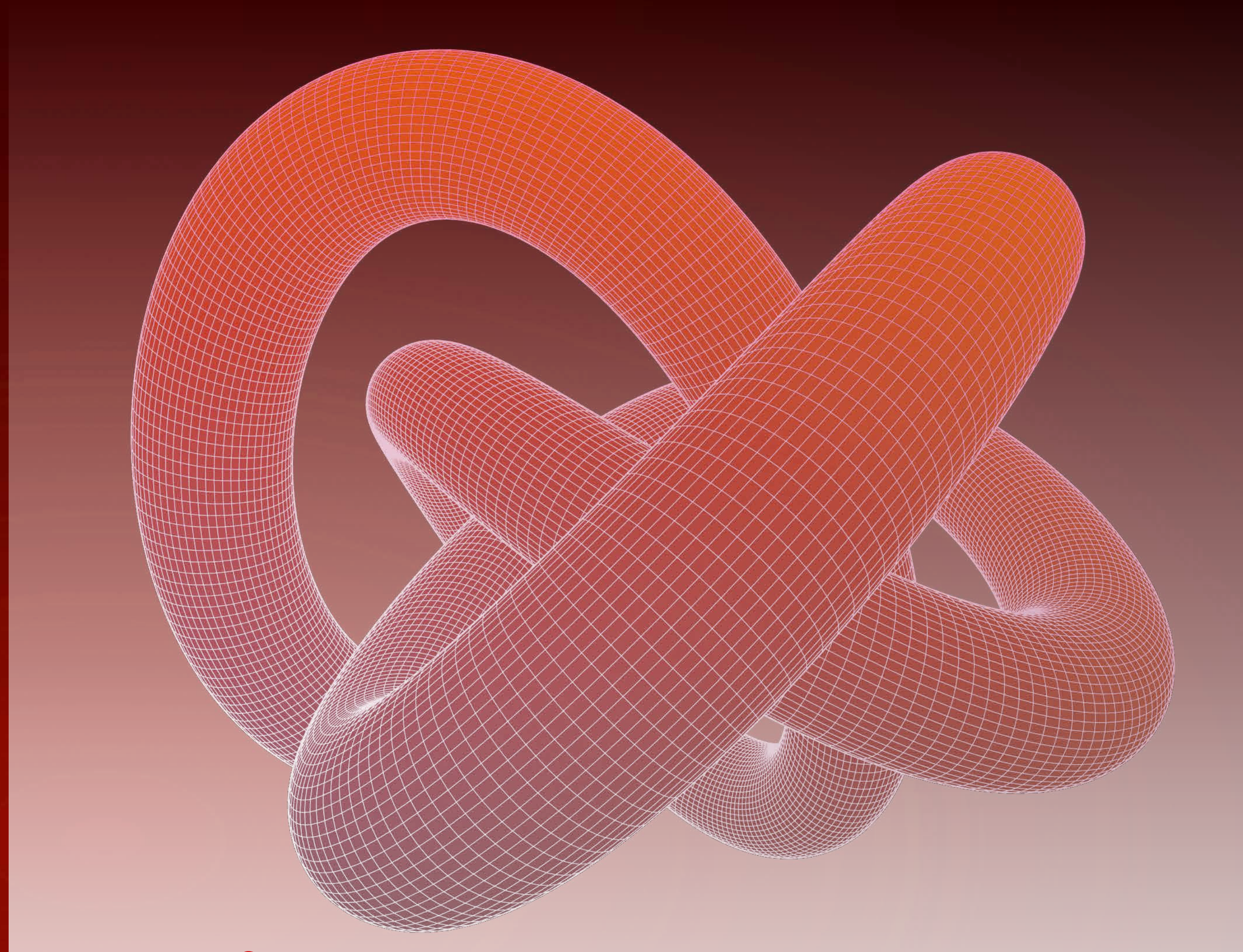

Categorifying quantum invariants

The discovery of the Jones polynomial in 1984 marked an important point in the development of the low-dimensional topology, and over the last 37 years many new invariants of knots, links and 3-dimensional manifolds, known as quantum invariants, have been discovered. We spoke to Professor Anna Beliakova about her work on categorification of quantum invariants, research which brings together elements of several different fields.

The process of categorification can be broadly thought of as a way of building new mathematical structures, analogous to constructing a house by first laying the foundations, then putting in the walls and subsequently adding more walls between those that are already there. In mathematics, a category is a set of objects, together with a set of morphisms between them, and then further morphisms can be added later on between the existing ones. "For example, think of points as objects, with lines connecting these points as a first layer of morphisms. The so-called 2-morphisms are surfaces between these lines – then we can bring a 3-dimensional object between these 2-dimensional surfaces and so on," explains Anna Beliakova, Professor of Mathematics at the University of Zurich. As the Principal Investigator of a research project based at her university, Professor Beliakova is working to categorify quantum invariants. "When we categorify, we essentially bring these mathematical objects one categorical level higher," she outlines.

Categorification

This research builds on earlier work in the field of knot theory, in which researchers consider knots which form a closed loop and so cannot be undone by pulling, stretching or other deformations, unlike say a shoelace or a tie for example. Researchers can project a knot onto a plane and so effectively replace this 3-dimensional object by its planar diagram. However, there are infinitely many diagrams representing different projections of the same knot. Hence, diagrams cannot help to tell knots apart. "A knot invariant is a mathematical object (e.g. a polynomial) assigned to a diagram which remains the same under deformations. Such invariants are our main tool to distinguish knots," says Professor Beliakova.

The first knot invariant was discovered in 1923 by Alexander, yet very little progress was made over the following 60 years. The discovery of the Jones polynomial by Vaughan Jones in 1984 led to the establishment of the theory of quantum invariants and ushered in a period of fairly rapid progress. "A large number of new invariants have since been uncovered. The use of a technique called skein relations enables us to compute these link invariants," outlines Professor Beliakova.

Further important progress was made when the Russian-American mathematician Mikhail Khovanov developed what has come to be known as the Khovanov homology in the late '90s, the second recombination of quantum invariants. "A polynomial with positive coefficients can be categorified as a graded vector space. However, the Jones polynomial may have negative coefficients and hence chain complexes are needed to categorify it," explains Professor Beliakova.

A chain complex is a collection of Abelian groups with a differential between them that squares to zero. "For us an important fact is that the homology of the Khovanov chain complex is a knot invariant, which is functorial, which means that surfaces bounded by knots in the 4-ball induce maps between Khovanov homologies of these knots," outlines Professor Beliakova. "This functoriality is essential for the proof that the Khovanov homology detects the unknot, the fact that after almost 40 years of study is still unknown about the Jones polynomial."

Of her own research Professor Beliakova says: "We are revealing new structures in Khovanov type complexes. For example, we discovered an action of some groups on these chain complexes. Surprisingly enough, this purely algebraic action may have deep topological implications."

In the meantime, many other quantum link invariants have been categorified. A real challenge is the categorification of the 3-manifold invariants. In fact, there is a deep intrinsic relationship between links and 3-dimensional manifolds: every 3-manifold can be obtained by surgery on a link, the surgery consists of taking out a neighbourhood of a link in the 3-sphere and then regluing it back differently. This procedure can also be imitated algebraically resulting in quantum 3-manifold invariants built out of quantum link invariants. During this process the polynomial variable is sent to a p-th root of unity. "A categorification of this construction is probably one of the most challenging open problems in quantum topology," explains Professor Beliakova. "It requires a use of chain complexes in which the p-th power of the differential is zero. We know very little about them".

Annular Khovanov homology and invariants of 4-manifolds

Professor Beliakova is also pursuing several other strands of research. The theory of so-called quantum annular Khovanov homology has been developed by Professor Beliakova and her colleagues, Dr. Krzysztof Putyra and Professor Stephan Wehrli, using so-called horizontal traces in some bicategories. "This provides new functorial invariants for annular links" says Professor Beliakova. The wider aim in Professor Beliakova's research is to essentially uncover new structures in low-dimensional topology and significant progress is being made. "We hope that ultimately our work will help to categorify quantum 3-manifold invariants and reveal their topological properties," she continues.

Recently Professor Beliakova, in collaboration with Dr. Marco De Renzi, discovered a new class of invariants of smooth 4-dimensional manifolds. In dimension four it may happen that the same topological 4-manifold has infinitely many different smooth structures, and hence, sensitive invariants are needed to distinguish between them.

Aside from that, the work of Professor Beliakova and Dr. Marco De Renzi opens up other research directions. For example, it may now be possible to access the Andrews-Curtis conjecture, a conjecture which is thought to be false, but has not yet been proven to be so. "In order to prove that it's false, a counterexample is required," says Professor Beliakova. With an invariant that has been developed in the course of her research, Professor Beliakova hopes to provide such counterexamples and make tangible progress on this specific conjecture. "The conjecture has been open since 1965 and belongs to the field of combinatorial group theory, but I hope now we will make progress through our approach, bringing together elements from many different fields, including representation theory and low-dimensional topology," she continues.

We can consider 3-dimensional manifolds as maps between 2-dimensional ones. We can extend this by building more and more morphisms, and going into higher and higher categorical levels.

CATEGORIFICATION OF QUANTUM 3-MANIFOLD INVARIANTS

Project Objectives
The aims of the project are to develop a theory of traces in higher categories, use it for a construction of annular Khovanov homology — a link homology theory that is functorial with respect to the annular link cobordism, define new topological invariants of smooth 4-manifolds and study their properties and finally, push forward categorifications of quantum 3-manifold invariants.

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