EQUIVARIANT BURNSIDE GROUPS: STRUCTURE AND OPERATIONS

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ABSTRACT. We introduce and study functorial and combinatorial constructions concerning equivariant Burnside groups.

1. INTRODUCTION

Let G be a finite group, and k a field of characteristic zero containing all roots of unity of order dividing G. In this paper, we continue the study of a new invariant in G-equivariant birational geometry over k, the *equivariant Burnside group*

$$\mathrm{Burn}_n(G),$$

introduced in [7], and building on [6], [5], [8], and [3].

The class of an n-dimensional G-variety in this group is computed on an appropriate smooth G-birational model X, called *standard form*: after a sequence of G-equivariant blowups we may assume that [12]:

- there exists a Zariski open $U \subset X$ such that the G-action on U is free,
- the complement $X \setminus U$ is a G-invariant simple normal crossing divisor,
- for every $g \in G$ and every irreducible component D of $X \setminus U$, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

The standard form is preserved under G-equivariant blowups with smooth centers which have normal crossings with respect to the components of D. Moreover, the stabilizer of every x on such X is an *abelian* subgroup of G [12, Thm. 4.1]. On such a model, the class of $X \circlearrowleft G$ is defined by:

$$[X \circlearrowleft G] := \sum_{H \subseteq G} \sum_{F} \mathfrak{s}_F \in \mathrm{Burn}_n(G), \tag{1.1}$$

with summation over conjugacy classes of *abelian* subgroups $H \subseteq G$ and strata $F \subseteq X$ with *generic* stabilizer H; the symbol

$$\mathfrak{s}_F := (H, N_G(H)/H \circlearrowright k(F), \beta_F(X)) \tag{1.2}$$

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records the action of the normalizer $N_G(H)$ of H on $k(F)$, the product of the function fields of the components of F, as well as the generic normal bundle representation $\beta_F(X)$ of H. The class in (1.1) takes values in a quotient of the free abelian group generated by such symbols, by certain *blow-up* relations, spelled out in [7, Definition 4.2], and ensuring that this expression is a well-defined G-birational invariant [7, Theorem 5.1].

In [3], we presented first geometric applications of this invariant. Here, we continue to explore *functorial* and *combinatorial* properties of $\mathrm{Burn}_n(G)$. We introduce and study:

- filtrations on $\mathrm{Burn}_n(G)$,
- the restriction homomorphism
$$\mathrm{Burn}_n(G) \to \mathrm{Burn}_n(G'),$$
where $G' \subset G$ is any subgroup,
- products,
- a combinatorial analog $\mathcal{BC}_n(G)$ of $\mathrm{Burn}_n(G)$, obtained by forgetting *field-theoretic* information, while keeping only discrete invariants encoded in a symbol (1.2).

One of the motivating problems in this field is to distinguish equivariant birational types of (projectivizations of) linear actions (see, e.g., [11], [4]). A sample question, raised in [2, Section 8], is: *Are there isomorphic finite subgroups of* PGL_3 *which are not conjugate in the plane Cremona group?*

Examples of equivariantly nonbirational representations considered in [13] required that G contains an abelian p-subgroup of rank equal to the dimension of the representation. Our formalism yields new examples without the rank condition on the group, see Example 5.3.

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2. Generalities

We adopt notational conventions from [7]:

- G is a finite group,
- k is a field of characteristic 0, containing roots of unity of order $|G|$,
- $H \subseteq G$ is an abelian subgroup, with character group
$$H^\vee := \mathrm{Hom}(H, k^\times),$$

- $\mathrm{Bir}_d(k)$ is the set of birational equivalence classes of d-dimensional algebraic varieties over k , i.e., the set of finitely generated fields of transcendence degree d over k; we identify a field with its isomorphism class in $\mathrm{Bir}_d(k)$,
- $\mathrm{Alg}_N(K_0)$ is the set of isomorphism classes of Galois algebras K over $K_0 \in \mathrm{Bir}_d(k)$ for the group
$$N := N_G(H)/H,$$
satisfying

 Assumption 1: the composite homomorphism
$$\mathrm{H}^1(N_G(H), K^\times) \to \mathrm{H}^1(H, K^\times)^N \to H^\vee \tag{2.1}$$
is surjective (see [7, Section 2] for more details).
- More generally, for a subgroup $M \subset N$ we denote by $\mathrm{Alg}_M(K_0)$ the set of isomorphism classes of M-Galois algebras K/K_0 (i.e., Galois algebras K over K_0 for the group M), such that $\mathrm{Ind}_M^N(K)$ satisfies Assumption 1. Of particular interest is
$$Z := Z_G(H)/H \subseteq N = N_G(H)/H.$$

Lemma 2.1. *Let* $K \in \mathrm{Alg}_N(K_0)$*. Then*
$$K \cong \mathrm{Ind}_Z^N(K')$$
for some $K' \in \mathrm{Alg}_Z(K_0)$*.*

Proof. With notation of Assumption 1, we have trivial H-action on $K^\times$, so
$$\mathrm{H}^1(H, K^\times) = \mathrm{Hom}(H, K^\times).$$
Writing $K \cong K^1 \times \cdots \times K^\ell$, where each K^i is a field, as rightmost map in (2.1) we take
$$\mathrm{H}^1(H, K^\times) \to \mathrm{H}^1(H, (K^1)^\times) \cong H^\vee.$$
Projection $K^\times \to (K^1)^\times$ is equivariant for the subgroup $Y \subseteq N$, defined by the condition of sending K^1 to K^1, where the action on $\mathrm{H}^1(H, (K^1)^\times)$ is given just by conjugation on H. Assumption 1 implies that the conjugation action is trivial, i.e., $Y \subseteq Z$. So the result holds with
$$K' = \mathrm{Ind}_Y^Z(K^1).$$
□

Remark 2.2. Assumption 1, for an N-Galois algebra K/K_0, of the form $\mathrm{Ind}_Z^N(K')$, where K'/K_0 is a Z-Galois algebra, may be expressed as the surjectivity of
$$\mathrm{H}^1(Z_G(H), K'^\times) \to H^\vee. \tag{2.2}$$

Given this, the proof of [7, Prop. 2.2] supplies an equivalence of categories between

- H-Galois algebras over étale K_0-algebras and
- $Z_G(H)$-Galois algebras with equivariant homomorphism from K';

in particular, there is then a $Z_G(H)$-Galois algebra L/K_0 with homomorphism $K' \to L$, compatible with the structure of Galois algebra for the group Z, respectively $Z_G(H)$. Assumption 1 is also implied by the existence of such a Galois algebra L and homomorphism $K' \to L$, as we see by using the Hochschild-Serre spectral sequence and Hilbert's Theorem 90 to identify $\mathrm{H}^1(Z_G(H), K'^\times)$ with $\mathrm{H}^1(H \times Z_G(H), L^\times)$. This allows us to view Assumption 1 as a lifting problem of Galois cohomology

$$\mathrm{H}^1(\mathrm{Gal}_{K_0}, Z_G(H)) \to \mathrm{H}^1(\mathrm{Gal}_{K_0}, Z)$$

and remark that the machinery of nonabelian cohomology (cf. [10, §1.3.2]) supplies an obstruction to lifting in $\mathrm{H}^2(\mathrm{Gal}_{K_0}, H)$.

We now recall the definition of the *equivariant Burnside group*

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$$

following [7, Section 4]: it is a $\mathbb{Z}$-module, generated by symbols

$$\mathfrak{s} := (H, N \circlearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup,
- $K \in \mathrm{Alg}_N(K_0)$, with $K_0 \in \mathrm{Bir}_d(k)$, and $d \le n$,
- $\beta = (b_1, \dots, b_{n-d})$, a sequence of nonzero elements of the character group $H^\vee$, that generate $H^\vee$.

The sequence of characters β determines a faithful $(n-d)$-dimensional representation of H over k, with trivial space of invariants. As every $(n-d)$-dimensional representation of H over k splits as a sum of one-dimensional representations, any faithful $(n-d)$-dimensional representation of H over k determines a sequence of characters, generating $H^\vee$, up to order. The ambiguity of order gives us the first of several relations that we impose on symbols:

(O): $(H, N \circlearrowright K, \beta) = (H, N \circlearrowright K, \beta')$ if β' is a reordering of β.

The further relations are **conjugation** and **blowup** relations:

(C): $(H, N \circlearrowright K, \beta) = (H', N' \circlearrowright K, \beta')$, when $H' = gHg^{-1}$ and $N' = N_G(H')/H'$, with $g \in G$, and β and β' are related by conjugation by g.

(B1): $(H, N \circlearrowright K, \beta) = 0$ when $b_1 + b_2 = 0$.

(B2): $(H, N \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, N \circlearrowright K, \beta_1) + (H, N \circlearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (b_1, b_2 - b_1, b_3, \dots, b_{n-d}), \quad \beta_2 := (b_2, b_1 - b_2, b_3, \dots, b_{n-d}), \tag{2.3}$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{N} \circlearrowright \overline{K}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

with

$$\overline{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-d}), \quad \bar{b}_i \in \overline{H}^\vee,$$

and $\overline{K}$ carries the action described in Construction **(A)** in [7, Section 2], applied to the character $b_1 - b_2$.

We permit ourselves to write a symbol in the form

$$(H, M \circlearrowright K, \beta) \tag{2.4}$$

with a subgroup $M \subset N$ and $K \in \mathrm{Alg}_M(K_0)$, with

$$(H, M \circlearrowright K, \beta) := (H, N \circlearrowright \mathrm{Ind}_M^N(K), \beta).$$

We further allow K_0 to be a *product* of fields; then (2.4) will denote the corresponding sum of symbols, one for each factor.

By Lemma 2.1, any symbol in $\mathrm{Burn}_n(G)$ is of the form

$$(H, Z \circlearrowright K, \beta),$$

with $K \in \mathrm{Alg}_Z(K_0)$. In this notation, Construction **(A)** has a compact formulation. Applied to a single character $b \in H^\vee$, this yields the subgroup

$$\overline{H} := \ker(b) \subset H$$

and the symbol

$$(\overline{H}, Z_G(H)/\overline{H} \circlearrowright K(t), \bar{\beta}),$$

where a $Z_G(H)$-action on $K(t)$ arises by lifting b via (2.2) and is trivial on $\overline{H}$, and $\bar{\beta}$ is obtained from β by applying the map $H^\vee \to \overline{H}^\vee$, as above.

Remark 2.3. Construction **(A)** may be applied to a collection of characters, yielding the same outcome as when applied iteratively, one character at a time.

A G-action on X in standard form always satisfies

Assumption 2: The stabilizers for the G-action on X are abelian, and for every H and F in (1.1) the composite homomorphism

$$\mathrm{Pic}^G(X) \to \mathrm{H}^1(N_G(H), k(F)^\times) \to H^\vee$$

is surjective, where the first map is given by restriction and the second is the map from Assumption 1, with $K = k(F)$.

Note that Assumption 2 implies Assumption 1, for every H and every $N_G(H)/H \circlearrowright k(F)$ (see [7, Rmk. 3.2(i)]).

A variant, that will occur below, is the requirement of surjectivity, when we restrict to a given subgroup of $\mathrm{Pic}^G(X)$. Given this, we will say that Assumption 2 holds for the given subgroup of $\mathrm{Pic}^G(X)$.

3. Filtrations

In this section, we explore additional combinatorial constructions on equivariant Burnside groups $\mathrm{Burn}_n(G)$, reflecting the geometry of the G-action on strata with given generic stabilizers.

Definition 3.1. A G-prefilter is a collection $\mathbf{H}$ of pairs (H, Y) consisting of an abelian subgroup $H \subseteq G$ and a subgroup

$$Y \subseteq Z = Z_G(H)/H,$$

such that $\mathbf{H}$ is closed under conjugation, i.e., for $(H, Y) \in \mathbf{H}$ we have

$$(gHg^{-1}, gYg^{-1}) \in \mathbf{H}, \qquad \text{for all } g \in G.$$

Definition 3.2. Given a G-prefilter $\mathbf{H}$, we let

$$\mathrm{Burn}_n^{\mathbf{H}}(G)$$

be the quotient of $\mathrm{Burn}_n(G)$ by the subgroup generated by classes of the form

$$(H, Y \circlearrowright K, \beta),$$

where $K \in \mathrm{Alg}_Y(K_0)$ is a *field*, and

$$(H, Y) \notin \mathbf{H}.$$

Proposition 3.3. *Let* $\mathbf{H}$ *be a* G*-prefilter such that if* $(H, Y) \in \mathbf{H}$*, with* H *nontrivial, then* $(\langle H, g\rangle, Y/\langle \bar{g}\rangle) \in \mathbf{H}$ *for all* $g \in Z_G(H)$ *satisfying*

$$\bar{g} \in Y \qquad \text{and} \qquad Y \subseteq Z_G(g)/H.$$

Then $\mathrm{Burn}_n^{\mathbf{H}}(G)$ *is generated by triples*

$$(H, Y \circlearrowright K, \beta),$$

where $K \in \mathrm{Alg}_Y(K_0)$ *is a field and*

$$(H, Y) \in \mathbf{H},$$

subject to relations **(O)**, **(C)**, **(B1)**, *and* **(B2)** *applied to these triples.*

Proof. For any

$$(H, Y \circlearrowright K, \beta)$$

with $K \in \mathrm{Alg}_Y(K_0)$ a field, the term Θ_2 from **(B2)**, when nontrivial, consists of a subgroup $\ker(b)$ of H for some $b \in H^\vee$, a field $K(t)$ with action of the pre-image of Y in $Z_G(H)/\ker(b)$, and a sequence of characters. If $(H, Y) \notin \mathbf{H}$, then by hypothesis the pair consisting of $\ker(b)$ and the pre-image of Y is not in $\mathbf{H}$. This observation establishes the proposition, since **(B1)** involves just one triple, and in Θ_1 from **(B2)** the group and algebra do not change. □

Example 3.4.

- For G abelian, we have
$$\mathrm{Burn}_n^G(G) = \mathrm{Burn}_n^{\{(G,\mathrm{triv})\}}(G),$$
where the left side was introduced in [7, §8]: this is the quotient of $\mathrm{Burn}_n(G)$ by all triples whose first entry is a proper subgroup of G.
- For $\mathbf{H}$ consisting of all (H, Y) with H nontrivial cyclic and Y noncyclic, and k algebraically closed, $\mathrm{Burn}_2^{\mathbf{H}}(G)$ appeared in [3, §7.4].

Remark 3.5. One can additionally suppress the field information, which will lead to *combinatorial* analogues of Burnside groups. We will explore this in Section 8.

4. Nontrivial generic stabilizers

In this section, we introduce a version of the equivariant Burnside group, relevant for considerations of actions with *nontrivial* generic stabilizer.

Let G be a finite group. A variant of the equivariant Burnside group takes the additional data of a finite index set

$$I \subset \mathbb{N}.$$

The *equivariant indexed Burnside group*

$$\mathrm{Burn}_{n,I}(G),$$

is defined as a quotient of the $\mathbb{Z}$-module generated by symbols

$$(H \subseteq H', N' \circlearrowright K, \beta, \gamma),$$

where

- $H \subseteq H'$ are abelian subgroups of G,
- $N' := N_{N_G(H)}(H')/H'$,
- $K \in \mathrm{Alg}_{N'}(K_0)$, with $K_0 \in \mathrm{Bir}_d(k)$, and $d \le n - |I|$,
- $\beta = (b_1, \dots, b_{n-d-|I|})$, a sequence of nonzero characters of H', trivial upon restriction to H, that generate $(H'/H)^\vee$,
- $\gamma = (c_i)_{i\in I}$ is a sequence of elements of $H'^\vee$, such that the images of c_i in $H^\vee$ generate $H^\vee$.

As in Section 2, we permit ourselves to write a symbol in the form

$$(H \subseteq H', M' \circlearrowright K, \beta, \gamma),$$

where $M' \subset N'$ is a subgroup. Every symbol may be expressed as

$$(H \subseteq H', Z' \circlearrowright K, \beta, \gamma), \qquad Z' := Z_G(H')/H'.$$

(Notice that $Z_G(H') = Z_{N_G(H)}(H')$.)

These symbols are subject to relations:

(O): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = (H \subseteq H', N' \circlearrowright K, \beta', \gamma)$ if β' is a reordering of β.

(C): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = (gHg^{-1} \subseteq gH'g^{-1}, gN'g^{-1} \circlearrowright K, \beta', \gamma')$ for $g \in G$, with β and β', respectively γ and γ', related by conjugation by g.

(B1): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = 0$ when $b_1 + b_2 = 0$.

(B2): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = \Theta_1 + \Theta_2$, where Θ_1 and Θ_2 are as in Section 2, with H prepended and γ, respectively $\bar{\gamma}$, appended to the corresponding symbols.

Remark 4.1. By analogy with Remark (2.2), we may express Assumption 1, for the Galois algebra K, as the surjectivity of the middle vertical map

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^1(Z_G(H')/H, K^\times) & \to & \mathrm{H}^1(Z_G(H'), K^\times) & \to & \mathrm{H}^1(H, K^\times)^{Z_G(H')/H} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (H'/H)^\vee & \longrightarrow & H'^\vee & \longrightarrow & H^\vee & \longrightarrow & 0 \end{array}$$

Here, the top row comes from the Hochschild-Serre spectral sequence. In a symbol, we have β generating the left-hand group in the bottom row, while γ is a sequence of characters of H', whose images generate $H^\vee$. Consequently, β and γ together generate $H'^\vee$. Thus we have a homomorphism

$$\psi_I \colon \mathrm{Burn}_{n,I}(G) \to \mathrm{Burn}_n(G),$$

sending $(H\subseteq H', Z'\circlearrowright K,\beta,\gamma)$ to

$$(H', Z'\circlearrowright K, \beta\cup\gamma)$$

when γ is a sequence of nontrivial characters, otherwise to 0.

In order to explain the relevance of this definition, we introduce a map which converts some of the characters in γ to a transcendental extension of the Galois algebra. Let

$$J\subseteq I$$

be a subset. Given a symbol $(H\subseteq H', Z'\circlearrowright K,\beta,\gamma)$, we use J to define subgroups

$$\overline{H}' := \bigcap_{i\in I\setminus J}\ker(c_i)\subseteq H',$$
$$\overline{H} := H\cap \overline{H}'\subseteq H.$$

Then we define

$$\omega_{I,J}\colon \operatorname{Burn}_{n,I}(G)\to \operatorname{Burn}_{n,J}(G),$$

by applying Construction **(A)** when possible:

$$(H\subseteq H', Z'\circlearrowright K,\beta,\gamma)\mapsto(\overline{H}\subseteq\overline{H}', Z_G(H')/\overline{H}'\circlearrowright K((t_i)_{i\in I\setminus J}),\bar\beta,\bar\gamma),$$

where $\bar\gamma=(\bar c_j)_{j\in J}$, when all of the characters of $\bar\beta$ are nonzero, and

$$(H\subseteq H', Z'\circlearrowright K,\beta,\gamma)\mapsto 0,\qquad \text{otherwise.}$$

This is compatible with relations: the only one that is nontrivial to check is **(B2)**, where Θ_1 maps to $\overline{\Theta}_1$, as we see by dividing into cases according to the vanishing of $\bar b_1$ or $\bar b_2$, or their equality, and Θ_2 maps to $\overline{\Theta}_2$, as we see using Remark 2.3.

We recall the setting of [7, Defn. 5.4]: Let X be a smooth projective variety of dimension n, with a generically free action of G, satisfying Assumption 2. Let D_1, …, D_ℓ be G-stable divisors, with

$$D_I := \bigcap_{i\in I} D_i,\qquad \text{for } I\subseteq\mathcal{I}:=\{1,\dots,\ell\},\quad D_\emptyset = X.$$

We suppose, for notational simplicity, that for every I the generic stabilizers of the components of D_I belong to a single conjugacy class of subgroups, and take H_I to be a representative. Then to $I\subseteq M\subseteq\mathcal{I}$ we attach the following class in $\operatorname{Burn}_{n,I}(G)$:

$$\chi_{I,M}(X\circlearrowleft G,(D_i)_{i\in\mathcal{I}}) := \sum_{H'\supseteq H_I}\ \sum_{\substack{W\subset D_I\\ \text{generic stabilizer } H'\\ \{i\in\mathcal{I}\,|\,W\subset D_i\}=M}} (H_I\subseteq H', N'\circlearrowright k(W),\beta,\gamma),$$

where

- the first sum is over conjugacy class representatives H' of abelian subgroups of $N_G(H_I)$, containing H_I,
- the second sum is over $N_{N_G(H_I)}(H')$-orbits of components W with generic stabilizer H', contained in components of D_I with generic stabilizer H_I and satisfying $\{i\in\mathcal{I}\,|\,W\subset D_i\}=M$,
- $\beta=\beta_W(D_I)$ encodes the normal bundle to W in D_I, and
- $\gamma=(c_i)_{i\in I}$, the characters coming from D_i with $i\in I$.

Then

$$[\mathcal{N}_{D_I/X}\circlearrowleft G]^{\mathrm{naive}}=\sum_{I\subseteq M\subseteq\mathcal{I}}\ \sum_{M\setminus I\subseteq J\subseteq M}\psi_{I\cap J}(\omega_{I,I\cap J}(\chi_{I,M}(X\circlearrowleft G,(D_i)_{i\in\mathcal{I}}))),$$

where the terms with $J=\emptyset$ contribute $[\mathcal{N}^{\circ}_{D_I/X}\circlearrowleft G]^{\mathrm{naive}}$. This provides some insight to [7, Lemma 5.7].

5. Fibrations

In this section, we define a projectivized version of the equivariant indexed Burnside group and use it to give a formula for the class in $\mathrm{Burn}_n(G)$ of the projectivization of a sum of line bundles.

Let G be a finite group and $I\subset\mathbb{N}$ a *nonempty* finite index set. The *equivariant projectively indexed Burnside group*

$$\mathrm{Burn}_{n,\mathbb{P}(I)}(G)$$

is defined with generators and relations as in Section 4, where

- β consists of $n-d-|I|+1$ characters (so $d\le n-|I|+1$),
- the *differences* of pairs of characters of γ should generate $H^\vee$,
- and there is an additional relation:

(P): If $\gamma'-\gamma$ is a constant sequence then

$$(H\subseteq H',N'\circlearrowright K,\beta,\gamma)=(H\subseteq H',N'\circlearrowright K,\beta,\gamma').$$

We define

$$\omega_{\mathbb{P}(I),J}\colon\ \mathrm{Burn}_{n,\mathbb{P}(I)}(G)\to\mathrm{Burn}_{n,J}(G),$$

for a *proper* subset

$$J\subsetneq I,$$

by

- choosing $i_0\in I\setminus J$,
- applying **(P)** to get a representative symbol

  $$(H\subseteq H',N'\circlearrowright K,\beta,\gamma)$$

  with $\gamma_{i_0}=0$, and

- applying $\omega_{I\setminus\{i_0\},J}$ to the class of $(H \subseteq H', N' \circlearrowright K, \beta, (c_i)_{i\in I\setminus\{i_0\}})$ in $\mathrm{Burn}_{n,I\setminus\{i_0\}}(G)$.

Let X be a smooth projective variety over k. Assume that X carries a G-action, and let $L_0, \dots, L_r$ be G-linearized line bundles on X. The next statement examines the condition, for G to act generically freely on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$, so that Assumption 2 satisfied.

Lemma 5.1. *Let X be a smooth projective variety over k with a G-action and G-linearized line bundles $L_0, \dots, L_r$. Let H be the stabilizer at the generic point of a component of X, and let us denote the $N_G(H)$-orbit of the component by X'. The following are equivalent.*

(i) *The N-action on X' satisfies Assumption 2, and H is abelian with $H^\vee$ spanned by the differences of characters determined by $L_0, \dots, L_r$.*
(ii) *The G-action on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ is generically free and satisfies Assumption 2.*
(iii) *The G-action on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ is generically free and satisfies Assumption 2 for $L_0, \dots, L_r$, together with the G-linearized line bundles on X associated with N-linearized line bundles on X'.*

The statement is inspired by [7, Lemma 7.3].

Proof. The action of G on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ is generically free if and only if the action of $N_G(H)$ on $\mathbb{P}(L_0|_{X'} \oplus \cdots \oplus L_r|_{X'})$ is generically free. The latter has generic stabilizer $\bigcap_{i=1}^r \ker(b_i - b_0)$. Thus the condition on H in (i) is equivalent to the condition of generically free action in (ii) and in (iii). We assume this from now on.

An N-linearized line bundle on X' determines an $N_G(H)$-linearized line bundle on X', with trivial H-action. An $N_G(H)$-linearized line bundle on X' determines, and is determined by, G-linearized line bundle on X; this is the meaning of the associated line bundles in (iii). Conversely, a G-linearized line bundle on X which restricts to an $N_G(H)$-linearized line bundle on X' with trivial H-action, gives rise to an N-linearized line bundle on X'.

We start by showing (i) implies (iii), using the interpretation of Assumption 2 in terms of the representability of a certain morphism from the quotient stack to a product of copies of $B\mathbb{G}_m$. Given (i), we have such a representable morphism

$$[X'/N] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m.$$

Correspondingly, the fibers of the composite morphism

$$[X/G] \to [X'/N] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m$$

all have constant stabilizer group H. The condition in (i) implies that the H-representation given by $b_0, \dots, b_r$ is faithful. With $r+1$ additional factors $B\mathbb{G}_m$ we get a representable morphism from $[X/G]$, hence also from $\mathbb{P}(L_0 \oplus \dots \oplus L_r)$.

Since trivially (iii) implies (ii), it remains only to show (ii) implies (i). Generally, a line bundle on a projective bundle is isomorphic to the pullback of a line bundle from the base, twisted by a power of the tautological line bundle. Two vector bundles, one obtained from the other by tensoring by a line bundle, have isomorphic projectivizations, the tautological line bundle of one obtained from the other by tensoring by the pullback of the line bundle from the base. A sum of line bundles, after such tensoring, may be brought in a form with trivial ith factor, for any i, and this way see that any power of the tautological line bundle on $\mathbb{P}(L_0 \oplus \dots \oplus L_r)$, restricted to the open $U_i \subset \mathbb{P}(L_0 \oplus \dots \oplus L_r)$ defined by nonvanishing on the component L_i, is identified with a line bundle pulled back from the base; all of these assertions are valid in an equivariant setting. With the notation of Assumption 2 for $\mathbb{P}(L_0 \oplus \dots \oplus L_r)$, we always have $\mathrm{Spec}(k(F)) \subset U_i$ for some i. So, (ii) implies that the G-action on $\mathbb{P}(L_0 \oplus \dots \oplus L_r)$ satisfies Assumption 2 for $\mathrm{Pic}^G(X)$. Since

$$\mathbb{P}(L_0 \oplus \dots \oplus L_r) \to X$$

admits equivariant sections, we deduce that some finite collection of G-linearized line bundles on X determines a representable morphism

$$[X/G] \to B\mathbb{G}_m \times \dots \times B\mathbb{G}_m.$$

Replacing each by a tensor product with combinations of $L_0, \dots, L_r$, we obtain a G-linearized line bundle on X that comes from an N-linearized line bundle on X'. The corresponding morphism

$$[X'/N] \to B\mathbb{G}_m \times \dots \times B\mathbb{G}_m$$

is representable, and thus we have (i). □

Proposition 5.2. *Let X be a smooth projective variety of dimension $n-r$ over k with a G-action and G-linearized line bundles $L_0, \dots, L_r$. We assume the conditions and adopt the notation of Lemma 5.1. We define $I := \{0, \dots, r\}$ and the following class in* $\mathrm{Burn}_{n,\mathbb{P}(I)}(G)$*:*

$$\xi(X \circlearrowleft G, (L_i)_{i\in I}) := \sum_{H' \supseteq H} \sum_{\substack{W \subset X' \\ \text{generic stabilizer } H'}} (H \subseteq H', N' \circlearrowright k(W), \beta, \gamma),$$

where

- *the first sum is over abelian subgroups H' of G that contain H, up to conjugacy in $N_G(H)$,*

- *the second sum is over $N_{N_G(H)}(H')$-orbits of components $W \subset X'$ where the generic stabilizer is H',*
- *$\beta = \beta_W(X')$ encodes the normal bundle to W in X', and*
- *$\gamma = (c_i)_{i\in I}$, the characters coming from L_i with $i \in I$.*

Then

$$[\mathbb{P}(L_0 \oplus \cdots \oplus L_r) \circlearrowleft G] = \sum_{J \subsetneq I} \psi_J(\omega_{\mathbb{P}(I),J}(\xi(X \circlearrowleft G, (L_i)_{i\in I})))$$

in $\mathrm{Burn}_n(G)$.

Proof. We identify each contribution to $[\mathbb{P}(L_0 \oplus \cdots \oplus L_r) \circlearrowleft G]$ as

$$V = \varphi_J^{-1}(W),$$

for some W in the definition of $\xi(X \circlearrowleft G, (L_i)_{i\in I})$, where φ_J denotes the projection to X from the projectivization of $\bigoplus_{i\in I\setminus J} L_i$. Then,

$$(H \subseteq H', N' \mathrel{\subset\!\!\!\!\cdot}\; k(W), \beta, \gamma) \in \mathrm{Burn}_{n,\mathbb{P}(I)}(G)$$

maps under $\psi_J \circ \omega_{\mathbb{P}(I),J}$ to

$$(\overline{H}', N_{N_G(H)}(\overline{H}') \mathrel{\subset\!\!\!\!\cdot}\; k(V), \beta_V(X)).$$

□

Example 5.3. Let $G := C_5 \times \mathfrak{S}_3$, acting on $X := \mathbb{P}^1$ via an irreducible 2-dimensional representation of $\mathfrak{S}_3$. We take L_0 to be trivial and L_1 to be the twist of $\mathcal{O}_{\mathbb{P}^1}(1)$ by a nontrivial character χ of C_5. Then we have the situation of Lemma 5.1 with $H = C_5$ and $N = \mathfrak{S}_3$, and the conditions of the lemma are satisfied. We have

$$\begin{aligned}\xi(X \circlearrowleft G, (L_0, L_1)) = {}& (C_5 \subseteq C_5, \mathfrak{S}_3 \curvearrowright k(\mathbb{P}^1), \emptyset, (0, \chi)) \\ &+ (C_5 \subseteq C_5 \times \langle(1,2)\rangle, \mathrm{triv} \curvearrowright k, (0,1), (0, (\chi, 0))) \\ &+ (C_5 \subseteq C_5 \times \langle(1,2)\rangle, \mathrm{triv} \curvearrowright k, (0,1), (0, (\chi, 1))) \\ &+ (C_5 \subseteq C_5 \times \mathfrak{A}_3, \mathfrak{S}_3/\mathfrak{A}_3 \curvearrowright k \times k, (0,1), (0, (\chi, 1))).\end{aligned}$$

The outcome of Proposition 5.2 is

$$\begin{aligned}[\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] = {}& (\mathrm{triv}, G \curvearrowright k(\mathbb{P}^1)(t), \emptyset) + (\langle(1,2)\rangle, C_5 \overset{\chi}{\curvearrowright} k(t), 1) \\ &+ (C_5, \mathfrak{S}_3 \curvearrowright k(\mathbb{P}^1), \chi) + (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \curvearrowright k, ((0,1), (\chi, 0))) \\ &+ (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \curvearrowright k, ((0,1), (\chi, 1))) \\ &+ (C_5 \times \mathfrak{A}_3, \mathfrak{S}_3/\mathfrak{A}_3 \curvearrowright k \times k, ((0,1), (\chi, 1))) \\ &+ (C_5, \mathfrak{S}_3 \curvearrowright k(\mathbb{P}^1), -\chi) + (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \curvearrowright k, ((0,1), (-\chi, 0))) \\ &+ (C_5 \times \langle(1,2)\rangle, \mathrm{triv} \curvearrowright k, ((0,1), (-\chi, 1))) \\ &+ (C_5 \times \mathfrak{A}_3, \mathfrak{S}_3/\mathfrak{A}_3 \curvearrowright k \times k, ((0,1), (-\chi, 1))).\end{aligned}$$

In the notation of Section 3 we observe that the G-prefilter

$$\mathbf{H} := \{(C_5, \mathfrak{S}_3)\}$$

satisfies the condition of Proposition 3.3. Upon projection

$$\mathrm{Burn}_2(G) \to \mathrm{Burn}_2^{\mathbf{H}}(G)$$

(see Definition 3.2), we obtain the class

$$(C_5, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), \chi) + (C_5, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), -\chi) \in \mathrm{Burn}_2^{\mathbf{H}}(G).$$

This class is nonzero. Moreover, it is different for $\chi \in \{\pm 1\}$ as compared to $\chi \in \{\pm 2\}$.

Geometrically, the situation above arises as follows: Consider the 3-dimensional representation $W_\chi = 1 \oplus (V \otimes \chi)$ of G, sum of a trivial 1-dimensional representation and twist by χ of the standard 2-dimensional representation V of $\mathfrak{S}_3$. This gives a generically free action of G on $\mathbb{P}^2 = \mathbb{P}(W_\chi)$, with a G-fixed point $\mathfrak{p}$. To bring the G-action into a form where Assumption 2 is satisfied, we need to blow up $\mathfrak{p}$, and

$$[\mathbb{P}(W_\chi) \circlearrowleft G] = [\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] \in \mathrm{Burn}_2(G).$$

6. Products

Let G' and G'' be finite groups. Define a product map

$$\mathrm{Burn}_{n'}(G') \times \mathrm{Burn}_{n''}(G'') \to \mathrm{Burn}_{n'+n''}(G' \times G'').$$

On symbols, it is given by

$$((H', Z' \circlearrowright K', \beta'), (H'', Z'' \circlearrowright K'', \beta'')) \mapsto (H, Z \circlearrowright K, \beta), \qquad (6.1)$$

where

- $H = H' \times H''$,
- $Z = Z' \times Z''$,
- $K = K' \otimes_k K''$, with the natural action of Z,
- $\beta = \beta' \cup \beta''$.

Proposition 6.1. *The product map* (6.1) *is well-defined, and satisfies*

$$([X' \circlearrowleft G'], [X'' \circlearrowleft G'']) \mapsto [X' \times X'' \circlearrowleft G' \times G''].$$

Proof. The map clearly respects relations. The only point to remark is that in **(B2)**, the condition for nontriviality of Θ_2 holds for β' if and only if it holds for $\beta = \beta' \cup \beta''$. $\square$

7. Restrictions

Let G be a finite group and $G' \subset G$ a subgroup. A G-action on a quasiprojective variety X induces an action of G', and thus it is natural to propose the existence of a restriction homomorphism from $\mathrm{Burn}_n(G)$ to $\mathrm{Burn}_n(G')$, acting by

$$[X \circlearrowleft G] \mapsto [X \circlearrowleft G']. \tag{7.1}$$

In this section we establish the existence and uniqueness of this homomorphism.

Example 7.1. Suppose H is an abelian subgroup of G, contained in G'. Symbols, identified in $\mathrm{Burn}_n(G)$ by relation **(C)**, might no longer be identified in $\mathrm{Burn}_n(G')$. E.g., with $G = \mathfrak{D}_4$ and $G' = C_4$ the restriction of $(G', G/G' \circlearrowright k \times k, 1) \in \mathrm{Burn}_1(G)$ to $\mathrm{Burn}_1(G')$ has to be a sum of two symbols with distinct characters:

$$(G', G/G' \circlearrowright k \times k, 1) \mapsto (G', \mathrm{triv} \circlearrowright k, 1) + (G', \mathrm{triv} \circlearrowright k, 3).$$

Theorem 7.2. *For all $n \geq 0$, there exists a unique homomorphism of abelian groups*

$$\mathrm{res}^G_{G'} : \mathrm{Burn}_n(G) \to \mathrm{Burn}_n(G').$$

compatible with (7.1)*.*

Proof. By Lemma 2.1, it suffices to consider symbols of the form

$$\mathfrak{s} = (H, Z \circlearrowright K, \beta).$$

When we act by conjugation by some element of G, we obtain an equivalent symbol, where H is replaced by a conjugate, the corresponding centralizer quotient replaces Z, and conjugation is used to form from β a sequence of characters of the conjugate of H. By conjugation we have a transitive action of G on a set $\mathfrak{S}$ of symbols, where $\mathfrak{s} \in \mathfrak{S}$ has stabilizer $Z_G(H)$. The restriction of the action to G' consists of finitely many orbits; in the formula below the sum is over orbit representatives

$$\mathfrak{s}' = (H', Z' \circlearrowright K, \beta'),$$

such that the restriction $\beta'|_{H'\cap G'}$ of β' to $H' \cap G'$ has trivial space of invariants; here, Z' denotes $Z_G(H')/H'$. Then we define the restriction to G' by

$$\mathfrak{s} \mapsto \sum_{\mathfrak{s}'} (H' \cap G', (Z_G(H') \cap G')/(H' \cap G') \circlearrowright K, \beta'|_{H'\cap G'});$$

this map respects relations. Uniqueness follows from [7, Rmk. 5.16]. $\square$

As an application of the restriction construction, we obtain a map

$$\mathrm{Burn}_{n'}(G) \times \mathrm{Burn}_{n''}(G) \to \mathrm{Burn}_{n'+n''}(G),$$

using the product construction in Section 6 with $G' = G'' = G$, followed restriction to the diagonal

$$G \subset G \times G.$$

This map on Burnside groups sends

$$([X' \circlearrowleft G], [X'' \circlearrowleft G]) \mapsto [X' \times X'' \circlearrowleft G].$$

8. Combinatorial analogs

Here we define and study a quotient

$$\mathrm{Burn}_n(G) \to \mathcal{BC}_n(G)$$

to a *combinatorial* version of the equivariant Burnside group, by forgetting the information about the Galois algebra.

Definition 8.1. The *combinatorial symbols group*

$$\mathcal{BC}_n(G)$$

is the $\mathbb{Z}$-module, generated by symbols

$$(H, Y, \beta)$$

with H abelian, $Y \subseteq Z_G(H)/H$, and β a sequence of nonzero elements generating $H^\vee$, of length at most n, modulo relations:

(O): $(H, Y, \beta) = (H, Y, \beta')$ if β' is a reordering of β.

(C): $(H, Y, \beta) = (gHg^{-1}, gYg^{-1}, \beta')$ for $g \in G$, with β and β' related by conjugation by g.

(B1): $(H, Y, \beta) = 0$ when $b_1 + b_2 = 0$.

(B2): $(H, Y, \beta) = \Theta_1 + \Theta_2$, where Θ_1 and Θ_2 are as in Section 2, i.e.,

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y, \beta_1) + (H, Y, \beta_2), & \text{otherwise,} \end{cases}$$

with β_1 and β_2 as in (2.3), and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{Y}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

where $\overline{H} = \ker(b_1 - b_2)$, $\overline{Y}$ is the pre-image of Y in $Z_G(H)/\overline{H}$, and $\bar{\beta}$ consists of the restrictions to $\overline{H}$ of the characters of β.

Proposition 8.2. *The map sending the class of a triple*

$$(H, Y \circlearrowright K, \beta) \in \mathrm{Burn}_n(G),$$

for fields $K \in \mathrm{Alg}_Y(K_0)$*,* $K_0 \in \mathrm{Bir}_d(k)$*, with* $d \le n$*, to*

$$[k' : k](H, Y, \beta) \in \mathcal{BC}_n(G),$$

where k' *is the algebraic closure of* k *in* K_0*, gives a surjective homomorphism*

$$\mathrm{Burn}_n(G) \to \mathcal{BC}_n(G).$$

Proof. This is clear from the description of the relations in $\mathrm{Burn}_n(G)$ from Section 2. ☐

Definition 8.3. Given a G-prefilter $\mathbf{H}$, we let

$$\mathcal{BC}_n^{\mathbf{H}}(G)$$

be the quotient of $\mathcal{BC}_n(G)$ by the subgroup generated by classes (H, Y, β) with $(H, Y) \notin \mathbf{H}$.

Exactly as in Section 3 we have

Proposition 8.4. *Let* $\mathbf{H}$ *be a* G*-prefilter, satisfying the hypothesis of Proposition 3.3. Then* $\mathcal{BC}_n^{\mathbf{H}}(G)$ *is generated by symbols* (H, Y, β) *for* $(H, Y) \in \mathbf{H}$*, subject to relations* **(O)***,* **(C)***,* **(B1)***, and* **(B2)** *applied to these symbols.*

Additionally, upon passage to the combinatorial analogue we also have the other structures developed in this paper:

- equivariant (projectively) indexed combinatorial Burnside group;
- product map;
- restriction homomorphisms.

Example 8.5. Suppose that G is abelian.

- We have (cf. [7, §8])

$$\mathcal{B}_n(G) = \mathcal{BC}_n^{(G,\mathrm{triv})}(G),$$

  where $\mathcal{B}_n(G)$ is the symbols group from [5].
- There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Burn}_n(G) & \longrightarrow & \mathcal{BC}_n(G) \\ \downarrow & & \downarrow \\ \mathrm{Burn}_n^G(G) & \longrightarrow & \mathcal{B}_n(G) \end{array}$$

  (The factor factor $[k' : k]$ in Proposition 8.2 matches the similar factor in [7, Prop. 8.1].)

9. Applications

As a first application of the formalism in [7] for nonabelian groups, we gave in [3] an example of $G = C_2 \times \mathfrak{S}_3$-actions on $\mathbb{P}^2$ and a quadric surface $Q \subset \mathbb{P}^3$, which were distinguished by the respective classes in $\mathrm{Burn}_2(G)$. The actions are *stably* G-equivariantly rational [9].

Here we give a further application, for $G = \mathfrak{S}_4$, acting on $\mathbb{P}^2$ and a del Pezzo surface of degree 6. This example was treated in [1], via birational rigidity techniques (Noether inequality).

We recall basic facts about the subgroup lattice of $G = \mathfrak{S}_4$:

- Conjugacy classes of nonabelian subgroups: $\mathfrak{S}_3$, $\mathfrak{D}_4$, $\mathfrak{A}_4$,
- Conjugacy classes of abelian subgroups: trivial, even $\mathbb{Z}/2$, odd $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, even $\mathfrak{K}_4 \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$, odd $\mathfrak{K}_4$.

First action: On $\mathbb{P}^2$, we consider the projectivization of the standard 3-dimensional representation V_3, with respect to basis

$$(-1,1,1,-1), \quad (1,-1,1,-1), \quad (1,1,-1,-1),$$

given by the 4 matrices below.

$$\sigma := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \tau := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \lambda_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here σ and τ generate $\mathfrak{S}_3$ and λ_1, λ_2 the even $\mathfrak{K}_4$.

The restriction of V_3 to $\mathfrak{D}_4$ decomposes as 1-dimensional plus irreducible 2-dimensional representation. Each $\mathfrak{D}_4 \subset G$, gives a distinguished line and a point; together these form a triangle. We blow up the 3 points to get a hexagon of lines, which form two triples of disjoint lines, each line has faithful Klein 4-group action and generic stabilizer even $\mathbb{Z}/2$. The intersection points of the lines have stabilizer even $\mathfrak{K}_4$. We blow up these intersection points. The result is a wheel of 12 rational curves:

$$\begin{array}{ccccccccccc} D_1 & — & R_1 & — & D_1' & — & R_2 & — & D_2 & — & R_3 \\ | & & & & & & & & & & | \\ R_6 & — & D_3' & — & R_5 & — & D_3 & — & R_4 & — & D_2' \end{array}$$

Each rational curve has generic stabilizer even $\mathbb{Z}/2$, and their intersection points have stabilizer even $\mathfrak{K}_4$. The 12 curves form 3 orbits: $\{D_1, D_2, D_3\}$

and $\{D_1', D_2', D_3'\}$ consist of lines with generic stabilizer even $\mathbb{Z}/2$ and faithful $\mathfrak{K}_4$-action, and the lines in the G-orbit $\{R_1, \dots, R_6\}$ have generic stabilizer even $\mathbb{Z}/2$ and a nontrivial $\mathbb{Z}/2$-action.

The restriction of V_3 to $\mathfrak{S}_3$ also decomposes into a 1-dimensional and an irreducible 2-dimensional representation. Looking at G-orbits, we find a G-orbit of 4 distinguished $\mathfrak{S}_3$-lines, which intersect in 6 points with odd $\mathfrak{K}_4$ stabilizer. We also have a G-orbit of 4 distinguished points with $\mathfrak{S}_3$-stabilizer. We blow up the points with odd $\mathfrak{K}_4$-stabilizer and also those with $\mathfrak{S}_3$-stabilizer. We get a G-orbit of 6 lines, each with $\mathbb{Z}/2$-action and generic stabilizer odd $\mathbb{Z}/2$, as well as an orbit of 4 exceptional curves with $\mathfrak{S}_3$-action.

Second action: Let X be a del Pezzo surface of degree 6 given by

$$x_0y_0z_0 = x_1y_1z_1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

We write the action of G in coordinates

$$x := x_0/x_1, \quad y := y_0/y_1, \quad z := z_0/z_1.$$

Then, $\mathfrak{S}_3 = \langle \sigma, \tau \rangle$ acts by permuting x, y, z; λ_1 changes signs on x and z, and λ_2 changes signs on x and y.

There are three orbits of points, of length 4, with stabilizer $\mathfrak{S}_3$ (see [1, Lemma 1.3]). Blowing these up, we obtain 3 G-orbits of $\mathfrak{S}_3$-lines. These do not contribute to $[X \circlearrowleft G]$. There are also two G-orbits of points with $\mathfrak{D}_4$-stabilizers, these points are precisely the intersection points of the 6 lines at infinity, i.e., in the locus

$$\{x_0 = 0\} \cup \{y_0 = 0\} \cup \{z_0 = 0\}.$$

These lines have generic stabilizer even $\mathbb{Z}/2$ and a nontrivial $\mathbb{Z}/2$-action, and they form a single G-orbit. After we blow up the two orbits of 3 points points, we obtain precisely the wheel configuration we described above.

To summarize, the difference

$$[X \circlearrowleft G] - [\mathbb{P}^2 \circlearrowleft G]$$

is a symbol

$$(\text{odd } \mathbb{Z}/2, \mathbb{Z}/2 \circlearrowright k(t), (1)) \tag{9.1}$$

corresponding to a G-orbit of 6 lines with generic stabilizer odd $\mathbb{Z}/2$ and nontrivial $\mathbb{Z}/2$-action. By a computation, analogous to the determination of $\mathrm{Burn}_2(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$ in [3, §5.4], the class (9.1) is nontrivial in $\mathrm{Burn}_2(G)$.

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