A COMPACTIFICATION OF THE SPACE OF MAPS FROM CURVES

BUMSIG KIM, ANDREW KRESCH, AND YONG-GEUN OH

Abstract. We construct a new compactification of the moduli space of maps from pointed nonsingular projective stable curves to a nonsingular projective variety with prescribed ramification indices at the points. It is shown to be a proper Deligne-Mumford stack equipped with a natural virtual fundamental class.

1. Introduction

1.1. **Overview and motivation.** Let $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i \in \mathbb{Z}_{\geq 1}$. In this paper, we construct a new compactification $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$ of the space of all maps f from genus g, n-pointed nonsingular projective curves $(C, p_1, \ldots, p_n)$ to a nonsingular projective variety X, representing class $\beta \in A_1(X)/\sim^{\mathrm{alg}}$ such that:

- f is unramified everywhere except possibly at $p_1, \ldots, p_n$, where the ramification indices of f are $\mu_1, \ldots, \mu_n$, respectively.
- $f(p_i)$, $i = 1, \ldots, n$, are pairwise distinct.

Here we follow the convention that the ramification index of f at a point $p \in C$ is 1 if f is unramified at p.

The boundary of $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$ consists of suitable maps, still keeping the conditions in a sense, from n-pointed prestable genus g curves to Fulton-MacPherson degeneration spaces of X. A Fulton-MacPherson degeneration space is, by definition, a fiber of the "universal" family $X[m]^+ \to X[m]$ of the Fulton-MacPherson configuration space $X[m]$ of m distinct labeled points in X, for some integer m (Definition 2.1.1). The elements in the moduli space are called stable ramified maps (Definition 3.1.1). They can be considered as stable log-unramified maps. Stable ramified maps are always finite maps. We show that the moduli space $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$ is a proper Deligne-Mumford stack over an algebraically closed base field $\mathbf{k}$ of characteristic zero, carrying a natural virtual fundamental class (Theorem 4.2.3 and Section 5). Therefore, we will be able to define ramified Gromov-Witten invariants (Section 5). This compactification incorporates four known spaces in algebraic geometry: Fulton-MacPherson's configuration space, Kontsevich's stable map compactification, Harris-Mumford's admissible cover compactification, and

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Li's compactification of the moduli of stable relative maps. All of the four will play key roles in our construction.

This paper is motivated by the third author's discussions with Fukaya on a similar compactification in the context of almost Kähler (or symplectic) geometry. A further motivation is a conjectural link with BPS counts due to Pandharipande (Section 5.2).

The main advantage of the algebro-geometric method exploited in this paper is the systematic use of the wonderful property of the universal families of Fulton-MacPherson spaces [6, 17], the notion of (log) admissible maps [8, 14, 19, 26], the deformation theory for such maps [15, 23], as well as the now-standard tools in Gromov-Witten theory in the algebro-geometric category [4, 7, 12, 16]. In [10], a variant of this new compactification is shown to be a smooth irreducible proper Deligne-Mumford stack, compactifying the space of maps from elliptic curves to a projective space.

1.2. **Conventions.** Let $\mathbf{k}$ be a base field, which is algebraically closed and of characteristic zero. Every space will be a Noetherian $\mathbf{k}$-scheme unless otherwise specified. We often use $W|_T$ or h^*W to denote the fiber product $W \times_S T$ of algebraic spaces, where $h : T \to S$ is the map in the fiber square. Except where otherwise mentioned, R will denote the DVR $\mathbf{k}[[t]]$, with field of fractions $K = k((t))$. The extension over R of an object over K will, without explicit mention, entail the passage to a finite extension of K, by which R gets replaced by its integral closure in the extension field.

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2. Stacks of Fulton-MacPherson degeneration spaces

2.1. **FM degeneration spaces.** Let X be a nonsingular variety of dimension $r \geq 1$. Denote by $X[n]$ the Fulton-MacPherson configuration space of n distinct labeled points in X. We refer to the paper [6] for the constructions and the basic properties of the configuration space. The space $X[n]$ has the "universal family"

$$\pi_{X[n]} : X[n]^+ \to X[n]$$

with disjoint sections

$$\sigma_i : X[n] \to X[n]^+$$

for $i = 1, \ldots, n$. The universal space $X[n]^+$ is an iterated blowup of $X[n] \times X$ along smooth centers. Hence, there are two natural projections: $\pi_{X[n]}$ as mentioned above, and

$$\pi_X : X[n]^+ \to X$$

from the second projection.

Definition 2.1.1. *A pair*

$$(\pi_{\mathcal{W}/S} : \mathcal{W} \to S,\, \pi_{\mathcal{W}/X} : \mathcal{W} \to X)$$

of morphisms is called a Fulton-MacPherson degeneration space of X over a scheme S *(for short, an* FM space of X over S*) if:*

- $\mathcal{W}$ *is an algebraic space.*
- *There exists an étale surjective map $T \to S$ from a scheme T, an integer $n > 0$, and a fiber square*

$$\begin{array}{ccc} \mathcal{W}|_T & \longrightarrow & X[n]^+ \\ \downarrow & & \downarrow {\scriptstyle \pi_{X[n]}} \\ T & \longrightarrow & X[n] \end{array}$$

such that the pullback of $\pi_{\mathcal{W}/X}$ to $\mathcal{W}|_T$ coincides with the composite $\mathcal{W}|_T \to X[n]^+ \to X$.

Furthermore, when S is $\operatorname{Spec}\mathbf{k}$*, we will simply say that $\mathcal{W}$ is an* FM space of X. *When n is specified, we will call $\mathcal{W}$ a* level-n FM space.

Example 2.1.2. Let $S = \operatorname{Spec} R = \operatorname{Spec} \mathbf{k}[[t]]$, and let g be a morphism from S to $X \times X$. Suppose that only the closed point $p \in S$ hits the diagonal Δ of $X \times X$ under the map g; then there is a unique lift $\tilde{g} : S \to X[2] = \mathrm{Bl}_\Delta X \times X$. Let k be the intersection number of S and the exceptional divisor of $X[2]$, and let $q = g(p)$. By definition, $X[2]^+$ is the blowup of $X[2] \times X$ along the proper transform of the small diagonal $\Delta_{\{1,2,3\}}$ of $X \times X \times X$. We see, by direct computation, that $\tilde{g}^* X[2]^+$ is isomorphic, as an FM space over S, to the pullback of the blowup $\mathrm{Bl}_{(p,q)} S \times X$ of $S \times X$ at the point (p, q), under the base change

$$S \to S$$
$$t \mapsto at^k + o(t^k)$$

for some nonzero $a \in \mathbf{k}$.

One of the goals of Section 2 is to construct the Artin stack which parameterizes FM spaces of X. To do so, we will recall some elementary properties of $X[n]$, $X[n]^+$, and $\pi_{X[n]}$. Using a variant of the FM configuration spaces, we will see that the stack is described by means of a smooth groupoid scheme, and it will follow that the stack is algebraic.

2.2. **Basic properties of $\pi_{X[n]} : X[n]^+ \to X[n]$.** In [6], the space $X[n]$ is constructed by an iterated blowup of X^n along nonsingular subvarieties Δ_I for all subsets $I \subset N := \{1, 2, \ldots, n\}$ with $|I| \geq 2$, where Δ_I is the proper transform of the diagonal

$$\{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j, \ \forall\, i, j \in I\}.$$

The blowup order of centers is suitably taken. For convenience, we use the following notation.

Notation: Let V_1 be the blowup of a nonsingular variety V_0 along a nonsingular closed subvariety Z. If Y is an irreducible subvariety of V_0, we will use the same notation Y to denote

- the total transform of Y, if $Y \subset Z$;
- the proper transform of Y, otherwise.

This abuse of notation causes no confusion as long as it is clear to which space Y belongs.

The universal family $X[n]^+$ is an iterated blowup

$$X[n]^+ := Y_{n-1} \to Y_{n-2} \to \cdots \to Y_0 := X[n] \times X.$$

The intermediate space Y_{k+1} is the blowup of Y_k along all disjoint nonsingular subvarieties Δ_{I^+} for $I \subset N$ and $|I| = n - k$, where $I^+ := I \cup \{n+1\}$.

2.3. **A local description of $\pi_{X[n]}$.** Notice that in any stage Y_k, Δ_{I^+} is the transverse intersection $\Delta_I \cap \Delta_{a^+}$ if $I \subset N$, $2 \leq |I| \leq n-k$ and $a \in I$, where $a^+ := \{a, n+1\}$. Also note that in each Y_k, the intersection of Δ_{a^+} with every fiber F_k of the natural projection $Y_k \to X[n]$ is transverse. These observations provide a local description of the projections $\pi_{k+1} : Y_{k+1} \to X[n]$ as follows. If p is a singular point of the projection π_{k+1}, then with respect to suitable coordinates the induced map on completed local rings has the form

$$\begin{gathered}\widehat{\mathcal{O}}_{\pi_{X[n]}(p)} \cong \mathbf{k}[[t_1, \ldots, t_{rn}]] \to \widehat{\mathcal{O}}_p \cong \mathbf{k}[[t_1, \ldots, t_{rn}, z_1, \ldots, z_{r+1}]]/(z_1 z_2 - t_1) \\ t_i \mapsto t_i,\end{gathered}$$

and the completed local ring of the fiber F_k at the point corresponding to p can be identified with $\mathbf{k}[[z_1, z_1 z_3, \ldots, z_1 z_{r+1}]]$.

Projective Tangent Bundle Map. Since the general fiber of $\pi_{X[n]}$ is just X there is an induced rational map from $\mathbb{P}(T(X[n]^+/X[n])|_{(X[n]^+)^{\mathrm{sm}}})$, the projectivization of the relative tangent sheaf restricted to the locus of smooth points of $\pi_{X[n]}$, to $\mathbb{P}(TX)$. From the local description it follows that this is actually a morphism

$$\mathbb{P}(T(X[n]^+/X[n])|_{(X[n]^+)^{\mathrm{sm}}}) \to \mathbb{P}(TX).$$

On the level of fibers it says that if W is an FM degeneration space and p is a smooth point of W, then $\mathbb{P}(T_p W)$ is identified naturally with $\mathbb{P}(T_{\pi_{W/X}(p)} X)$.

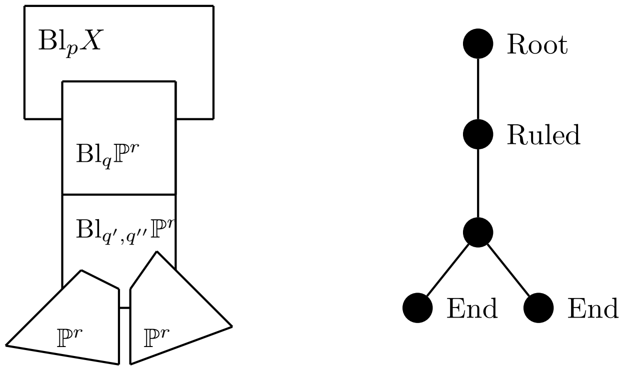


FIGURE 1. An FM degeneration space and its corresponding tree graph.

This identification is independent of the choice of the isomorphism of W and a fiber of the universal family $\pi_{X[n]} : X[n]^+ \to X[n]$.

2.4. **Trees of FM degeneration spaces.** Notice that an FM space of $X := \mathbb{P}^1_{\mathbf{k}}$ without markings over $\mathbf{k}$ is a connected genus 0 nodal curve with a distinguished component $\mathbb{P}^1_{\mathbf{k}}$. In that case there is a natural dual tree graph corresponding to the prestable curve. Likewise, for an FM space W of a nonsingular variety X, we associate a tree whose vertices (resp. edges) correspond one-to-one to components of W (resp. W^{sing}); see Figure 1. The distinguished vertex corresponding to the component, a blowup of X, is called the *root* of the tree. For a given vertex, the number of edges of the minimal chain connecting the root and the vertex will be called the *level* of the corresponding component of W. Let us call a component of W an *end* if it is not the root component and the valance of its vertex is 1. A component of W will be called *ruled* if it is not the root and its vertex has valance 2. The descendants of a component with associated vertex are by definition the components of higher level for which the minimal chain connecting the root and the associated vertex contains v. For each non-root component Y of W, we denote by $D_+(Y)$ the divisor of Y corresponding to the edge which connects the vertex of Y to the vertex at lower level. The $+$ sign in $D_+(Y)$ is explained by the fact that the intersection number of any curve on Y and the divisor is strictly positive unless the curve lies also on the component one level higher. Here the intersection number is taken in the nonsingular variety Y. Similarly, we define $D_-(Y)$ when Y is a ruled component. Non-root components of W will also be called *screens*.

2.5. **Automorphisms.** An FM degeneration W of X over $\mathrm{Spec}\,\mathbf{k}$ is called a stable degeneration in [6] since every ruled (resp. end) screen contains at least one (resp. two) marked point(s). We will forget about markings, and consider the automorphism groups that then arise. Define an automorphism of W/X to be an automorphism ψ of W fixing X, that is, $\pi_{W/X} \circ \psi = \pi_{W/X}$.

Since there are no markings in W, the automorphism group $\mathrm{Aut}(W, X)$ is nontrivial if there is an end component. For example, the group is $\mathbb{G}_a^r \rtimes \mathbb{G}_m$ if the tree of W has only two vertices. Note that each end (resp. ruled) component Y is (non-canonically) isomorphic to $\mathbb{P}^r_{\mathbf{k}}$ (resp. the blowup of $\mathbb{P}^r_{\mathbf{k}}$ at a point) with a marked divisor $D_+(Y)$ (resp. $D_\pm(Y)$). The automorphism group of an end (resp. ruled) Y fixing $D_+(Y)$ (resp. $D_\pm(Y)$) is a subgroup, isomorphic to $\mathbb{G}_a^r \rtimes \mathbb{G}_m$ (resp. $\mathbb{G}_m$), of $\mathrm{Aut}(W, X)$, where notation $\mathbb{G}_m$ (resp. $\mathbb{G}_a$) is for the multiplicative (resp. additive) group $\mathbf{k}^\times$ (resp. $\mathbf{k}$). So, for general (W, X), we see that

$$\mathrm{Aut}(W, X) \cong \Bigg(\prod_{W_i\colon \text{ruled}} \mathbb{G}_m \Bigg) \times \Bigg(\prod_{W_i\colon \text{ends}} \mathbb{G}_a^r \rtimes \mathbb{G}_m \Bigg)$$

More generally, for an FM degeneration space $\mathcal{W}$ of X over S we may analogously define $\mathrm{Aut}(\mathcal{W}/S, X)$, at least as a presheaf of groups. In case $\mathcal{W} \to S$ is projective, one may use relative Hilbert schemes (for example, see [11]) to conclude that $\mathrm{Aut}(\mathcal{W}/S, X)$ is represented by a group scheme over S. It is at least an algebraic space in general, as a consequence of the results in the rest of this section.

2.6. **Operations.** *Forgetting Markings.* Since only the labeling of points matters, we also use notation $X[M]$ for $X[m]$ if M is a set of cardinality m. For example, $X[N]$ will be used instead of $X[n]$, where $N = \{1, \ldots, n\}$. There is a natural iterated blow-down map $X[N]$ to $X[J] \times X^{N\setminus J}$ if $J \subset N$. By similar reasoning, there is a natural blow-down map $X[N]^+ \to X[J]^+ \times X^{N\setminus J}$. To see the latter, we may assume that $J = N - 1$, that is, $\{1, \ldots, n-1\}$. Then, by the results of L. Li on rearrangements of centers [17], $X[N]^+$ coincides with the iterated blowup of $X[J]^+ \times X$ along Δ_T, $n \in T$, $|T| \geq 2$, where $|T| \geq 3$ whenever $n+1 \in T$. ($X[N]^+$ is an iterated blowup of $X^J \times X^2$ along centers which form a building set $\{\Delta_T : T \subset N+1,\, T \neq \{a, n+1\},\, a \in N\}$. Then using a building set order satisfying $\Delta_{T_1} \prec \Delta_{T_2}$ if $n \in T_2 \setminus T_1$ we see that $X[N]^+$ is the iterated blowup of $X[J]^+ \times X$ along Δ_T, $n \in T$, $|T| \geq 2$, where $|T| \geq 3$ whenever $n+1 \in T$.) Combined with projections, we obtain natural forgetful maps $X[N] \to X[J]$ and $X[N]^+ \to X[J]^+$.

By forgetting a point labeled by, say, $n+1$, there is a natural commutative diagram

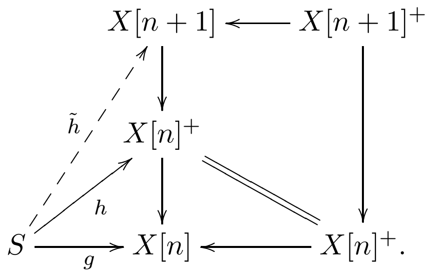
$$\begin{array}{ccc} X[N+1]^+ & \xrightarrow{\pi_+} & X[N]^+ \\ \big\downarrow{\scriptstyle \pi_{X[N+1]}} & & \big\downarrow{\scriptstyle \pi_{X[N]}} \\ X[N+1] & \xrightarrow{\pi} & X[N] \end{array}$$

It induces a map $\tilde{\pi}_+ : X[N+1]^+ \to X[N+1] \times_{X[N]} X[N]^+$.

Lemma 2.6.1. *The map $\tilde{\pi}_+$ is an isomorphism over the open locus of $X[N+1]$ where $\pi_+ \circ \sigma_{n+1}$ meets neither any σ_i, $i = 1, \ldots, n$ nor the relative singular locus of $X[N]^+/X[N]$.*

Proof. We note that Δ_{i+} in $X[N+1]^+$ coincides with the proper transform of Δ_{i+} in $X[N]^+ \times X$ for $i = 1, \ldots, n$. Hence $\pi_+ \circ \sigma_i = \sigma_i \circ \pi$, and the result is established, for otherwise there would be an unstable locus in $X[N+1]^+/X[N+1]$, which would be a contradiction. □

Lifting. Let g be a map from S to $X[n]$ and let h be a section of $g^*X[n]^+$. Assume that the image of h meets neither the relative singular locus of $X[n]^+/X[n]$ nor any sections $\sigma_i \circ g$, $i \in N$. Then there is a unique lift $\tilde{h} : S \to X[n+1]$ of h, since $X[n+1]$ is the blowup of $X[n]^+$ along the images of sections σ_i. We note that the S-scheme $g^*X[n]^+$ is canonically isomorphic to the S-scheme $\tilde{h}^*X[n+1]^+$ preserving σ_i, $i = 1, \ldots, n$, due to Lemma 2.6.1 and the diagram



This diagram is commutative except the trapezoid. The trapezoid however commutes if it is restricted to the image $\Delta_{\{n+1,n+2\}}$ of the section σ_{n+1}. This implies that $\sigma_{n+1} \circ \tilde{h}$ coincides with h when $\tilde{h}^*X[n+1]^+$ is identified with $g^*X[n]^+$. The iterated operation of liftings will be a key tool in Section 4.

Proposition 2.6.2. *Let $f_i : S \to X[n]$ be a morphism of* $\mathbf{k}$*-schemes. Suppose that there is an isomorphism between $f_i^*X[n]^+$, $i = 1, 2$, fixing X and preserving the n induced sections. Then $f_1 = f_2$.*

Proof. This can be seen by the universal property of Theorem 4 in [6] and the natural identification $\mathcal{I}_{\Delta_J}|_x \cong ((T_xX)^J/T_x)^*$, where $\mathcal{I}_{\Delta_J}$ is the ideal sheaf of diagonal Δ_J of X^J and $x \in X = \Delta_J \subset X^J$, for $J \subset N$, $|J| \geq 2$. (See page 195 of [6] for a friendly explanation of the universal property and its relation with screens.) Let f denote f_i followed by the map $X[N] \to X^N$. Then, étale locally at a point which is mapped into Δ_S in X^J, we can express the data $f^*\mathcal{I}_{\Delta_J} \to \mathcal{L}_J \cong \mathcal{O}_S$ by sending $x_{i,j} - x_{i',j}$ to $\sigma_{i,j} - \sigma_{i',j}$ for $i, i' \in J \subset N$, $j = 1, \ldots, r$, where $x_{i,j}$ are (copies of) coordinates of the i-th component X of X^n, and $\sigma_{i,j}$, $j = 1, \ldots, r$ are fiberwise direction coordinates of $X[J]^+$, pulled back via the composite of the sections σ_i with the forgetting map

$$\tilde{\pi}_+ \colon X[N]^+ \to X[N] \times_{X[J]} \mathrm{Bl}_{\Delta_{J+}}(X[J] \times X).$$

We are thus reduced to showing that the given isomorphism induces an isomorphism between the $f_i^*(X[N] \times_{X[J]} \mathrm{Bl}_{\Delta_{J+}}(X[J] \times X))$, $i = 1, 2$. This follows from the claim that for any $S \to X[N]$ the following base change property holds:

$$(1_S \times \tilde{\pi}_+)_* \mathcal{O}_{S \times_{X[N]} X[N]^+} \cong \mathcal{O}_{S \times_{X[J]} \mathrm{Bl}_{\Delta_{J^+}}(X[J] \times X)},$$
$$R^p(1_S \times \tilde{\pi}_+)_* \mathcal{O}_{S \times_{X[N]} X[N]^+} = 0 \qquad \text{for } p > 0.$$

(The assertion about direct image would suffice, but the following reduction step uses also the vanishing of the higher direct images.) It suffices to treat the case $J = N - 1$, and since $\tilde{\pi}_+$ is a proper morphism of flat $X[N]$-schemes, a cohomology-and-base-change argument as in §II.5 of [20] allows us to reduce to the case $S = \operatorname{Spec} \mathbf{k}$. Then $1_S \times \tilde{\pi}_+$ is (when it is not an isomorphism) a proper morphism of FM degeneration spaces contracting a ruled component or an end component. In either case, the argument proceeds using the Theorem on Formal Functions: for the direct image, by a computation in local coordinates, and for the higher direct images, as in the proof of Theorem I.9.1(ii) of [3]. □

2.7. **Spaces $X[n:m]$ and $X[n:m]^+$.** In this subsection we define a compactification $X[n,m]$ of the configuration space of pairs of black colored n ordered points in X and red colored m ordered points in X. Let $N + M$ denote $\{1_b, \ldots, n_b\} \sqcup \{1_r, \ldots, m_r\}$, a collection of n "black numbers" and m "red numbers". We will call the labels of N black and the labels of M red. Then we want to compactify

$$\Big(X^N \setminus \bigcup_{B \subset N\,:\,|B|=2} \Delta_B\Big) \times \Big(X^M \setminus \bigcup_{R \subset M\,:\,|R|=2} \Delta_R\Big)$$

allowing red points to collide with black points but not allowing any two points with the same color to coincide. A "universal family" can be constructed by an iterated blowup of $X[n,m] \times X$ along smooth centers. The centers will be proper transforms of suitable diagonals in the morphism $X[n,m] \times X \to X^n \times X^m \times X$. Those diagonals are

- Δ_{I^+} with $|I_{\text{black}}||I_{\text{red}}| \geq 2$, $I_{\text{black}} := I \cap N$ and $I_{\text{red}} := I \cap M$;
- Δ_{B^+} with $|B| \geq 2$, $B \subset N$;
- Δ_{R^+} with $|R| \geq 2$, $R \subset M$,

where $A^+ = A \cup \{n + m + 1\}$.

The detail of the construction is as follows. First, let $X[1,1] = X^2$. Now $X[n,m]$ and its universal family will be defined inductively. Start with $X[n,m] \times X$ and blow it up along Δ_{I^+} and then Δ_{B^+} and Δ_{R^+}. By [17], any order of the blowups along centers of each type will give the same outcome as long as the blowup along $\Delta_{I_1^+}$ is taken before the blowup along $\Delta_{I_2^+}$ whenever $|I_1| > |I_2|$. Denote by $X[n,m]^+$ the result of the blowups. Then $\Delta_{i_b^+}$ and $\Delta_{i_r^+}$ provide sections σ_{i_b} and σ_{i_r} of $X[n,m]^+ \to X[n,m]$. Now define $X[n,m+1]$ to be the blowup of $X[n,m]^+$ along all $\Delta_{\{b,r\}^+}$ and then all Δ_{r^+}. We mark the last label as $(m+1)_r$. Similarly, define $X[n+1,m]$

to be the blowup of $X[n,m]^+$ along all $\Delta_{\{b,r\}^+}$ and then all Δ_{b^+}. We mark the last label as $(n+1)_b$. In fact, $X[n,m]$ is the closure of

$$X^{n+m} \setminus \bigcup_{\Delta} \Delta$$

in

$$X^{n+m} \times \prod_{\Delta} \mathrm{Bl}_{\Delta} X^{n+m}$$

where Δ runs over all diagonals except those of type $\Delta_{b,r}$, $b \in N$ and $r \in M$. The claims above can be justified by directly modifying the arguments in [6] or by L. Li's general approach to the wonderful compactification [17].

Define $X[n:m]$ to be the maximal open subset of $X[n,m]$ such that the restriction of the universal family $X[n,m]^+$ to the subset is still stable after forgetting red markings but keeping the black markings and vice versa. Here "stable" means by definition that every fiber has only the trivial automorphism fixing X and the remaining marked points. Denote by $X[n:m]^+$ the restriction of the fibration $X[n,m]^+ \to X[n,m]$ to $X[n:m]$.

Lemma 2.7.1. *The image of a map $g : S \to X[n_1, n_2]$ is in $X[n_1 : n_2]$ if and only if $g^* X[n_1, n_2]^+$ is isomorphic to $g_i^* X[n_i]^+$ preserving n_i-sections and fixing X, for $i = 1, 2$, where g_i is the composite $S \xrightarrow{g} X[n_1, n_2] \to X[n_i]$.*

Proof. For $i = 1, 2$, there is the naturally induced map

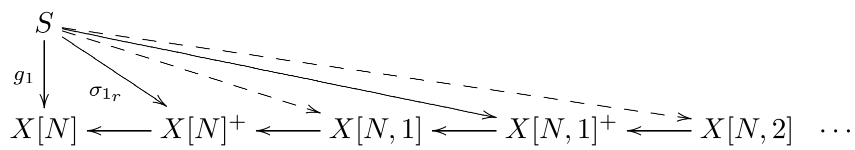
$$h_i : X[n_1, n_2]^+ \to X[n_1, n_2] \times_{X[n_i]} X[n_i]^+$$

over $X[n_1, n_2]$. Then $X[n_1 : n_2]$ is the maximal open subset of $X[n_1, n_2]$ over which h_1 and h_2 are isomorphisms. □

Proposition 2.7.2. *Consider $g_1 : S \to X[N]$, $N = \{1_b, \dots, n_b\}$ with extra sections σ_i, $i \in M = \{1_r, \dots, m_r\}$, of $g_1^* X[N]^+$ such that the extra sections meet neither each other nor the relative singular locus of $g_1^* X[N]^+/S$. Then it induces a unique map $g : S \to X[N, M]$ such that canonically, $g^* X[N,M]^+ \cong g_1^* X[N]^+$ preserving N-labeled sections and fixing X; the extra sections coincide with the sections from the second label M.*

Furthermore, if each geometric fiber of $g_1^* X[N]$ is stable with respect to markings by σ_i, $i \in M$, then canonically $g^* X[N,M]^+ \cong g_2^* X[M]^+$ preserving M-labeled sections and fixing X, where g_2 is the natural composite $S \xrightarrow{g} X[N,M] \to X[M]$.

Proof. The proof of the first statement follows from the inductive use of the argument similar to the one given in §2.6.



The second statement is an immediate consequence of Lemma 2.7.1. □

2.8. **Definition of $\mathfrak{X}[n]$.** Let $\mathfrak{X}[n]$ denote the stack of FM spaces of level n over (Sch/$\mathbf{k}$). It follows from Propositions 2.6.2 and 2.7.2 that $\mathfrak{X}[n] \cong [X[n:n] \rightrightarrows X[n]]$, the stackification of the prestack associated to a groupoid scheme

$$X[n:n] \overset{t}{\underset{s}{\rightrightarrows}} X[n].$$

The groupoid scheme is equipped with obvious maps s and t, diagonal map $e : X[n] \to X[n:n]$, "composition" $m : X[n:n]\,{}_t\!\times_s X[n:n] \to X[n:n]$, and exchange map $i : X[n:n] \to X[n:n]$. The stack is algebraic and smooth by Proposition 4.3.1 in [13], which requires that s and t are smooth and the relative diagonal $(s,t) : X[n:n] \to X[n] \times X[n]$ is separated and quasi-compact; these conditions are easily checked.

There is also a variant $X[n:n]_m$ of $X[n:n]$ for $m \le n$, defined as a subscheme by the condition $\sigma_{i_b} = \sigma_{i_r}$ for $i = 1, \ldots, m$, or alternatively by a blowup construction. This leads to the stack $\mathfrak{X}[n]_m \cong [X[n:n]_m \rightrightarrows X[n]]$ of FM spaces with m sections, locally defined by the given m sections plus $n-m$ additional sections. There is the forgetful morphism $\mathfrak{X}[n]_m \to X[m]$.

In the particular case $n = m+1$, there is another variant $\mathfrak{X}[m+1]'_m$ of FM spaces with m sections and marked component, locally defined by the given sections plus one more lying on the marked component. As a groupoid, $\mathfrak{X}[m+1]'_m \cong [X[m+1:m+1]'_m \rightrightarrows X[m+1]]$ where $X[m+1:m+1]'_m$ is the open subscheme of $X[m+1:m+1]_m$ where the black and red $(m+1)$-st sections lie on the same component.

There is, further, the open substack $\mathfrak{X}[m+1]''_m$ of $\mathfrak{X}[m+1]'_m$ where the marked component is that of the m-th section. It is also an open substack of $\mathfrak{X}[m+1]_m$, and if we define $X[m+1]''$ to be the locus in $X[m+1]$ where the m-th and $(m+1)$-st sections lie on the same component, with $X[m+1:m+1]''_m$ the common pre-image in $X[m+1:m+1]'_m$, then $\mathfrak{X}[m+1]''_m \cong [X[m+1:m+1]''_m \rightrightarrows X[m+1]'']$. A further open substack is $X[m] \cong [X[m+1:m+1]'''_m \rightrightarrows X[m+1]''']$ where the FM space with m sections is itself stable. Here $X[m+1]'''$ is the complement in $X[m+1]''$ of the divisor $\Delta_{m,m+1}$ where the m-th and $(m+1)$-st sections have come together. We will also use the forgetful morphism $\mathfrak{X}[m+1]'_m \to \mathfrak{X}[2]_1$ forgetting the first $m-1$ sections, given via groupoids as

$$[X[m+1:m+1]'_m \rightrightarrows X[m+1]] \to [X[2:2]_1 \rightrightarrows X[2]],$$

as well as the isomorphism $\mathfrak{X}[m+1]''_m \cong X[m] \times [\mathbb{A}^1/\mathbb{G}_m]$ given by forgetful morphism (first factor) and divisor $\Delta_{m,m+1} \subset X[m+1]''$ mentioned above (second factor). The multiplicative group $\mathbb{G}_m$ acts in the standard way on $\mathbb{A}^1$ with quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ parametrizing pairs consisting of a line bundle with a regular section; note that an effective Cartier divisor canonically determines such a pair.

3. The Stack of Stable Ramified Maps

3.1. **Stable ramified maps.** We introduce a generalized notion of stable maps. For the basic definitions and properties of stable maps, the reader may see, for example, [7]. From now on, we assume that X is a nonsingular projective variety over $\mathbf{k}$. Let

$$NE_1(X) \subset A_1(X)/\sim_{\mathrm{alg}}$$

denote the semigroup of effective curve classes modulo algebraic equivalences. Given $\beta \in NE_1(X)$, $g, n \in \mathbb{Z}_{\geq 0}$, and $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i \in \mathbb{Z}_{\geq 1}$, we define:

Definition 3.1.1. *A triple*

$$((C, p_1, \ldots, p_n), \pi_{W/X} : W \to X, f : C \to W)$$

is called a stable map with μ-ramification from an n-pointed, genus g curve to an FM degeneration space W of X, representing class β *(for short,* a (g, β, μ)-stable ramified map*) if:*

- $(C, p_1, \ldots, p_n)$ *is an n-pointed, genus g prestable curve over* $\mathbf{k}$.
- $\pi_{W/X} : W \to X$ *is an FM degeneration of X over* $\mathbf{k}$.
- *The pushforward $(\pi_X \circ f)_*[C]$ of the fundamental class $[C]$ is β.*
- *The following four conditions are satisfied:*

(1) Prescribed Ramification Index Condition:
  - *The smooth locus C^{sm} of C coincides with the inverse image $f^{-1}(W^{\mathrm{sm}})$ of the smooth locus W^{sm}.*
  - *$f|_{C^{\mathrm{sm}}}$ is unramified everywhere possibly except at p_i.*
  - *At p_i the ramification index*

$$\mathrm{length}(\mathfrak{m}_{p_i}/\mathfrak{m}_{f(p_i)}\mathcal{O}_{p_i}) + 1$$

  of f is exactly μ_i.

(2) Distinct Points Condition: *$f(p_i)$, $i = 1, \ldots, n$ are pairwise distinct points of W.*

(3) The Admissibility Condition: *At every nodal point p of C, there are identifications $\widehat{\mathcal{O}}_{f(p)} \cong \mathbf{k}[[z_1, \ldots, z_{r+1}]]/(z_1 z_2)$ and $\widehat{\mathcal{O}}_p \cong \mathbf{k}[[x, y]]/(xy)$, so that $\hat{f}^*$ sends z_1 to x^m and z_2 to y^m for some positive integer m.*

(4) Stability Condition:
  - *For each ruled component W_r of W, there is either an image of a marking in W_r (that is, $f(p_i) \in W_r$ for some i) or a non-fiber image $f(D) \subset W_r$ of an irreducible component D of C.*
  - *For each end component $W_e \cong \mathbb{P}^r$ of W, there are either images of two distinct markings in W_e or a non-line image $f(D) \subset W_e$ of an irreducible component D of C.*

Lemma 3.1.2. *Let W be a target of a (g, β, μ)-stable ramified map $f : C \to W$. Then the number of components of W is bounded above by an integer depending only on X, β, g and n. So is the number of components of C.*

Proof. First assume that $W = X$ is the projective line $\mathbb{P}^1_{\mathbf{k}}$, C is a genus g nodal curve not necessarily connected, d is a positive integer, and $f : C \to W$ is a degree d map whose restriction to each connected component is stable. Then the sum of the number of nodal points of C, ramification points of f, and marked points is less than or equal to $2d - 2 + 2g + 2n$.

Now we come to the general case that W is an FM-degeneration space of a nonsingular projective variety X. Choose an embedding $X \hookrightarrow \mathbb{P}^N_{\mathbf{k}}$. Let W_k, $W_{\geq k}$ be the set of all level-k resp. level-k-and-higher screens, $C_{\geq k}$ the preimage of $W_{\geq k}$ under f, and C_k the stabilization of $C_{\geq k} \to W_{\geq k} \to W_k$. By considering a general projection to $\mathbb{P}^1_{\mathbf{k}}$ such that the composite $C_0 \to W \to X \dashrightarrow \mathbb{P}^1_{\mathbf{k}}$ is well-defined, stable, and collapses no components not already collapsed by the map to X, we see that $|W_1| \leq 2d - 2 + 2g + 2n$, where d is the degree of β. Note that the total degree of C in *each* level-one screen (understood as degree in $\mathbb{P}^r_{\mathbf{k}}$ after blowdown) is less than or equal to d. By the Admissibility Condition and the Stability Condition, the sum of the total degree and the number of marked points of C in *each* level-two screen is less than or equal to $d+n-1$. By the same reasoning, the sum of the total degree and the number of marked points of C in a level-k screen is less than or equal to $d+n-k+1$. Therefore the level of any screen can be at most $d+n$. This fact combined with the bound $|W_k| \leq 2(d-1+g+n)\prod_{1\leq i\leq k-1} 2(d+g+n-i)$ leads to a bound on the total number of components of W, and since each component supports the image of at most d components of C, also a bound on the number of components of C. □

3.2. **The families of stable ramified maps.** Given $\beta \in NE_1(X)$, $g, n \in \mathbb{Z}_{\geq 0}$, and $\mu = (\mu_1, \ldots, \mu_n)$ where $\mu_i \in \mathbb{Z}_{\geq 1}$, we define a family version of stable ramified maps.

Definition 3.2.1. *A triple*

$$((\pi : \mathcal{C} \to S, \{p_1, \ldots, p_n\}), (\pi_{\mathcal{W}/S} : \mathcal{W} \to S, \pi_{\mathcal{W}/X} : \mathcal{W} \to X), f : \mathcal{C} \to \mathcal{W})$$

is called an S-family of stable maps with μ-ramification from n-pointed, genus g curves to an FM degeneration space $\mathcal{W}$ of X, representing class β *if:*

- $(\pi : \mathcal{C} \to S, \{p_1, \ldots, p_n\})$ *is a family of* n*-pointed, genus* g *prestable curves over* S*.*
- $(\pi_{\mathcal{W}/S} : \mathcal{W} \to S, \pi_{\mathcal{W}/X} : \mathcal{W} \to X)$ *is an FM degeneration of* X *over* S*.*
- *The data form a commutative diagram*

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\quad f \quad} & \mathcal{W} & \xrightarrow{\pi_{\mathcal{W}/X}} & X \\ & {\scriptstyle \pi}\searrow & \swarrow {\scriptstyle \pi_{\mathcal{W}/S}} & & \\ & & S & & \end{array}$$

such that over each geometric point of S*, it is a* (g, β, μ)*-stable ramified map.*

- Prescribed Ramification Index Condition: *f has ramification index μ_i at p_i.*
- The Admissibility Condition: *For any geometric point t of S, if p is a nodal point of C_t and two isomorphisms are given as*

$$\widehat{\mathcal{O}}_{f(p)} \cong \widehat{\mathcal{O}}_{\pi_S(p)}[[z_1, \ldots, z_{r+1}]]/(z_1 z_2 - s), \ \textit{for some } s \in \widehat{\mathcal{O}}_{\pi_S(p)}$$

and

$$\widehat{\mathcal{O}}_p \cong \widehat{\mathcal{O}}_{\pi_S(p)}[[x, y]]/(xy - s'), \ \textit{for some } s' \in \widehat{\mathcal{O}}_{\pi_S(p)}$$

then

$$\hat{f}^*(z_1) = \alpha_1 x^m, \ \hat{f}^*(z_2) = \alpha_2 y^m$$

for some units α_i in $\widehat{\mathcal{O}}_p$ with $\alpha_1\alpha_2 \in \widehat{\mathcal{O}}_{\pi_S(p)}$ and a positive integer m.

Remark 3.2.2. Let $\mathcal{C}$ and $\mathcal{W}$ be as in above. Then we say that $f : \mathcal{C} \to \mathcal{W}$ has ramification index ℓ at a smooth point $p : S \to \mathcal{C}$ if we have equality of sheaves of ideals

$$f^*(\mathcal{I}_{f(p(S))})\widehat{\mathcal{O}}_{p(s)} = \mathcal{I}^{\ell}_{p(S)}\widehat{\mathcal{O}}_{p(s)}$$

for all $s \in S$.

Remark 3.2.3. Admissibility Condition, which is called Predeformability Condition in [14], was introduced and studied by J. Li in his construction of stable relative map and relative Gromov-Witten invariants. When the target is one-dimensional, the notion of admissibility was introduced in [8] and well studied in [19] by log structures ([9]); the analogous log structures in case of higher-dimensional target are studied in [22]. As explained in §3.7 in [19] and in Simplification 1.7 in [15], we may let $\alpha_1 = \alpha_2 = 1$ in Definition 3.2.1 for a suitable isomorphism $\widehat{\mathcal{O}}_p \cong \widehat{\mathcal{O}}_{\pi_S(p)}[[x, y]]/(xy - s')$.

From now on, we use abbreviations of the imposed conditions on stable ramified maps, for example, PRIC for Prescribed Ramification Index Condition.

Lemma 3.2.4. (Tangent Line Map Condition) *Let f be a family of stable ramified maps as in Definition 3.2.1. PRIC and AC together imply that there is a natural extension*

$$\mathbb{P}(Tf) : \mathcal{C} \to \mathbb{P}(TX)$$

of the projectivization of the induced map $T(\mathcal{C}/S)^{\mathrm{sm}} \to T(\mathcal{W}/S)^{\mathrm{sm}}$ between tangent bundles.

Proof. This is a local property. PRIC (resp. AC) will imply that $\mathbb{P}(Tf)$ is well-defined at smooth points (resp. at singular points) of $\pi : \mathcal{C} \to S$. First, at a smooth point p of $\mathcal{C}/S$, let x be a S-relative uniformizing parameter, locally defining the section p_i if $p = p_i$ for some i. Since the image of p is a smooth point q, we have also $t_1, \ldots, t_r$, S-relative uniformizing parameters at q. The submodule $(f^*dt_1, \ldots, f^*dt_r)$ of the stalk of $\Omega_{\mathcal{C}/S}$ at p is generated by $x^{m-1}dx$ if m is the ramification index of f at p. Hence

$$[x^{-(m-1)}f^*dt_1, \ldots, x^{-(m-1)}f^*dt_r]$$

is regular at p since there is no jump of the ramification indices at p. In other words, $f^*\Omega_{\mathcal{W}/S} \to \Omega_{\mathcal{C}/S}(-\sum(m_i - 1)p_i)$ is surjective on the smooth locus of $\mathcal{C}/S$.

At a singular point of $\mathcal{C}/S$, the map f satisfies AC:

$$A[[z_1, \ldots, z_{r+1}]]/(z_1z_2 - s) \to A[[x, y]]/(xy - s')$$

sending $z_1 \mapsto \alpha_1 x^m,\ z_2 \mapsto \alpha_2 y^m$, where $A = \widehat{\mathcal{O}}_{\pi(p)}$. Using the local description of $\pi_{X[n]}$ and the projective tangent map given in §2.3, we may assume that $W = Y_{k+1}$ and $F_k = X$; and we compute that $\mathbb{P}(Tf)$ is

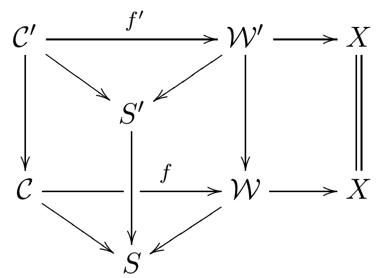
$$[m, mz_3 + x\frac{\partial z_3}{\partial x}, \ldots, mz_{r+1} + x\frac{\partial z_{r+1}}{\partial x}]$$

for $x \neq 0$ and

$$[m, mz_3 - y\frac{\partial z_3}{\partial y}, \ldots, mz_{r+1} - y\frac{\partial z_{r+1}}{\partial x}]$$

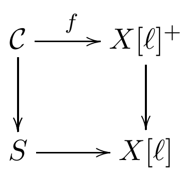
for $y \neq 0$. Since $x\,\partial/\partial x = -y\,\partial/\partial y$ is a regular derivation of $A[[x, y]]/(xy - s')$, it follows that the rational map $\mathbb{P}(Tf) : \mathcal{C} \dashrightarrow \mathbb{P}(TX)$ is well-defined also at nodal points. □

3.3. **Construction of $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$.** Define a category $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$ of (g, β, μ)-stable ramified maps to FM degenerations of X; it is a CFG over the étale site (Sch/$\mathbf{k}$). A morphism is a commutative diagram



preserving markings, where the squares to each side of $S' \to S$ are cartesian. It is straightforward to see that this CFG is a stack.

Definition 3.3.1. *An S-family of* stable μ-ramified maps to a fixed target *$X[\ell]^+/X[\ell]$ from n-pointed genus g curves is a stable n-pointed genus g map $(\mathcal{C} \to S, \{p_1, \ldots, p_n\}, f : \mathcal{C} \to X[\ell]^+)$ with a commutative diagram*



satisfying PRIC, DPC, AC, and SC.

Let β be a curve class in $NE_1(X)$ and denote also by β, the induced class in $NE_1(X[\ell]^+)$ using any canonical inclusion $X \subset X[\ell]^+$ as a general fiber of $X[\ell]^+ \to X[\ell]$.

Proposition 3.3.2. *The stack $\overline{\mathcal{M}}_{g,n}(X[\ell]^+/X[\ell], \beta)^\mu$ of stable μ-ramified maps to a fixed target $X[\ell]^+/X[\ell]$ from n-pointed genus g curves, representing class β, is a separated finite-type Deligne-Mumford stack over $\mathbf{k}$.*

Proof. Since the moduli of (g, β)-stable maps to $X[\ell]^+$ is a proper Deligne-Mumford stack, it is enough to show that, given a family f of stable maps over S, representing class β, there is a locally closed subscheme Z of S such that: for any $T \to S$, the pullback $f|_T$ is a T-family of stable ramified maps (with fixed target $X[\ell]^+/X[\ell]$) if and only if $T \to S$ factors through Z.

First, take the maximal open locus S_1 of S where f does not send any component of any geometric fiber of $\mathcal{C}$ to the relative singular locus of $X[\ell]^+/X[\ell]$, and furthermore DPC is satisfied. Then there is a natural closed subscheme S_2 of S_1 representing a functor of admissible stable maps. This can be shown by the proof of Theorem 2.11 in [14] (or one may use §3C in [19]). Now take the maximal open locus S_3 of S_2 where $f^*\Omega^\dagger_{X[\ell]^+/X[\ell]} \to \Omega^\dagger_{\mathcal{C}_{S_3}/S_3}$ is surjective, where $\Omega^\dagger_{X[\ell]^+/X[\ell]}$ and $\Omega^\dagger_{\mathcal{C}_{S_3}/S_3}$ are the sheaves of relative log differentials induced from the log structures of the boundary divisors. In order to take care of PRIC, introduce relative uniformizing parameters z_j, $j = 1, \ldots, r$ of $\mathcal{O}_{f(p_i(t))}$ at $f(p_i(t))$ where t is a point of an étale chart of S_3, and let z be a relative parameter of $\mathcal{O}_{p_i(t)}$ at $p_i(t)$. Then, define the closed subscheme $S_{4,i}$ of S_3 by equations $a_{0,j} = 0, \ldots, a_{\mu_i - 1,j} = 0$, for all j, where $f^*(z_j) = a_{0,j} + a_{1,j}z + \cdots \in \mathcal{O}_{p_i}$, $a_{k,j} \in \mathcal{O}_t$. Then take the maximal open subscheme S_5 of $\bigcap_i S_{4,i}$, where restricted to every geometric fiber $\mathcal{C}_t$, f has ramification order exactly μ_i at $p_i(t)$. Now Z is the maximal open subscheme of S_5 where SC is satisfied. □

Corollary 3.3.3. *The stack $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$ is a finite-type Deligne-Mumford stack.*

Proof. We apply Theorem 4.21 in [5], using that by Lemma 3.1.2 for sufficiently large ℓ we have $\overline{\mathfrak{U}}_{g,\mu}(X, \beta) \to \mathfrak{X}[\ell]$ and an isomorphism

$$\overline{\mathcal{M}}_{g,n}(X[\ell]^+/X[\ell], \beta)^\mu \cong \overline{\mathfrak{U}}_{g,\mu}(X, \beta) \times_{\mathfrak{X}[\ell]} X[\ell].$$

We need to check conditions:

(1) The diagonal map of the stack is representable, quasi-compact, separated and unramified. Using the isomorphism, the first three properties follow from Proposition 3.3.2 (see [2, Lemma C.5]); the last property follows from SC.

(2) There is a scheme U and a smooth surjective map $U \to \overline{\mathfrak{U}}_{g,\mu}(X, \beta)$. This again follows from Proposition 3.3.2. □

4. Properness

4.1. **Preliminary results.**

Lemma 4.1.1. *Let $(f : \mathcal{C} \to X[m]^+, p_1, \dots, p_n)$ be a stable ramified map over K with partial stabilization $(\mathcal{C}, p_1, \dots, p_n) \to (\mathcal{C}', p'_1, \dots, p'_n)$ of prestable curves, extension of the latter to $(\overline{\mathcal{C}}', \bar{p}'_1, \dots, \bar{p}'_n)$ over R, and chosen component $E \subset \overline{\mathcal{C}}'_0$. Then there exists open $\mathcal{U} \subset \overline{\mathcal{C}}' \times_{\operatorname{Spec} R} \overline{\mathcal{C}}'$ whose special fiber $\mathcal{U}_0$ satisfies $\emptyset \neq \mathcal{U}_0 \subset E \times E$, such that for any sections $(\sigma_1, \sigma_2) : \operatorname{Spec} R \to \mathcal{U}$, adding these to obtain $\operatorname{Spec} K \to X[m+2]$ and forgetting the first m sections yields $\mathcal{C} \to X[m]^+|_{\operatorname{Spec} K} \cong X[m+2]^+|_{\operatorname{Spec} K} \to X[2]^+$ such that the stabilization $(\mathcal{C}'' \to X[2]^+, p''_1, \dots, p''_n, \sigma''_1, \sigma''_2)$ extends over R to $\overline{\mathcal{C}}'' \to X[2]^+$ with the extended σ''-sections landing in the same component of $\overline{\mathcal{C}}''_0$, and the induced $\overline{\mathcal{C}}' \dashrightarrow X[2]^+$ restricts to a nonconstant map on E.*

Proof. We easily reduce to the case $\mathcal{C}' = \mathcal{C}$, then we write $(\overline{\mathcal{C}}, \bar{p}_1, \dots, \bar{p}_n)$ for $(\overline{\mathcal{C}}', \bar{p}'_1, \dots, \bar{p}'_n)$. There exists open $\mathcal{V} \subset \overline{\mathcal{C}} \times_{\operatorname{Spec} R} \overline{\mathcal{C}}$ disjoint from the diagonal of $\overline{\mathcal{C}}$, the sections $\bar{p}_i$ on both factors, and the pre-image in $\mathcal{C} \times_{\operatorname{Spec} K} \mathcal{C}$ of the diagonal of $X[m]^+$, the m sections, and the relative singular locus of $X[m]^+/X[m]$ on both factors, with special fiber $\mathcal{V}_0$ nonempty and contained in $E \times E$, such that there is a morphism

$$\mathcal{V} \to X[m+2]$$

extending the obvious one on $\operatorname{Spec} K \times_{\operatorname{Spec} R} \mathcal{V}$. By symmetry the generic point of $\mathcal{V}_0$ maps into $X[m+2]''$, so by further shrinking $\mathcal{V}$ preserving the condition $\emptyset \neq \mathcal{V}_0 \subset E \times E$ we may suppose $\mathcal{V}$ irreducible and smooth over $\operatorname{Spec} R$ with image contained in $X[m+2]''$. By construction the image of $\operatorname{Spec} K \times_{\operatorname{Spec} R} \mathcal{V}$ is contained in $X[m+2]'''$, which means that the divisor $\Delta_{m+1,m+2}$ of $X[m+2]''$ pulls back to $e \cdot \mathcal{V}_0$ for some integer $e \geq 0$.

We claim that $\mathcal{V} \times_{\operatorname{Spec} R} \mathcal{C} \to X[2]^+$ extends to a map

$$\phi : \mathcal{V} \times_{\operatorname{Spec} R} \overline{\mathcal{C}} \dashrightarrow X[2]^+$$

well-defined on an open subset containing the points (u, u', u) and (u, u', u') for general $(u, u') \in \mathcal{V}_0$. Given this, we then may define $\mathcal{U}$ by deleting from $\mathcal{V}$ the points $(u, u') \in \mathcal{V}_0$ for which ϕ is not defined at (u, u', u) or at (u, u', u'), then the desired conclusion holds since the extension $\overline{\mathcal{C}}'' \to X[2]^+$ mentioned in the statement of the lemma may be obtained by resolving the indeterminacy of $\phi \circ ((\sigma_1, \sigma_2) \times 1_{\overline{\mathcal{C}}}) : \overline{\mathcal{C}} \dashrightarrow X[2]^+$ and stabilizing.

To prove the claim, we let $\mathcal{B}$ denote the image of $\mathcal{V}$ by the first projection morphism to $\overline{\mathcal{C}}$. There is a unique morphism $\mathcal{B} \to X[m+1]$ through which the composite morphism $\mathcal{V} \to X[m+2]'' \to X[m+1]$ factors. Then for $\mathcal{B} \to X[m+1] \times [\mathbb{A}^1/\mathbb{G}_m]$ on the second factor given by the divisor $e \cdot \mathcal{B}_0$, if we let $\mathcal{W} \to \mathcal{B}$ denote an FM space corresponding to it under the isomorphism $\mathfrak{X}[m+2]''_{m+1} \cong X[m+1] \times [\mathbb{A}^1/\mathbb{G}_m]$, then there is a morphism $X[m+2]^+|_{\mathcal{V}} \to \mathcal{W}$, fitting into a cartesian diagram with $\mathcal{V} \to \mathcal{B}$, compatible with $m+1$ sections and fixing X. By stability the restriction $X[m+2]^+|_{\operatorname{Spec} K \times_{\operatorname{Spec} R} \mathcal{V}} \to$

$\operatorname{Spec} K \times_{\operatorname{Spec} R} \mathcal{W}$ is compatible with the unique morphisms to $X[m]^+|_{\operatorname{Spec} K}$ preserving m sections and fixing X. Forgetting the first m sections, we obtain a level-2 FM space $\mathcal{Z}$ over $\mathcal{B}$ and morphism $\mathcal{W} \to \mathcal{Z}$ compatible with $X[m+2]^+|_{\mathcal{V}} \to X[2]^+|_{\mathcal{V}}$. Then we obtain $\mathcal{B} \times_{\operatorname{Spec} R} \overline{\mathcal{C}} \dashrightarrow \mathcal{Z}$ compatible with $\mathcal{V} \times_{\operatorname{Spec} R} \overline{\mathcal{C}} \dashrightarrow X[2]^+|_{\mathcal{V}}$ in the sense that if $\mathcal{D} \subset \mathcal{B} \times_{\operatorname{Spec} R} \overline{\mathcal{C}}$ is open on which the map to $\mathcal{Z}$ is well-defined, then $\mathcal{V} \times_{\operatorname{Spec} R} \overline{\mathcal{C}} \dashrightarrow X[2]^+|_{\mathcal{V}}$ will be well-defined on $\mathcal{V} \times_{\mathcal{B}} \mathcal{D}$, together fitting into a commutative diagram. We may take $\mathcal{D}$ to contain the generic point of $E \times E$, then we clearly have $(u, u', u') \in \mathcal{V} \times_{\mathcal{B}} \mathcal{D}$ for general $(u, u') \in \mathcal{V}_0$. □

Lemma 4.1.2. *Let $(f : \mathcal{C} \to X[m]^+, p_1, \ldots, p_n)$ be a stable ramified map over K such that the usual stable map compactification has a component E of the closed fiber mapped into a relative singular locus of $X[m]^+/X[m]$. Then there exists a neighborhood $\mathcal{U}$ of the generic point of E, such that for any section $\sigma : \operatorname{Spec} R \to \mathcal{U}$, we have $\mathcal{C} \to X[m]^+|_{\operatorname{Spec} K} \cong X[m+1]^+|_{\operatorname{Spec} K}$ whose extension fits into a commutative diagram*

$$\begin{array}{ccc} \overline{\mathcal{C}}' & \longrightarrow & X[m+1]^+ \\ {\scriptstyle c}\downarrow & & \downarrow \\ \overline{\mathcal{C}} & \longrightarrow & X[m]^+ \end{array}$$

of stable maps over R, where c is a contraction, such that the lifted section $\sigma' : \operatorname{Spec} R \to \overline{\mathcal{C}}'$ meets the component $E' \subset \overline{\mathcal{C}}'_0$ corresponding to $E \subset \overline{\mathcal{C}}_0$.

Proof. As in the proof of Lemma 4.1.1, there exists open $\mathcal{V} \subset \overline{\mathcal{C}}$ and morphism $\mathcal{V} \to X[m+1]$ lifting $\operatorname{Spec} R \to X[m]$, such that the induced $\mathcal{V} \to \mathfrak{X}[m+1]'_m$ factors up to 2-isomorphism through $\operatorname{Spec} R$. The last conclusion follows from a description of $\mathfrak{X}[m+1]'_m$ in a neighborhood of a point where the marked component is ruled with no sections. Taking $I \subset \{1, \ldots, m\}$ to be the set of sections on all the descendants of the marked component, and $g = 0$ to be a defining equation for $\Delta_I \subset X[m]$ in a neighborhood U of the image point, then an open neighborhood in $\mathfrak{X}[m+1]'_m$ may be identified with the stack in which an object consists of morphism to U, pair of line bundles each with regular section, and trivialization of the tensor product of the line bundles taking the product of the sections to the function g. Then there is the extended FM space $\mathcal{W} \to \operatorname{Spec} R$, and we may take $\mathcal{U}$ to be the complement in $\mathcal{V}$ of the indeterminacy locus of $\overline{\mathcal{C}} \dashrightarrow \mathcal{W}$. □

4.2. **Valuative criteria.** We want to show that the stack $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$ of stable ramified maps is proper over $\mathbf{k}$. We do this using the valuative criterion for properness stated in [5], by first verifying the existence of extensions as indicated, then checking that the valuative criterion for separatedness is satisfied.

Proposition 4.2.1. *Given an n-pointed stable ramified map $(f, \mathbf{p})$ over the quotient field K of a DVR R, there is a stable ramified map extension*

$(\tilde{f}, \tilde{\mathbf{p}})$ of $(f, \mathbf{p})$, namely, $(\tilde{f}, \tilde{\mathbf{p}})$ is a stable ramified map defined over a DVR R' which is the integral closure of R in a finite extension K' of K and $(\tilde{f}|_{K'}, \tilde{\mathbf{p}}) \cong (f, \mathbf{p})|_{K'}$.

It is enough to prove the above when $R = \mathbf{k}[[t]]$. Let $R = \mathbf{k}[[t]]$ from now on in this section.

Proof of Proposition 4.2.1 when $\dim X = 1$. Let $\mathcal{W}/K$ be the pullback of the universal family $X[m]^+$ by a map $\operatorname{Spec} K \to X[m]$. Then, using the image of the marked points under $\mathcal{C} \to \mathcal{W} \to X[m]^+$, there is an induced map $\operatorname{Spec} K \to X[m : n]$. Thus, we may assume that $\mathcal{W}$ is the pullback of $X[n]^+$ under $g : \operatorname{Spec} K \to X[n]$ such that $f \circ p_i = \sigma_i \circ g$. Now take the usual stable map limit of this $f : \mathcal{C} \to X[n]^+$. The extension is a stable ramified map as shown in the proof of Theorem 4 in [8] using Abhyankar's Lemma (which removes any possible dimension 1 branch locus in the special fiber), the purity of the branch locus (which removes any possible dimension 0 branch locus), and the universal covering of an A_l-singularity (which shows the admissibility). □

Local Analysis. Let $(\pi : \mathcal{C} \to \operatorname{Spec} R, \{p_1, \ldots, p_n\}, f : \mathcal{C} \to X)$ be an *ordinary stable map* over $\operatorname{Spec} R$ such that:

- Restricted to the generic fiber $\mathcal{C}_K$, f is a stable μ-ramified map.
- Restricted to the closed fiber $\mathcal{C}_0$, f is no longer μ-ramified.

Here $\mathcal{W} = X \times \operatorname{Spec} R$. Assume that $\dim X = r \geq 2$ and $\mathcal{C}$ is *irreducible.* Then only the following cases are possible unless some pair of markings have the same image over the closed point of $\operatorname{Spec} R$.

1. (Jump of a Ramification Index) There exists a smooth point p on a component of $\mathcal{C}_0$ non-contracted under f such that the ramification index jumps at p.
2. (Creation of a Nodal Point) There exists a nodal point p of $\mathcal{C}_0$ such that any component of $\mathcal{C}_0$ containing p does not contract under f.
3. (Contraction of a Component) There exists a contracted component E of $\mathcal{C}_0$ and there exist no pair of ramification markings approaching E.

Lemma 4.2.2. *In Case (1) or (2), the tangent line map*

$$\mathbb{P}(Tf) : \mathcal{C} \dashrightarrow \mathbb{P}(TX)$$

is not well-defined at p. In Case (3), $\mathbb{P}(Tf)(E)$ is not a point or there is a point $q \in E$ such that $\mathbb{P}(Tf)$ is not well-defined at q.

Proof. Case (1). Let m be the ramification index of $f|_{\mathcal{C}_0}$ at p, and let x, t be local parameters of $\mathcal{O}_p$, where t is the local parameter of R. Let $f = (f_1, \ldots, f_r)$ after introducing a local coordinate system of X at $f(p)$. Without loss of generality, we assume that the tangent line direction of the image curve of x-axis under $f(x, 0)$ is $[1, 0, \ldots, 0] \in \mathbb{P}^{r-1} = \mathbb{P}(T_{f(p)}X)$.

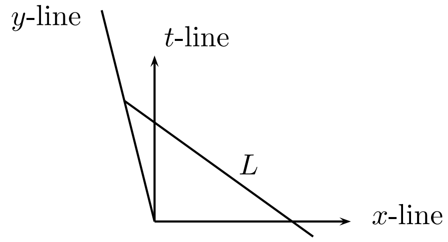


Figure 2. (δ, ν)-plane

Consider the equation $\frac{\partial f_1}{\partial x} = 0$. It defines a curve Z not containing the x-axis, but containing the origin $(x, t) = (0, 0)$. If the curve has at least two irreducible components, then f is generically unramified along one of the components by assumption of stable ramified maps. Hence along the component, $\mathbb{P}(Tf)$ approaches a point different from $[1, 0, \dots, 0]$. If the curve Z is irreducible, then along the curve, $\mathbb{P}(Tf)$ approaches a point different from $[1, 0, \dots, 0]$ since the generic point of the curve is a zero of multiplicity $m - 1$ of $\frac{\partial f_1}{\partial x}$, while the generic point cannot be a zero of multiplicity $m - 1$ or higher of the derivative $\frac{\partial f_i}{\partial x}$ for all $i \neq 1$ (otherwise, along the curve, f has ramification index at least m).

Case (2). The domain surface $\mathcal{C}$ is étale locally defined by the relation $xy = t^k$ at $(0, 0, 0)$ for some positive integer k, with $\pi(x, y, t) = t$. Furthermore, $f(x, y, t) \in R[[x, y]]^{\oplus r}$, $f(x, 0, 0) \neq 0$, and $f(0, y, 0) \neq 0$. We may suppose that $f(0, 0, t) = 0$. The argument is based on the set of lattice points (δ, ν) of monomials $x^\delta t^\nu$ ($\delta \geq -\nu/k$, $\nu \geq 0$) appearing in f (Figure 2).

There is a line L containing two distinct exponents $x^{\alpha_i} t^{\beta_i}$, $i = 1, 2$ of f with

$$\alpha_1 > 0, \qquad \beta_1 \geq 0, \qquad \alpha_2 < 0, \qquad \beta_2 \geq -k\alpha_2,$$

and with all exponents of f contained in the closed half-plane bounded below by L. We define $\ell = -(\beta_1 - \beta_2)/(\alpha_1 - \alpha_2)$. Notice that $\ell < k$, since the half-plane contains a lattice point corresponding to a power of y. If, among all exponents $x^\alpha t^\beta$, $(\alpha, \beta) \in L$, there is some pair for which the coefficient vectors of f are linearly independent, then $\mathbb{P}(Tf)(x, y, t)$ approaches different points along paths

$$(x, y) = (ct^\ell, c^{-1} t^{k-\ell})$$

as $c \in \mathbf{k}^\times$ varies. In this case, $\mathbb{P}(Tf)(p)$ is not well-defined.

Otherwise there is a constant tangent direction along such paths, which without loss of generality we take to be $[1, 0, \dots, 0] \in \mathbb{P}^{r-1} = \mathbb{P}(T_{f(p)} X)$. Putting

$$x = zt^\ell, \qquad y = z^{-1} t^{k-\ell},$$

we have

$$f(zt^\ell, z^{-1}t^{k-\ell}, t) = t^{\frac{\alpha_1\beta_2-\alpha_2\beta_1}{\alpha_1-\alpha_2}} g(z,t)$$

for some $g = (g_1, \ldots, g_r) \in \mathbf{k}[z,z^{-1}][[t]]^{\oplus r}$ with $g_i(z,0) = 0$ for $i \geq 2$, while $g_1(z,0)$ is a Laurent polynomial containing at least one positive and one negative power of z. It follows that there exists $c \in \mathbf{k}^\times$ such that $\frac{\partial g_1}{\partial z}$ vanishes at $(c,0)$. We are in the situation of Case (1): f is generically unramified, and $\mathbb{P}(Tf)$ approaches a point different from $[1,0,\ldots,0]$, on any irreducible component of the curve defined by $\frac{\partial g_1}{\partial z} = 0$.

Case (3). Set $p = f(E)$. We suppose that $\mathbb{P}(Tf)(E)$ is the point $[1,0,\ldots,0] \in \mathbb{P}^{r-1} = \mathbb{P}(T_pX)$. We may choose a projection to $\mathbb{P}^1$ from X, defined in a neighborhood of p, so that with p' the image of p in $\mathbb{P}^1$ the image in $T_{p'}\mathbb{P}^1$ of $(1,0,\ldots,0) \in T_pX$ is different from zero, the composite $f' : \mathcal{C} \to \mathbb{P}^1$ is well-defined, stable, and over $\operatorname{Spec} K$ is a stable ramified map, and the only irreducible components of $\mathcal{C}_0$ mapping by f' to p' are those mapping to p by f. Now we consider the stable ramified limit $\mathcal{C}' \to \mathcal{W}$ over $\operatorname{Spec} R$ of $\mathcal{C}' \to \mathbb{P}^1$ over $\operatorname{Spec} K$. There exists a component $E' \subset \mathcal{C}'_0$ over $E \subset \mathcal{C}_0$ (i.e. mapping into E via stabilization $\mathcal{C}' \to \mathcal{C}$) containing a smooth point that is a ramification point of $\mathcal{C}'_0 \to \mathcal{W}_0$ and is not the limit of any of the p_i. A pair of general sections approaching E' gives rise to a new target space $X[2]^+$ and a new stable map limit $\widetilde{\mathcal{C}} \to X[2]^+$.

The hypothesis concerning $\mathbb{P}(Tf)(E)$ implies that the image of every irreducible component of $\widetilde{\mathcal{C}}_0$ over E meeting $(X[2]^+)^{\mathrm{sm}}$ is a line intersecting $(X[2]^+)^{\mathrm{sing}} = \mathbb{P}^{r-1}$ at $[1,0,\ldots,0]$. Now, considering the new target space $\mathbb{P}^1[2]^+$ with its stable map limit $\widetilde{\mathcal{C}}'$, on which there is the component $\widetilde{E}'$ corresponding to $E' \subset \mathcal{C}'_0$, we see using Lemma 4.1.1 that there is a corresponding component $\widetilde{E} \subset \widetilde{\mathcal{C}}_0$, over E, having image a line in $X[2]^+$ and a ramification point which is not the limit of any of the p_i. We conclude by the argument of Case (1). □

Proof of Proposition 4.2.1 when $\dim X \geq 2$. We assume that the target $\mathcal{W}$ is $X[m]^+|_{\operatorname{Spec} K}$ for some given $\operatorname{Spec} K \to X[m]$. By Lemma 3.2.4 we have the tangent line map $\mathbb{P}(Tf) : \mathcal{C} \to \mathbb{P}(TX)$ and hence its ordinary stabilization with the n markings

$$\mathbb{P}(Tf)' : \mathcal{C}' \to \mathbb{P}(TX)$$

where $\mathcal{C}'$ is a suitable contraction of $\mathcal{C}$. With stable map extension $\overline{\mathcal{C}}' \to \mathbb{P}(TX)$ (with n markings) we apply Lemma 4.1.1 to each component E_i of $\overline{\mathcal{C}}'_0$ to get open sets $\mathcal{U}^{(i)} \subset \overline{\mathcal{C}}' \times \overline{\mathcal{C}}'$ where we may suppose $\mathcal{U}^{(i)}_0 \subset (E_i^{\mathrm{sm}} \times E_i^{\mathrm{sm}}) \setminus \Delta_{E_i^{\mathrm{sm}}}$ for $i = 1, \ldots, c$ where c denotes the number of components of $\overline{\mathcal{C}}'_0$. There is an open subset $\mathcal{U}$ of the product of the $\mathcal{U}^{(i)}$, with $\mathcal{U}_0 \neq \emptyset$, on which we may apply Proposition 2.7.2 to obtain $\operatorname{Spec} K \to X[m,2c]$ with $X[m,2c]^+|_{\operatorname{Spec} K} \cong \mathcal{W}$.

Arguing as in the proof of Lemma 4.1.1, we see that for a general section $\operatorname{Spec} R \to \mathcal{U}$ the stable map extension $\overline{\mathcal{C}}'' \to X[2c]^+ \times \mathbb{P}(TX)$ of the map $\mathcal{C} \to X[2c]^+ \times \mathbb{P}(TX)$ (the composite $\mathcal{C} \to X[m,2c]^+ \to X[2c]^+$ on the first factor, and the map $\mathbb{P}(Tf)$ on the second factor) has the property that the composite $\overline{\mathcal{C}}'' \to X[2c]^+ \times \mathbb{P}(TX) \to X[2c]^+$ remains stable. Hence the same holds if we extend using $n+2c$ sections; we therefore obtain the stable map extension $\overline{\mathcal{C}}'' \to X[n+2c]^+$.

Over K we have $X[m,n+2c]^+ \to X[n+2c]^+$ contracting only ruled components; correspondingly on $\mathcal{C} \to \mathcal{C}'$ each contracted component has a line non-fiber image of some end rational component of $\mathcal{C}$, joined by a chain of $\mathbb{P}^1$'s to a component that survives in $\mathcal{C}'$. Let j denote the total number of collapsed ruled components. We may put back these collapsed components by adding j sections. Indeed, if we let $\mathcal{I}$ denote the set of subsets of $N \sqcup \{1,\dots,2c\}$ describing the image of $\operatorname{Spec} K \to X[n+2c]$, then the target $X[m,n+2c]^+|_{\operatorname{Spec} K}$ is contained in a closed substack of $\mathfrak{X}[n+2c+j]_{n+2c}$ isomorphic to

$$\left(\bigcap_{I\in\mathcal{I}} \Delta_I\right) \times (B\mathbb{G}_m)^j,$$

where $B\mathbb{G}_m$ denotes the classifying stack of $\mathbb{G}_m$. Hence we have $\mathfrak{X}[n+2c+j]^+_{n+2c}|_{\operatorname{Spec} K} \cong \mathcal{W}$, determined up to an element of $(\mathbf{k}((t))^\times)^j$. The isomorphism class over $\operatorname{Spec} R$ is therefore determined by j integers. These are uniquely determined by the requirement that each line image of a rational end component mentioned above should in the limit tend to a non-fiber image, with at least one line per component tending in the limit to a non-fiber image not contained in the relative singular locus. For such an R-point of $\mathfrak{X}[n+2c+j]^+_{n+2c}$ we consider a lift over $\operatorname{Spec} K$ to $X[n+2c+j]$ which factors through $X[m:n+2c+j]$ (compatibly with $\operatorname{Spec} K \to X[m]$) and use this to determine a stable map limit $\overline{\mathcal{C}}''' \to X[n+2c+j]^+$ over R.

Using Lemmas 4.1.2 and 4.2.2 it follows from the stability condition on $\overline{\mathcal{C}}''' \to X[n+2c+j]^+$ that in the special fiber there are no contracted components. Some finite number of components may be mapped to lines in relative singular loci of $X[n+2c+j]^+/X[n+2c+j]$. If the number of such components is k then we obtain by repeated use of Lemma 4.1.2 a target $\widetilde{\mathcal{W}} := X[n+2c+j+k]^+|_{\operatorname{Spec} R}$ with isomorphism $\widetilde{\mathcal{W}}|_{\operatorname{Spec} K} \cong \mathcal{W}$, such that the stable map limit $\widetilde{C} \to \widetilde{\mathcal{W}}$ fulfills the conditions of the proposition. Indeed, AC is automatically satisfied by Proposition 2.2 of [14]; PRIC holds by Lemma 4.2.2; we have DPC because the n sections are among those defining the target $\widetilde{\mathcal{W}}$; and SC holds by construction, using Lemma 4.2.2. □

Finally we come to the main result of the paper.

Theorem 4.2.3. *The stack $\overline{\mathfrak{U}}_{g,\mu}(X,\beta)$ of (g,β,μ)-stable ramified maps to FM degeneration spaces of X is a proper Deligne-Mumford stack over* $\mathbf{k}$.

Proof. It remains only to verify the valuative criterion for separatedness; then properness follows from Proposition 4.2.1. Assume that for $i = 1, 2$ we have stable ramified maps $f_i : \mathcal{C}_i \to \mathcal{W}_i$, with $\mathcal{W}_i = g_i^* X[n_i]^+$ for some $g_i : \operatorname{Spec} R \to X[n_i]$, and an isomorphism of stable ramified maps over K, i.e., pair of isomorphisms (φ, ψ)

$$\varphi : \mathcal{C}_1|_{\operatorname{Spec} K} \to \mathcal{C}_2|_{\operatorname{Spec} K}, \qquad \psi : \mathcal{W}_1|_{\operatorname{Spec} K} \to \mathcal{W}_2|_{\operatorname{Spec} K}$$

satisfying

$$\psi \circ f_1|_{\operatorname{Spec} K} = f_2|_{\operatorname{Spec} K} \circ \varphi$$

where ψ fixes X and φ preserves the n sections of the $\mathcal{C}_i$. By the uniqueness of the extension of ordinary stable maps, it suffices to verify that targets $\mathcal{W}_1$ and $\mathcal{W}_2$ are isomorphic by an extension of ψ. To do so, we use the Tangent Line Map Condition.

By Lemma 3.2.4, we have the tangent line map

$$\mathbb{P}(Tf_i) : \mathcal{C}_i \to \mathbb{P}(TX)$$

and hence its ordinary stabilization with the n markings

$$\mathbb{P}(Tf_i)' : \mathcal{C}'_i \to \mathbb{P}(TX)$$

where $\mathcal{C}'_i$ is a suitable contraction of $\mathcal{C}_i$. Since the n-pointed stable maps $\mathbb{P}(Tf_1)'$ and $\mathbb{P}(Tf_2)'$ are equivalent over K, they are equivalent over R. Hence there is an isomorphism $\varphi' : \mathcal{C}'_1 \to \mathcal{C}'_2$ satisfying that:

- the diagram

$$\begin{array}{ccc} \mathcal{C}_1|_K & \xrightarrow{\varphi} & \mathcal{C}_2|_K \\ \downarrow & & \downarrow \\ \mathcal{C}'_1|_K & \xrightarrow{\varphi'|_K} & \mathcal{C}'_2|_K \end{array}$$

  commutes;
- $\mathbb{P}(Tf_1)' = \mathbb{P}(Tf_2)' \circ \varphi'$; and
- the n markings are preserved under φ'.

We consider all components of the closed fiber of $\mathcal{C}'_1$; for each such component, there is the corresponding one in the closed fiber of $\mathcal{C}_1$. Consider two general points on each such component and two sections passing through those points. These sections together with the n markings form $n + 2c$ sections of $\mathcal{C}_1 \to \operatorname{Spec} R$, where c is the number of components of the closed fiber of $\mathcal{C}'_1$. By φ', we obtain the corresponding sections of $\mathcal{C}_2 \to \operatorname{Spec} R$. Using DPC, we may assume that the images of those $n + 2c$ sections under f_i are pairwise distinct. Now applying Proposition 2.7.2 to g_i with those $n + 2c$ sections, we obtain an isomorphism of contractions of $\mathcal{W}_1$ and $\mathcal{W}_2$.

Let j denote the number of contracted components of $\mathcal{W}_i \to X[n + 2c]^+$ over K, and let us add j new sections as described in the proof of Proposition 4.2.1. Notice that the choice of expanded target, up to isomorphism, is determined completely by restriction over K of the given $(\mathcal{C}_i, \mathcal{W}_i)$, hence we obtain an isomorphism of contractions of $\mathcal{W}_i$ restricting to an isomorphism over K.

By the uniqueness of stable map extension, the number of components of the special fiber of $\mathcal{C}_i$ landing in the relative singular locus of $X[n+2c+j]^+$ is the same for $i = 1, 2$; let us call it k and now add k new sections of $\mathcal{C}_1 \to \operatorname{Spec} R$ and the corresponding ones of $\mathcal{C}_2 \to \operatorname{Spec} R$. We claim that the induced maps $\operatorname{Spec} R \to X[n_i, n+2c+j+k]$ are in fact in $X[n_i : n+2c+j+k]$. Stability is clear for any screen containing a non-line image, since it will then have two out of the $2c$ markings. It remains only to consider ruled screens containing none of the n markings and only line images. Then there is some non-fiber line image. If it is a limit of non-fiber line images on a ruled component over K then it will be stable by one of the j sections. Otherwise the non-fiber line maps to a line in a singular component of $X[n+2c+j]^+$, hence there will be one of the k sections. This shows that $\mathcal{W}_1$ and $\mathcal{W}_2$ are isomorphic under an extension of ψ. □

5. Ramified Gromov-Witten Invariants

5.1. **Obstruction theory.** The approach to relative obstruction theory suggested by J. Li at the beginning of §1.2 of [15] can be worked out in the case of the moduli stack of stable ramified maps using Olsson's deformation theory of log schemes [23]. If $(\mathcal{C} \to S, \mathcal{W} \to S, f : \mathcal{C} \to \mathcal{W})$ is a family of ramified stable maps then we have natural log structures $M^{\mathcal{C}/S}$ on $\mathcal{C}$, $M^{\mathcal{W}/S}$ on $\mathcal{W}$ and $N^{\mathcal{C}/S}$ and $N^{\mathcal{W}/S}$ on S making $(\mathcal{C}, M^{\mathcal{C}/S}) \to (S, N^{\mathcal{C}/S})$ and $(\mathcal{W}, M^{\mathcal{W}/S}) \to (S, N^{\mathcal{W}/S})$ log smooth morphisms. Following §3B of [19] there is a canonical log structure N on S, associated to the monoid pushout $N^{\mathcal{C}/S} \oplus_{N'} N^{\mathcal{W}/S}$ where N' is the submonoid of $N^{\mathcal{C}/S} \oplus N^{\mathcal{W}/S}$ generated by $(m \cdot \log(s'), \log(s))$ for every node of the geometric fibers of $\mathcal{C} \to S$, and if we let $(\mathcal{C}, M)$ denote the log scheme obtained as the fiber product $(\mathcal{C}, M^{\mathcal{C}/S}) \times_{(S, N^{\mathcal{C}/S})} (S, N)$ then there is, canonically, $f^* M^{\mathcal{W}/S} \to M$ making

$$\begin{array}{ccc} (\mathcal{C}, M) & \longrightarrow & (\mathcal{W}, M^{\mathcal{W}/S}) \\ \downarrow & & \downarrow \\ (S, N) & \longrightarrow & (S, N^{\mathcal{W}/S}) \end{array}$$

a commutative diagram of fine log schemes [9].

Proposition 5.1.1. *Let $S = \operatorname{Spec} A$ and let I be an A-module. Consider a square zero extension B of A by I. Let $f : \mathcal{C} \to \mathcal{W}$ be a stable ramified map over S. Let $(\tilde{\mathcal{C}} \to \tilde{S}, \{\tilde{p}_1, \dots, \tilde{p}_n\})$ and $(\tilde{\mathcal{W}} \to \tilde{S}, \tilde{\mathcal{W}} \to X)$ be extensions of $(\mathcal{C} \to S, \{p_1, \dots, p_n\})$ and $(\mathcal{W} \to S, \mathcal{W} \to X)$ over $\tilde{S} = \operatorname{Spec} B$, let $\tilde{q}_i : S \to \mathcal{W}$ be extensions of $q_i := f \circ p_i$ for $i = 1, \dots, n$, and let $\tilde{N}$ be a fine log structure on $\tilde{S}$ extending the log structure N on S, together with morphisms $N^{\tilde{\mathcal{C}}/\tilde{S}} \to \tilde{N}$ and $N^{\tilde{\mathcal{W}}/\tilde{S}} \to \tilde{N}$ extending the ones over S. Then* *there is a natural element*

$$ob(f,I) \in H^1(\mathcal{C}, f^*T^\dagger_{\mathcal{W}}(-\mu_1 p_1 - \cdots - \mu_n p_n) \otimes_{\mathcal{O}_S} I)$$

of obstruction to extension to a stable ramified map $\tilde{f} : \tilde{\mathcal{C}} \to \tilde{\mathcal{W}}$ *over* $\tilde{S}$ *such that* $\tilde{N}'$ *and* $\tilde{N}' \to N^{\tilde{\mathcal{W}}/\tilde{S}}$ *are compatible with the given* $N^{\tilde{\mathcal{C}}/\tilde{S}} \to \tilde{N}$ *and* $N^{\tilde{\mathcal{W}}/\tilde{S}} \to \tilde{N}$*. When the obstruction vanishes, the extensions* $\tilde{f}$ *satisfying the compatibility are a torsor under*

$$H^0(\mathcal{C}, f^*T^\dagger_{\mathcal{W}}(-\mu_1 p_1 - \cdots - \mu_n p_n) \otimes_{\mathcal{O}_S} I).$$

Proof. We use Theorem 5.9 in [23]. To enforce PRIC, we replace $\mathcal{W}$ with the (non-separated) union

$$\mathcal{W} \cup_{\mathcal{W}\setminus\{q_1,\dots,q_n\}} \mathrm{Bl}_{\{q_1,\dots,q_n\}}\mathcal{W}$$

where the log structure is the standard one on $\mathcal{W}$ and the standard one plus the exceptional divisor on $\mathrm{Bl}_{\{q_1,\dots,q_n\}}\mathcal{W}$. Then f gets replaced by a map f' sending $\mathcal{C} \setminus \{p_1,\dots,p_n\}$ to $\mathcal{W}$ and by the lift to $\mathrm{Bl}_{\{q_1,\dots,q_n\}}\mathcal{W}$ in a neighborhood of p_i (which is well-defined by Remark 3.2.2), and near p_i we have an identification of log tangent sheaves $f'^*T^\dagger_{\mathrm{Bl}_{\{q_1,\dots,q_n\}}\mathcal{W}} \cong f^*T^\dagger_{\mathcal{W}}(-\mu_i p_i)$. □

By standard machinery we have a perfect obstruction theory ([4])

$$R\pi_*(f^*T^\dagger_{\mathcal{W}}(-\mu_1 p_1 - \cdots - \mu_n p_n))^\vee \to L^\bullet_{\overline{\mathfrak{U}}_{g,\mu}(X,\beta)/\mathfrak{B}}$$

relative over the base stack $\mathfrak{B}$ of curves (prestable n-pointed of genus g), FM spaces of X with n-tuples of smooth pairwise distinct points, fine log structures, and pairs of morphisms of log structures

$$(\mathcal{C} \to S, \mathcal{W} \to S, N, N^{\mathcal{C}/S} \to N, N^{\mathcal{W}/S} \to N).$$

By Theorems 1.1 and 4.6 and Corollary 5.25 of [21] and Proposition 2.11 of [23] the stack $\mathfrak{B}$ is algebraic and pure-dimensional, so by [4] there is a virtual fundamental class $[\overline{\mathfrak{U}}_{g,\mu}(X,\beta)]^{\mathrm{vir}}$. Alternative ways to define a virtual fundamental class are by adding auxiliary log structures ([10], see also [18]) or orbifold structures ([1]).

Using the natural morphism

$$Te_i : \overline{\mathfrak{U}}_{g,\mu}(X,\beta) \to \mathbb{P}(TX),$$

we define *ramified Gromov-Witten invariants* by

$$\int_{[\overline{\mathfrak{U}}_{g,\mu}(X,\beta)]^{\mathrm{vir}}} \prod \psi_i^{a_i} Te_i^*(\gamma_i),$$

where ψ_i are gravitational descendants associated to the i-th marking and γ_i are cohomology classes of $\mathbb{P}(TX)$. These invariants are deformation invariant just like the usual Gromov-Witten invariants. When $\mu = (1,\dots,1) = (1^n)$ we write $\overline{\mathfrak{U}}_{g,n}(X,\beta)$ for $\overline{\mathfrak{U}}_{g,\mu}(X,\beta)$ and speak of *unramified Gromov-Witten invariants.*

5.2. **Pandharipande's conjectures.** In this subsection let $\mathbf{k}$ be the field of complex numbers. For threefold targets, R. Pandharipande has proposed a conjectural link between unramified Gromov-Witten invariants and certain integer quantities that can be defined with usual Gromov-Witten theory. A consequence would be the following statement.

Conjecture 5.2.1. *Let X be a smooth projective three-dimensional algebraic variety and β a curve class on X with $\int_\beta c_1(T_X) > 0$. Then the unramified Gromov-Witten invariants on X defined by integral cohomology classes on X, without gravitational descendants, are integers.*

Let us call a curve class *locally Fano* if it has positive intersection number with the anticanonical class. Generalizing the BPS counts from string theory, Pandharipande defined numbers $n_{g,\beta}(\gamma_1, \ldots, \gamma_n)$ which for locally Fano β are given by

$$\sum_{g\geq 0} n_{g,\beta}(\gamma_1, \ldots, \gamma_n)\lambda^{2g-2}\left(\frac{\sin(\lambda/2)}{\lambda/2}\right)^{2g-2+\int_\beta c_1(T_X)} = \sum_{g\geq 0}\langle\gamma_1, \ldots, \gamma_n\rangle_{g,\beta}\lambda^{2g-2}$$

where the $\langle\gamma_1, \ldots, \gamma_n\rangle_{g,\beta}$ are usual Gromov-Witten invariants and λ is a formal variable (see [24, 25]). He conjectured that the $n_{g,\beta}(\gamma_1, \ldots, \gamma_n)$ are integers, provided that the γ_i are integral cohomology classes. This was proved in the locally Fano case by Zinger [27].

According to Pandharipande, the unramified Gromov-Witten invariants in locally Fano curve classes of cohomology classes on X should yield BPS counts directly. Notice that by Zinger's result, this statement would imply Conjecture 5.2.1.

Conjecture 5.2.2. *If β is a locally Fano curve class on a nonsingular projective threefold X, then for any cohomology classes $\gamma_1, \ldots, \gamma_n$ on X,*

$$\int_{[\overline{\mathfrak{U}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod e_i^*(\gamma_i) = n_{g,\beta}(\gamma_1, \ldots, \gamma_n).$$

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