_n(G)\otimes\mathbb{Q}.$$

Our first result is a refinement of [1, Prop. 3.2].

Theorem 2.1. *Let $n \geq 2$.*
(i) *Let p be a prime and $a \in (\mathbb{Z}/p\mathbb{Z})^\times$. The class*

$$[a, 0, \dots] + [-a, 0, \dots] \in \mathcal{B}_n(\mathbb{Z}/p\mathbb{Z})$$

is zero when $p \leq 5$, and is annihilated by $(p^2-1)/24$ when $p \geq 7$.
(ii) *Let $N > 1$ be an integer and $a \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then*

$$[a, 0, \dots] + [-a, 0, \dots] \in \mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})_{\mathrm{tors}},$$

the subgroup of torsion elements.

We start with a sequence of technical lemmas.

Lemma 2.2. *For a, $b \in (\mathbb{Z}/p\mathbb{Z})^\times$ we have*

$$[a, b] + [a, -b] = [a, 0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z}).$$

Proof. We write $b = ma$ with $m \in \{1, \dots, p-1\}$ and proceed by induction on m. The base case $m = 1$ is clear, since

$$[a, a] = [a, 0] \qquad \text{and} \qquad [a, -a] = 0.$$

The induction hypothesis, in combination with

$$[a, (m+1)a] = [a, ma] + [(m+1)a, -ma]$$

and

$$[a, -ma] = [a, -(m+1)a] + [(m+1)a, -ma],$$

gives the inductive step. □

Lemma 2.3. *For a, $b \in (\mathbb{Z}/p\mathbb{Z})^\times$ we have*

$$[a, 0] + [-a, 0] = [a, b] + [a, -b] + [-a, b] + [-a, -b]$$

in $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$, and this element is independent of a and b.

Proof. The equality holds by Lemma 2.2. The right-hand side is symmetric in a and b and, by the equality, is independent of b. Hence it is also independent of a. □

Lemma 2.4. *For a, $b \in (\mathbb{Z}/p\mathbb{Z})^\times$ with $a + b \neq 0$, we have*

$$[a, 0] = [a, b] + [-b, a+b] + [-a-b, a] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z}).$$

Proof. This follows from

$$[a, -b] = [-b, a+b] + [-a-b, a]$$

and $[a, b] + [a, -b] = [a, 0]$. □

Lemma 2.3 tells us that

$$\delta := [a, 0] + [-a, 0] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z}) \tag{2.1}$$

is independent of $a \in (\mathbb{Z}/p\mathbb{Z})^\times$.

Lemma 2.5. *For a, $b \in (\mathbb{Z}/p\mathbb{Z})^\times$ with $a+b \neq 0$, we have*

$$\begin{aligned}\delta = [a,b] + [-b,a+b] + [-a-b,a] \\ + [-a,-b] + [b,-a-b] + [a+b,-a] \in \mathcal{B}_2(\mathbb{Z}/p\mathbb{Z}).\end{aligned}$$

Proof. We add together

$$[a,0] = [a,b] + [-b,a+b] + [-a-b,a]$$

and

$$[-a,0] = [-a,-b] + [b,-a-b] + [a+b,-a],$$

and recognize δ on the left-hand side. □

Lemma 2.6. *We have in $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$:*

$$\sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} [a,a] = \frac{p-1}{2}\delta,$$

$$\sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} [a,-2a] = 0.$$

Proof. We pair summands indexed by a and $-a$ and use $[a,a] = [a,0]$ and the definition of δ to get the first equality. Also, from

$$[a,0] = [a,a] + [-a,2a] + [-2a,a]$$

follows the vanishing of pairs of summands in the second equality. □

Lemma 2.7. *Let β, β', $\beta'' \in (\mathbb{Z}/p\mathbb{Z})^\times \setminus \{-1\}$ with*

$$\beta' = -\beta^{-1} - 1 \qquad \text{and} \qquad \beta'' = -(\beta+1)^{-1}.$$

Then

$$\sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} [a,\beta a] + [a,\beta' a] + [a,\beta'' a] = \frac{p-1}{2}\delta.$$

Furthermore, if $\beta = \beta' = \beta''$ then

$$\sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} [a,\beta a] = \frac{p-1}{6}\delta.$$

Proof. We identify pairs of summands in the first expression with δ. We have $\beta = \beta' = \beta''$ if and only if β is a primitive cube root of unity. Then we may identify 6-tuples of summands with δ to get the second equality. □

Proof of Theorem 2.1. For (i), let $p \geq 5$. We partition $(\mathbb{Z}/p\mathbb{Z})^\times \setminus \{1,-1\}$ into $\{-2,-1/2\}$, the primitive cube roots of unity (which exist only when $p \equiv 1 \bmod 3$), and 6-element sets

$$\{\beta, \beta', \beta'', \beta^{-1}, \beta'^{-1}, \beta''^{-1}\},$$

with distinct β, β', β'' as above. We take a subset

$$I \subset (\mathbb{Z}/p\mathbb{Z})^\times \setminus \{1,-1\},$$

to consist of one of -2, $-1/2$, one primitive cube root of unity if it exists, and β, β', β'' from every 6-element set as above. Then δ from (2.1) satisfies

$$\begin{aligned}\frac{(p-1)(p-2)}{6}\delta &= \sum_{\beta=1}^{p-3}\sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times}[a,\beta a]\\ &= \sum_{\beta\in I}\sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times}[a,(\beta-1)a]+[a,(\beta^{-1}-1)a]\\ &= \sum_{\beta\in I}\sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times}[a,\beta a]\\ &= \frac{(p-1)(p-5)}{12}\delta.\end{aligned}$$

It follows that δ is annihilated by $(p^2-1)/12$. Next we prove annihilation by $(p^2-1)/8$, and thus by $(p^2-1)/24$ as claimed. This is an adaptation of the previous argument: take

$$J \subset (\mathbb{Z}/p\mathbb{Z})^\times \setminus \{1,-1\}$$

to consist of one square root of -1 when $p \equiv 1 \bmod 4$ as well as β and $-\beta$ from every 4-element set

$$\{\beta,-\beta,\beta^{-1},-\beta^{-1}\}.$$

From

$$\begin{aligned}\frac{(p-1)^2}{4}\delta &= \sum_{\beta=1}^{p-3}\sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times}[a,\beta a]\\ &= \sum_{\beta\in J}\sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times}[a,\beta a] = \frac{(p-1)(p-3)}{8}\delta\end{aligned}$$

we get the desired conclusion.

For (ii), we treat composite N, as in the proof of [1, Prop. 3.2]: We recall that, for a,b with $\gcd(a,b,N)=1$, we have

$$\langle a,b\rangle = \begin{cases}[a,b] & \text{when both } a,b\neq 0,\\ \frac{1}{2}[a,0] & \text{when } b=0.\end{cases}$$

In this case, we work with

$$\delta(a,b) := \langle a,b\rangle + \langle -a,b\rangle + \langle a,-b\rangle + \langle -a,-b\rangle \in \mathcal{B}_2(\mathbb{Z}/N\mathbb{Z}).$$

We observe that $\delta(a,b)$ satisfies the blow-up relation (**M**), thus

$$S := \sum_{a,b} \delta(a,b) = 2S.$$

It follows that $S=0$ in $\mathcal{B}_2(\mathbb{Z}/N\mathbb{Z})$. On the other hand, $\delta(a,b)$ is seen to be invariant under $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. This implies that $\delta(a,b)$ is torsion in $\mathcal{B}_2(\mathbb{Z}/N\mathbb{Z})$ (annihilated by the number of summands in S). Substituting $b=0$, we $[a,0]+[-a,0]=0$ in $\mathcal{B}_2(\mathbb{Z}/N\mathbb{Z})\otimes\mathbb{Q}$. □

The following theorem settles Conjectures 8 and 9 of [2]:

Theorem 2.8. *Let $n\geq 3$.*
(i) *Let p be a prime. Then*

$$[0,0,1,\dots]\in\mathcal{B}_n(\mathbb{Z}/p\mathbb{Z})$$

is zero when $p\leq 5$, and is annihilated by $(p^2-1)/24$ when $p\geq 7$.
(ii) *Let G be a finite abelian group. Any element of the form*

$$[0,0,\dots]\in\mathcal{B}_n(G)$$

is a torsion element.

Proof. For (ii) it suffices to consider cyclic $G=\mathbb{Z}/N\mathbb{Z}$. Theorem 2.1 (ii) gives, for $a\in(\mathbb{Z}/N\mathbb{Z})^\times$, that

$$[a,0,c\dots]+[-a,0,c,\dots]$$

is torsion in $\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$. Substituting $c=a$, and using that

$$[a,0,a,\dots]=[0,0,a,\dots]\in\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$$

and

$$[-a,0,a,\dots]=0\in\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$$

we obtain the result. We obtain (i) similarly, from Theorem 2.1 (i). □

Proof of Theorem 1.2. The assertion for μ^- follows immediately from the vanishing of all $[0,a_2,\dots,a_n]$, respectively $\langle 0,a_2,\dots,a_n\rangle$ in $\mathcal{B}_n^-(G)\otimes\mathbb{Q}$, respectively $\mathcal{M}_n^-(G)\otimes\mathbb{Q}$.

To obtain the assertion for μ, we combine Theorem 2.8 (ii) with analogous relations in $\mathcal{M}_n(G)$, stated at the beginning of Section 3 of [2], to show directly that μ induces an isomorphism after tensoring with $\mathbb{Q}$. □

3. Interpretation via lattices

As before, G is a finite abelian group G; we denote by A the character group of G. Our starting point is the free abelian group on triples

$$(\mathbf{L}, \chi, \Lambda),$$

where

- $\mathbf{L} \simeq \mathbb{Z}^n$ is an n-dimensional lattice,
- $\chi \in \mathbf{L} \otimes A$ is an element inducing, by duality, a surjection $\mathbf{L}^\vee \to A$,
- Λ is a basic cone, i.e., a simplicial cone spanned by a basis of $\mathbf{L}$.

Let $\mathbf{T}$ be the quotient of this group by the equivalence relation: two triples are equivalent if they differ by the action of $\mathrm{GL}_n(\mathbb{Z})$. There is a natural map

$$\begin{array}{ccc} \mathbf{T} & \to & \mathcal{S}_n(G), \\ (\mathbf{L}, \chi, \Lambda) & \mapsto & [a_1, \dots, a_n], \end{array}$$

defined by decomposing

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \quad a_i \in A, \tag{3.1}$$

where $\{e_1, \dots, e_n\}$ is a basis of Λ. The symmetry property (1.3) is precisely the ambiguity in the order of generating elements of Λ. Imposing scissor-type relations [2, (4.4)] on $\mathbf{T}$, we obtain a diagram

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{\psi} & \mathcal{M}_n(G) \\ \Big\downarrow \mathsf{s} & \nearrow_{\sim} & \\ \mathbf{T}/(\text{scissor-type relations}) & & \end{array}$$

We propose a similar group $\widetilde{\mathbf{T}}$, based on triples

$$(\mathbf{L}, \chi, \Lambda'),$$

where now Λ' is a smooth cone of *arbitrary* dimension (i.e., one spanned by part of a basis of $\mathbf{L}$), such that when we let $\mathbf{L}'$ denote the sublattice of $\mathbf{L}$ spanned by Λ', we have

$$\chi \in \mathrm{Im}(\mathbf{L}' \otimes A \to \mathbf{L} \otimes A). \tag{3.2}$$

Again, we impose the relations coming from the evident $\mathrm{GL}_n(\mathbb{Z})$-action. There is a natural map

$$\begin{array}{ccc} \widetilde{\mathbf{T}} & \to & \mathcal{S}_n(G), \\ (\mathbf{L}, \chi, \Lambda') & \mapsto & [a_1, \dots, a_n]. \end{array}$$

We introduce **Subdivision relations** on $\widetilde{\mathbf{T}}$:

(**S**) for a face Λ'' of Λ' of dimension at least 2,

$$\Lambda'' = \mathbb{R}_{\geq 0}\langle e_1, \dots, e_r\rangle \subset \Lambda' = \mathbb{R}_{\geq 0}\langle e_1, \dots, e_s\rangle,$$

consider the star subdivision $\Sigma^*_{\Lambda'}(\Lambda'')$, consisting of the $2^r - 1$ cones spanned by $e_1 + \cdots + e_r$, $e_{r+1}, \dots, e_s$, and all proper subsets of $\{e_1, \dots, e_r\}$. Then

$$(\mathbf{L}, \chi, \Lambda') = \sum_{\substack{\widetilde{\Lambda}' \in \Sigma^*_{\Lambda'}(\Lambda'') \\ \chi \in \mathrm{Im}(\widetilde{\mathbf{L}}' \otimes A \to \mathbf{L} \otimes A)}} (-1)^{\dim(\Lambda') - \dim(\widetilde{\Lambda}')} (\mathbf{L}, \chi, \widetilde{\Lambda}'), \tag{3.3}$$

$$(\mathbf{L}, \chi, \Lambda') = (\mathbf{L}, \chi, \Lambda), \tag{3.4}$$

for a basic cone Λ, having Λ' as a face.

We have:

$$\begin{array}{ccc} \widetilde{\mathbf{T}} & \xrightarrow{\quad \tilde{\psi} \quad} & \mathcal{B}_n(G) \\ \big\downarrow \tilde{\mathsf{s}} & \nearrow_{\sim} & \\ \widetilde{\mathbf{T}}/(\text{subdivision relations}) & & \end{array} \tag{3.5}$$

Lemma 3.1. *The subdivision relations are generated by* (3.3) *for* $r = 2$*, and* (3.4)*.*

Proof. As in the proof of [1, Prop. 2.1], we show inductively that the relations (3.3) for given $r > 2$ are generated by (3.3) with smaller values of r. ☐

In (3.5) we have an obvious map from $\mathcal{B}_n(G)$ to the quotient of $\widetilde{\mathbf{T}}$ by the subdivision relations, sending $[a_1, \dots, a_n]$ to a triple $(\mathbf{L}, \chi, \Lambda)$ with Λ a basic cone and χ given by the formula (3.1). It is readily verified that this respects the relation (1.4), and that the bottom map in (3.5) is an isomorphism.

As in [2, Section 4] we extend the definition of $\tilde{\psi}(\mathbf{L}, \chi, \Lambda')$ to the case of a simplicial cone Λ', satisfying (3.2) with $\mathbf{L}' = \mathbf{L} \cap \Lambda' \otimes \mathbb{R}$. We choose a subdivision by smooth cones and sum, with signs, the contributions from the cones, not contained in any proper face of Λ'. Here, as in (3.3), the signs are given by codimension, and contributions are only taken from summands satisfying the analogous condition to (3.2).

Now we can define Hecke operators

$$T_{\ell,r} : \mathcal{B}_n(G) \to \mathcal{B}_n(G),$$

where ℓ is a prime not dividing the order of G and $1 \leq r \leq n-1$, following the construction in [2, Section 6], as a sum over certain overlattices:

$$T_{\ell,r}(\tilde{\psi}(\mathbf{L}, \chi, \Lambda')) := \sum_{\substack{\mathbf{L} \subset \widehat{\mathbf{L}} \subset \mathbf{L} \otimes \mathbb{Q} \\ \widehat{\mathbf{L}}/\mathbf{L} \simeq (\mathbb{Z}/\ell\mathbb{Z})^r}} \tilde{\psi}(\widehat{\mathbf{L}}, \chi, \Lambda').$$

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