Construction of an Edwards' probability measure on

$\mathcal{C}(\mathbb{R}_+, \mathbb{R})$

Joseph Najnudel

February 20, 2010

Abstract

In this article, we prove that the measures $\mathbb{Q}_T$ associated to the one-dimensional Edwards' model on the interval $[0, T]$ converge to a limit measure $\mathbb{Q}$ when T goes to infinity, in the following sense: for all $s \geq 0$ and for all events Λ_s depending on the canonical process only up to time s, $\mathbb{Q}_T(\Lambda_s) \to \mathbb{Q}(\Lambda_s)$.

Moreover, we prove that, if $\mathbb{P}$ is Wiener measure, there exists a martingale $(D_s)_{s \in \mathbb{R}_+}$ such that $\mathbb{Q}(\Lambda_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\Lambda_s} D_s)$, and we give an explicit expression for this martingale.

Keywords: Edwards' model, polymer measure, Brownian motion, penalisation, local time.

1   Introduction and statement of the main theorems

Edwards' model is a model for polymers chains, which is defined by considering Brownian motion "penalised" by the "quantity" of its self-intersections (see also [4]). More precisely, for $d \in \mathbb{N}^*$, and $T > 0$, let $\mathbb{P}_T^{(d)}$ be Wiener measure on the space $\mathcal{C}([0, T], \mathbb{R}^d)$, and $(X_t^{(d)})_{t \in [0,T]}$ the corresponding canonical process. The d-dimensional Edwards' model on $[0, T]$ is defined by the probability measure $\mathbb{Q}_T^{(d),\beta}$ on $\mathcal{C}([0, T], \mathbb{R}^d)$ such that, very informally:

$$\mathbb{Q}_T^{(d),\beta} = \frac{\exp\left(-\beta \int_0^T \int_0^T \delta(X_s^{(d)} - X_u^{(d)}) \, ds \, du\right)}{\mathbb{P}_T^{(d)}\left[\exp\left(-\beta \int_0^T \int_0^T \delta(X_s^{(d)} - X_u^{(d)}) \, ds \, du\right)\right]} \cdot \mathbb{P}_T^{(d)} \tag{1}$$

where β is a strictly positive parameter, and δ is Dirac measure at zero.

(In this article, we always denote by $\mathbb{Q}[V]$ the expectation of a random variable V under the probability $\mathbb{Q}$).

Of course, (1) is not really the definition of a probability measure, since the integral with respect to Dirac measure is not well-defined. However, it has been proven that one can define rigorously the measure $\mathbb{Q}_T^{(d),\beta}$ for $d = 1, 2, 3$, by giving a meaning to (1) (for $d \geq 4$, the Brownian path has no self-intersection, so the measure $\mathbb{Q}_T^{(d),\beta}$ has to be equal to $\mathbb{P}_T^{(d)}$).

In particular, for $d = 1$, one has formally the equality:

$$\int_0^T \int_0^T \delta(X_s^{(1)} - X_u^{(1)}) \, ds \, du = \int_{-\infty}^{\infty} (L_T^y)^2 \, dy \tag{2}$$

where $(L_T^y)_{y \in \mathbb{R}}$ is the continuous family of local times of $(X_s^{(1)})_{s \leq T}$ (which is $\mathbb{P}_T^{(1)}$-almost surely well-defined).

2

Therefore, one can take the following (rigorous) definition:

$$\mathbb{Q}_T^{(1),\beta} = \frac{\exp\left(-\beta \int_{-\infty}^{\infty} (L_T^y)^2 \, dy\right)}{\mathbb{P}_T^{(1)}\left[\exp\left(-\beta \int_{-\infty}^{\infty} (L_T^y)^2 \, dy\right)\right]} \cdot \mathbb{P}_T^{(1)}.$$

Under $\mathbb{Q}_T^{(1),\beta}$, the canonical process has a ballistic behaviour; more precisely, Westwater (see [22]) has proven that for $T \to \infty$, the law of $\frac{X_T^{(1)}}{T}$ under $\mathbb{Q}_T^{(1),\beta}$ tends to $\frac{1}{2}(\delta_{b^*\beta^{1/3}} + \delta_{-b^*\beta^{1/3}})$, where δ_x is Dirac measure at x and b^* is a universal constant (approximately equal to 1.1).

This result was improved in [18] (see also [17]), where van der Hofstad, den Hollander and König show that $\frac{|X_T^{(1)}| - b^*\beta^{1/3}T}{\sqrt{T}}$ tends in law to a centered gaussian variable, which has a variance equal to a universal constant (approximately equal to 0.4; in particular, smaller than one).

Moreover, in [19], the authors prove large deviation results for the variable X_T under $\mathbb{Q}_T^{(1),\beta}$.

In dimension 2, the problem of the definition of Edwards' model was solved by Varadhan (see [16], [8], [10]). In this case, it is possible to give a rigorous definition of $I := \int_0^T \int_0^T \delta(X_s^{(2)} - X_u^{(2)}) \, ds \, du$, but this quantity appears to be equal to infinity. However, if one formally subtracts its expectation (i.e. consider the quantity: $I - \mathbb{P}_T^{(2)}[I]$), one can define a finite random variable which has negative exponential moments of any order; therefore, if we replace $\int_0^T \int_0^T \delta(X_s^{(2)} - X_u^{(2)}) \, ds \, du$ by this random variable in equation (1), we obtain a rigorous definition of $\mathbb{Q}_T^{(2),\beta}$. Moreover, this probability is absolutely continuous with respect to Wiener measure.

In dimension 3 (the most difficult case), subtracting the expectation (this technique is also called "Varadhan renormalization") is not sufficient to define Edwards' model.

3

However, by a long and difficult construction, Weswater (see [20], [21]) has proven that it is possible to define the probability $\mathbb{Q}_T^{(3),\beta}$; this construction has been simplified by Bolthausen in [1] (at least if β is small enough). Moreover, the measures $(\mathbb{Q}_T^{(3),\beta})_{\beta \in \mathbb{R}_+^*}$ are mutually singular, and singular with respect to Wiener measure.

The behaviour of the canonical process under $\mathbb{Q}_T^{(d),\beta}$, as $T \to \infty$, is essentially unknown for $d = 2$ and $d = 3$. One conjectures that the following convergence holds:

$$\mathbb{Q}_T^{(3),\beta} [|||X_T|||] \to D\, T^\nu$$

where $D > 0$ depends only on d and β, and where ν is equal to $3/4$ for $d = 2$ and approximately equal to 0.588 for $d = 3$ (see [17], Chap. 1).

At this point, we note that all the measures considered above are defined on finite interval trajectories (exactly, on $\mathcal{C}([0,T], \mathbb{R})$).

An interesting question is the following: is it possible to define Edwards' model on trajectories indexed by $\mathbb{R}_+$?

More precisely, if $\mathbb{P}^{(d)}$ is Wiener measure on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and $(X_s^{(d)})_{s \in \mathbb{R}_+}$ the corresponding canonical process, is it possible to define a measure $\mathbb{Q}^{(d),\beta}$ (for all $\beta > 0$) such that, informally:

$$\mathbb{Q}^{(d),\beta} = \frac{\exp\left(-\beta \int_0^\infty \int_0^\infty \delta(X_s^{(d)} - X_u^{(d)})\, ds\, du\right)}{\mathbb{P}^{(d)}\left[\exp\left(-\beta \int_0^\infty \int_0^\infty \delta(X_s^{(d)} - X_u^{(d)})\, ds\, du\right)\right]} \cdot \mathbb{P}^{(d)} \ ?$$

In this article, we give a positive answer to this question in dimension one. The construction of the corresponding measure is analogous to the construction given by Roynette, Vallois and Yor in their articles about penalisation (see [14], [11], [13], [12]).

More precisely, let us replace the notation $\mathbb{P}^{(1)}$ by $\mathbb{P}$ for the standard Wiener measure and the notation $(X_s^{(1)})_{s \in \mathbb{R}_+}$ by $(X_s)_{s \in \mathbb{R}_+}$ for the canonical process. If $(\mathcal{F}_s)_{s \in \mathbb{R}_+}$ is the natural filtration of X, and if for all $T \in \mathbb{R}_+$, the measure $\mathbb{Q}_T^\beta$ is defined by:

$$\mathbb{Q}_T^\beta = \frac{\exp\left(-\beta \int_{-\infty}^{\infty} (L_T^y)^2 \, dy\right)}{\mathbb{P}\left[\exp\left(-\beta \int_{-\infty}^{\infty} (L_T^y)^2 \, dy\right)\right]} \cdot \mathbb{P}$$

where $(L_T^y)_{T \in \mathbb{R}_+, y \in \mathbb{R}}$ is the jointly continuous version of the local times of X ($\mathbb{P}$-almost surely well-defined), the following theorem holds:

Theorem 1.1 *For all $\beta > 0$, there exists a unique probability measure $\mathbb{Q}^\beta$ such that for all $s \geq 0$, and for all events $\Lambda_s \in \mathcal{F}_s$:*

$$\mathbb{Q}_T^\beta(\Lambda_s) \underset{T \to \infty}{\longrightarrow} \mathbb{Q}^\beta(\Lambda_s). \tag{3}$$

Theorem 1.1 is the main result of our article.

Let us remark that if $\Lambda_s \in \mathcal{F}_s$ ($s \geq 0$) and $\mathbb{P}(\Lambda_s) = 0$, then $\mathbb{Q}_T^\beta(\Lambda_s) = 0$, since $\mathbb{Q}_T^\beta$ is, by definition, absolutely continuous with respect to $\mathbb{P}$. Hence, if Theorem 1.1 is assumed, $\mathbb{Q}^\beta(\Lambda_s)$ is equal to zero.

Therefore, the restriction of $\mathbb{Q}^\beta$ to $\mathcal{F}_s$ is absolutely continuous with respect to the restriction of $\mathbb{P}$ to $\mathcal{F}_s$, and there exists a $\mathbb{P}$-martingale $(D_s^\beta)_{s \geq 0}$ such that for all s:

$$\mathbb{Q}^\beta_{|\mathcal{F}_s} = D_s^\beta . \mathbb{P}_{|\mathcal{F}_s}.$$

In our proof of Theorem 1.1, we obtain an explicit formula for the martingale $(D_s^\beta)_{s \geq 0}$. However, we need to define other notations before giving this formula.

Let ν be the measure on $\mathbb{R}_+^*$, defined by $\nu(dx) = x \, dx$, and let $L^2(\nu)$ be the set of

5

functions g from $\mathbb{R}_+^*$ to $\mathbb{R}$ such that:

$$\int_0^\infty [g(x)]^2 \nu(dx) < \infty,$$

equipped with the scalar product:

$$\langle\, g|h\,\rangle = \int_0^\infty g(x)\, h(x)\, \nu(dx).$$

The operator $\mathcal{K}$ defined from $L^2(\nu) \cap \mathcal{C}^2(\mathbb{R}_+^*)$ to $\mathcal{C}(\mathbb{R}_+^*)$ by:

$$[\mathcal{K}(g)](x) = 2g''(x) + \frac{2\, g'(x)}{x} - xg(x) \tag{4}$$

is the infinitesimal generator of the process $2R$ killed at rate x at level x, where R is a Bessel process of dimension two; it is a Sturm-Liouville operator, and there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\nu)$, consisting of eigenfunctions of $\mathcal{K}$, with the corresponding negative eigenvalues: $-\rho_0 > -\rho_1 \geq -\rho_2 \geq -\rho_3 \geq ...$, where $\rho := \rho_0$ is in the interval $[2.18, 2.19]$.

Moreover, the functions $(e_n)_{n\in\mathbb{N}}$ are analytic and bounded (they tend to zero at infinity, faster than exponentially), and e_0 is strictly positive (these properties are quite classical, and they are essentially proven in [17], Chap. 2 and 3; see also [6]).

Now, for $l \in \mathbb{R}_+$, let us denote by $(Y_l^y)_{y\in\mathbb{R}}$ a process from $\mathbb{R}$ to $\mathbb{R}_+$ such that:

- $(Y_l^{-y})_{y\geq 0}$ is a squared Bessel process of dimension zero, starting at l.

- $(Y_l^y)_{y\geq 0}$ is an independent squared Bessel process of dimension two.

Moreover, let f be a continuous function with compact support from $\mathbb{R}$ to $\mathbb{R}_+$, and let M be a strictly positive real such that $f(x) = 0$ for all $x \notin [-M, M]$. We define the

following quantities:

$$A_+^{\beta,M}(f) = \int_0^\infty dl\, \mathbb{E}\left[e^{\int_{-\infty}^M \left[-\beta(Y_l^y + f(y))^2 + \rho\beta^{2/3}\, Y_l^y\right]\, dy}\, e_0(\beta^{1/3}\, Y_l^M)\right],$$

$$A_-^{\beta,M}(f) = A_+^{\beta,M}(\tilde{f}),$$

where $\tilde{f}$ is defined by $\tilde{f}(x) = f(-x)$, and

$$A^{\beta,M}(f) = A_+^{\beta,M}(f) + A_-^{\beta,M}(f).$$

With these notations, we can state the following theorem, which gives an explicit formula

for the martingale $(D_s^\beta)_{s \geq 0}$:

Theorem 1.2 *For all $\beta > 0$ and for all continuous and positive functions f with compact*

support, the quantity $A^{\beta,M}(f)$ is finite, different from zero, and does not depend on the

choice of $M > 0$ such that $f = 0$ outside the interval $[-M, M]$; therefore, we can write:

$A^\beta(f) := A^{\beta,M}(f)$. Moreover, for all $s \geq 0$, the density D_s^β of the restriction of $\mathbb{Q}^\beta$ to

$\mathcal{F}_s$, with respect to the restriction of $\mathbb{P}$ to $\mathcal{F}_s$, is given by the equality:

$$D_s^\beta = e^{\rho\beta^{2/3}\, s}\cdot \frac{A^\beta(L_s^{\bullet + X_s})}{A^\beta(0)}, \tag{5}$$

where $L_s^{\bullet + X_s}$ denotes the function F (which depends on the trajectory $(X_u)_{u \leq s}$) such that

$F(y) = L_s^{y + X_s}$ for all $y \in \mathbb{R}$.

Remark: The independence of $A^{\beta,M}(f)$ with respect to M (provided the support of f

is included in $[-M, M]$) can be checked directly by using the fact that

$$\left(\exp\left(\int_0^x \left[-\beta(Y_l^y)^2 + \rho\beta^{2/3}Y_l^y\right]\, dy\right)\, e_0(\beta^{1/3}\, Y_l^x)\right)_{x \geq 0} \tag{6}$$

is a martingale, property which can be easily proven by using the differential equation

satisfied by e_0.

For $l > 0$, $\mu \in \mathbb{R}$ and $v > 0$ let us now define the following quantity:

$$K_l^{(\mu)}(v) = \alpha_l(v) e^{\mu v} \mathbb{E}\left[e^{-2 \int_0^v V_u^{(l/2,v)} \, du} \right]. \tag{7}$$

where $\alpha_l(v) = \frac{l}{\sqrt{8\pi v^3}} e^{-l^2/8v}$ denotes the density of the first hitting time of zero of a Brownian motion starting at $l/2$ (or equivalently, the density of the last hitting time of $l/2$ of a standard Bessel process of dimension 3), and $(V_u^{(l/2,v)})_{u \leq v}$ is the bridge of a Bessel process of dimension 3 on $[0, v]$, starting at $l/2$ and ending at 0.

To simplify the notation, we set:

$$K_l(v) = K_l^{(0)}(v)$$

Moreover, let us consider, for $v > 0$, the function χ_v defined by:

$$\chi_v(l) = \frac{K_l(v)}{l} = \frac{1}{\sqrt{8\pi \, v^3}} e^{-l^2/8v} \mathbb{E}\left[e^{-2 \int_0^v V_u^{(l/2,v)} \, du} \right]. \tag{8}$$

for $l > 0$.

With these notations, Theorem 1.2 is a essentially a consequence of the two propositions stated below.

Proposition 1.3 *When T goes to infinity:*

$$e^{\rho T} \mathbb{P}\left[e^{-\int_{-\infty}^{\infty} [L_T^y + f(y)]^2 \, dy} \mathbb{1}_{X_T \in [0, M]} \right] \longrightarrow 0. \tag{9}$$

Proposition 1.4 *When T goes to infinity:*

$$e^{\rho T} \mathbb{P}\left[e^{-\int_{-\infty}^{\infty} [L_T^y + f(y)]^2 \, dy} \mathbb{1}_{X_T \geq M} \right] \longrightarrow K \, A_+^{1,M}(f) < \infty, \tag{10}$$

where $K \in \mathbb{R}_+^*$ is a universal constant (in particular, K does not depend on f and M). Moreover, for all $v > 0$, $\chi_v \in L^2(\nu)$ and the constant K is given by the formula:

$$K = \int_0^\infty e^{\rho v} \langle \chi_v | e_0 \rangle \, dv < \infty.$$

In the proof of these two propositions, we use essentially the same tools as in the papers by van der Hofstad, den Hollander and König. In particular, for $f = 0$, Propositions 1.3 and 1.4 are consequences of Proposition 1 of [18].

However, for a general function f, it is not obvious that one can deduce directly our results from the material of [18] and [19], since for $X_T > 0$, one has to deal with the family of local times of the canonical process on the intervals $\mathbb{R}_-$, $[0, X_T]$ and $[X_T, \infty)$ as for $f = 0$, but also on the support of f. Moreover, some typos in [18] make the argument as written incorrect. For this reason, we present a proof of this result in a different way than was done in [18].

The next sections of this article are organized as follows: in Section 2, we prove that Propositions 1.3 and 1.4 imply Theorems 1.1 and 1.2; in Section 3, we prove Proposition 1.3. The proof of Proposition 1.4 is splitted into two parts: the first one is given in Section 4; the second one, for which one needs some estimates of different quantities, is given in Section 6, after the proof of these estimates in Section 5. In Section 7, we make a conjecture on the behaviour of the canonical process under the limit measure $\mathbb{Q}^\beta$.

2   Proof of Theorems 1.1 and 1.2, assuming Propositions 1.3 and 1.4

Let us begin to prove the following result, which is essentially a consequence of Brownian scaling:

Proposition 2.1 *Let us assume Propositions 1.3 and 1.4. For any positive continuous function f with compact support included in $[-M, M]$, and for all*

$\beta > 0$:

$$e^{\rho\beta^{2/3}T}\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\right] \longrightarrow K\beta^{1/3}\,A^{\beta,M}(f) < \infty, \qquad (11)$$

when T goes to infinity.

Proof: Propositions 1.3 and 1.4 imply:

$$e^{\rho T}\,\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\mathbb{1}_{X_T\geq 0}\right] \xrightarrow[T\to\infty]{} K\,A_+^{1,M}(f) < \infty.$$

Now, $((L_T^{-y})_{y\in\mathbb{R}}, -X_T)$ and $((L_T^y)_{y\in\mathbb{R}}, X_T)$ have the same law; hence:

$$e^{\rho T}\,\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\mathbb{1}_{X_T\leq 0}\right] = e^{\rho T}\,\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^{-y}+f(y)]^2\,dy}\mathbb{1}_{-X_T\leq 0}\right]$$

$$= e^{\rho T}\,\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(-y)]^2\,dy}\mathbb{1}_{X_T\geq 0}\right]$$

$$\xrightarrow[T\to\infty]{} K\,A_+^{1,M}(\tilde{f}) = K\,A_-^{1,M}(f),$$

which is finite.

Therefore:

$$e^{\rho T}\,\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\right] \xrightarrow[T\to\infty]{} K\,A^{1,M}(f) < \infty.$$

Now, let us set: $\alpha = \beta^{1/3}$. By Brownian scaling, $(L_{T\alpha^2}^{y\alpha})_{y\in\mathbb{R}}$ and $(\alpha L_T^y)_{y\in\mathbb{R}}$ have the same law. Consequently:

$$e^{\rho\alpha^2 T}\,\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\right] = e^{\rho\alpha^2 T}\,\mathbb{P}\left[e^{-\alpha\int_{-\infty}^{\infty}[L_{T\alpha^2}^{y\alpha}+\alpha f(y)]^2\,dy}\right]$$

$$= e^{\rho\alpha^2 T}\,\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_{T\alpha^2}^z+\alpha f(z\alpha^{-1})]^2\,dz}\right]$$

$$\xrightarrow[T\to\infty]{} K\,A^{1,M\alpha}(f_\alpha) < \infty,$$

where f_α, defined by $f_\alpha(z) = \alpha f(z\alpha^{-1})$, has a support included in $[-M\alpha, M\alpha]$.

Therefore, Proposition 2.1 is proven if we show that $A^{1,M\alpha}(f_\alpha) = \alpha\,A^{\beta,M}(f)$.

Now, by change of variable and scaling property of squared Bessel processes:

$$A_+^{1,M\alpha}(f_\alpha) = \int_0^\infty dl\,\mathbb{E}\left[e^{\int_{-\infty}^{M\alpha}\left[-(Y_l^y+\alpha f(y\alpha^{-1}))^2+\rho Y_l^y\right]dy}e_0(Y_l^{M\alpha})\right]$$

10

$$= \int_0^\infty dl \, \mathbb{E}\left[e^{\alpha \int_{-\infty}^M \left[-(Y_l^{z\alpha} + \alpha f(z))^2 + \rho Y_l^{z\alpha} \right] dz} e_0(Y_l^{M\alpha}) \right]$$

$$= \int_0^\infty dl \, \mathbb{E}\left[e^{\beta \int_{-\infty}^M \left[-(Y_{l\alpha^{-1}}^z + f(z))^2 + \rho \alpha^{-1} Y_{l\alpha^{-1}}^z \right] dz} e_0(\alpha Y_{l\alpha^{-1}}^M) \right]$$

$$= \alpha \int_0^\infty dl \, \mathbb{E}\left[e^{\beta \int_{-\infty}^M \left[-(Y_l^z + f(z))^2 + \rho \alpha^{-1} Y_l^z \right] dz} e_0(\alpha Y_l^M) \right] = \alpha \, A_+^{\beta,M}(f). \tag{12}$$

By replacing f by $\tilde{f}$, one obtains:

$$A_-^{1,M\alpha}(f_\alpha) = \alpha \, A_-^{\beta,M}(f), \tag{13}$$

and by adding (12) and (13):

$$A^{1,M\alpha}(f_\alpha) = \alpha \, A^{\beta,M}(f),$$

which proves Proposition 2.1. $\qquad\square$

At this point, we remark that $A^\beta(f) := A^{\beta,M}(f)$ does not depend on M (as written in Theorem 1.2), since M does not appear in the left hand side of (11).

Now, let $T > s$ be in $\mathbb{R}_+$. One has, for all $y \in \mathbb{R}$:

$$L_T^y = L_s^y + \widetilde{L}_{T-s}^{y-X_s},$$

where $\widetilde{L}$ is the continuous family of local times of the process $(X_{s+u} - X_s)_{u \geq 0}$.

Therefore, for all $\beta > 0$:

$$\mathbb{P}\left[e^{-\beta \int_{-\infty}^\infty (L_T^y)^2 \, dy} | \mathcal{F}_s \right] \quad = \mathbb{P}\left[e^{-\beta \int_{-\infty}^\infty (L_s^y + \widetilde{L}_{T-s}^{y-X_s})^2 \, dy} | \mathcal{F}_s \right]$$

$$= \mathbb{P}\left[e^{-\beta \int_{-\infty}^\infty (L_s^{y+X_s} + \widetilde{L}_{T-s}^y)^2 \, dy} | \mathcal{F}_s \right].$$

Under $\mathbb{P}$ and conditionally on $\mathcal{F}_s$, $(L_s^{y+X_s})_{y \in \mathbb{R}}$ is fixed and by Markov property, $(X_{s+u} - X_s)_{u \geq 0}$ is a standard Brownian motion.

11

Hence, if we assume Propositions 1.3 and 1.4, we obtain, by using Proposition 2.1:

$$e^{\rho(T-s)\alpha^2}\, \mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\big|\mathcal{F}_s\right] \xrightarrow[T\to\infty]{} K\alpha\, A^\beta(L_s^{\bullet+X_s})$$

Moreover:

$$e^{\rho(T-s)\alpha^2}\, \mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\big|\mathcal{F}_s\right] \le e^{\rho(T-s)\alpha^2}\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(\widetilde{L}_{T-s}^y)^2\,dy}\big|\mathcal{F}_s\right]$$

$$= e^{\rho(T-s)\alpha^2}\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_{T-s}^y)^2\,dy}\right]$$

$$\le 2\,K\alpha\, A^\beta(0) < \infty$$

if $T-s$ is large enough. On the other hand:

$$e^{\rho T\alpha^2}\, \mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\right] \xrightarrow[T\to\infty]{} K\alpha\, A^\beta(0),$$

and for T large enough:

$$e^{\rho T\alpha^2}\, \mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\right] \ge \frac{K}{2}\alpha\, A^\beta(0).$$

Now, for all β and f, $A^\beta(f)$ is different from zero (as written in Theorem 1.2), since it is the integral of a strictly positive quantity. Therefore:

$$\frac{\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\big|\mathcal{F}_s\right]}{\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\right]} \xrightarrow[T\to\infty]{} e^{\rho\alpha^2 s}\,\frac{A^\beta(L_s^{\bullet+X_s})}{A^\beta(0)},$$

and for s fixed and T large enough:

$$\frac{\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\big|\mathcal{F}_s\right]}{\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\right]} \le 4\,e^{\rho\alpha^2 s} < \infty.$$

Consequently, for all $s\ge 0$ and $\Lambda_s\in\mathcal{F}_s$, by dominated convergence:

$$\mathbb{P}\left[\mathbb{1}_{\Lambda_s}\,\frac{\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\big|\mathcal{F}_s\right]}{\mathbb{P}\left[e^{-\beta\int_{-\infty}^{\infty}(L_T^y)^2\,dy}\right]}\right] \xrightarrow[T\to\infty]{} \mathbb{P}\left[\mathbb{1}_{\Lambda_s}\,e^{\rho\alpha^2 s}\,\frac{A^\beta(L_s^{\bullet+X_s})}{A^\beta(0)}\right].$$

Hence:

$$\mathbb{Q}_T^\beta(\Lambda_s) \xrightarrow[T\to\infty]{} \mathbb{P}(\mathbb{1}_{\Lambda_s}\,D_s^\beta),$$

where D_s^β is defined by the equation (5):

$$D_s^\beta = e^{\rho \beta^{2/3} s} \cdot \frac{A^\beta(L_s^{\bullet + X_s})}{A^\beta(0)}.$$

This convergence implies Theorems 1.1 and 1.2. $\qquad\qquad\square$

3 Proof of Proposition 1.3

If f is a continuous function from $\mathbb{R}$ to $\mathbb{R}_+$ with compact support included in $[-M, M]$, one has:

$$\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y + f(y)]^2 \, dy} \, \mathbb{1}_{X_T \in [0,M]}\right] \leq \mathbb{P}\left[e^{-\int_{-\infty}^{\infty}(L_T^y)^2 \, dy} \, \mathbb{1}_{X_T \in [0,M]}\right]$$

$$= \mathbb{P}\left[e^{-T^{3/2} \int_{-\infty}^{\infty}(L_1^y)^2 \, dy} \, \mathbb{1}_{X_1 \in [0,M/\sqrt{T}]}\right] \tag{14}$$

by scaling properties of Brownian motion.

Hence, the right hand side of (14) is decreasing with T, which implies (for $T > 1$):

$$e^{\rho T} \mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y + f(y)]^2 \, dy} \, \mathbb{1}_{X_T \in [0,M]}\right] \leq e^{\rho T} \int_{T-1}^{T} du \, \mathbb{P}\left[e^{-\int_{-\infty}^{\infty}(L_u^y)^2 \, dy} \, \mathbb{1}_{X_u \in [0,M]}\right]$$

$$\leq e^{\rho} \int_{T-1}^{T} du \, \mathbb{P}\left[e^{\int_{-\infty}^{\infty}\left[-(L_u^y)^2 + \rho L_u^y\right] dy} \, \mathbb{1}_{X_u \in [0,M]}\right]$$

by using the equality

$$\int_{\mathbb{R}} \rho L_u^y \, dy = \rho u.$$

By dominated convergence, Proposition 1.3 is proven if we show that:

$$\int_0^{\infty} du \, \mathbb{P}\left[e^{\int_{-\infty}^{\infty}\left[-(L_u^y)^2 + \rho L_u^y\right] dy} \, \mathbb{1}_{X_u \in [0,M]}\right] < \infty. \tag{15}$$

In order to estimate the left hand side of (15), we need the following lemma:

Lemma 3.1 *For every positive and measurable function G on $\mathbb{R} \times \mathcal{C}(\mathbb{R}, \mathbb{R}_+)$:*

$$\int_0^{\infty} \mathbb{P}\left[G(X_u, L_u^{\bullet})\right] du = \int_{\mathbb{R}} da \int_0^{\infty} dl \, \mathbb{E}\left[G(a, Y_{l,a}^{\bullet})\right]$$

where the law of the process $(Y^y_{l,a})_{y\in\mathbb{R}}$ is defined in the following way:

- for $a \geq 0$, $(Y^{-y}_{l,a})_{y\geq 0}$ is a squared Bessel process of dimension zero, starting at l.

- for $a \geq 0$, $(Y^y_{l,a})_{y\geq 0}$ is an independent inhomogeneous Markov process, which has the same infinitesimal generator as a two-dimensional squared Bessel process for $y \in [0,a]$ and the same infinitesimal generator as a zero-dimensional squared Bessel process for $y \geq a$.

- for $a \leq 0$, $(Y^y_{l,a})_{y\in\mathbb{R}}$ has the same law as $(Y^{-y}_{l,-a})_{y\in\mathbb{R}}$.

Proof: For $a \geq 0$, let B be a standard Brownian motion, $B^{(a)}$ an independent Brownian motion starting at a, and let us denote by $(\tau_l)_{l\geq 0}$ the inverse local time of B at level 0, and $T^{(a)}_0$ the first time when $B^{(a)}$ reaches zero.

By [7] and [3], for every process $(F_u)_{u\geq 0}$ on the space $\mathcal{C}(\mathbb{R}_+,\mathbb{R})$, which is progressively measurable with respect to the filtration $(\mathcal{F}_u)_{u\geq 0}$:

$$\int_0^\infty du\, \mathbb{E}[F_u(B)] = \int_0^\infty dl \int_{-\infty}^\infty da\, \mathbb{E}[F_{\tau_l+T^{(a)}_0}(Z^{(l,a)})], \qquad (16)$$

where $Z^{(l,a)}$ is a process such that $Z^{(l,a)}_r = B_r$ for $r \leq \tau_l$ and $Z^{(l,a)}_{\tau_l+T^{(a)}_0-s} = B^{(a)}_s$ for $s \leq T^{(a)}_0$. By applying (16) to the process defined by $F_u(X) = G(X_u, L^\bullet_u)$, and by using Ray-Knight theorems, one obtains Lemma 3.1. $\qquad\square$

An immediate application of this lemma is the following equality:

$$\int_0^\infty du\, \mathbb{P}\left[e^{\int_{-\infty}^\infty [-(L^y_u)^2+\rho L^y_u]\,dy}\,\mathbb{1}_{X_u\in[0,M]}\right]$$

$$= \int_0^\infty dl \int_0^M da\, \mathbb{E}\left[e^{\int_{-\infty}^\infty [-(Y^y_{l,a})^2+\rho Y^y_{l,a}]\,dy}\right]. \qquad (17)$$

14

In order to majorize this expression, let us prove another result, which is also used in the proof of Proposition 1.4:

Lemma 3.2 *For all $l > 0$, $\mu \in \mathbb{R}$ and for all measurable functions g from $\mathbb{R}_+$ to $\mathbb{R}_+$, the following equality holds:*

$$\mathbb{E}\left[e^{\int_0^\infty \left[-(Y_{l,0}^y)^2 + \mu Y_{l,0}^y\right] dy}\, g\left(\int_0^\infty Y_{l,0}^y\, dy\right)\right] = \int_0^\infty K_l^{(\mu)}(v)\, g(v)\, dv. \tag{18}$$

where $K_l^{(\mu)}(v)$ is defined by equation (7).

In particular:

$$\mathbb{E}\left[e^{\int_0^\infty \left[-(Y_{l,0}^y)^2 + \rho Y_{l,0}^y\right] dy}\right] = \bar{K}_l^{(\rho)}$$

where

$$\bar{K}_l^{(\rho)} = \int_0^\infty K_l^{(\rho)}(v)\, dv.$$

Moreover $\bar{K}_l^{(\rho)}$ is bounded by a universal constant and:

$$\int_0^\infty \bar{K}_l^{(\rho)}\, dl < \infty.$$

Proof: The process $Y_{l,0}$ is a local martingale with bracket given, for $y \geq 0$, by:

$$\langle Y_{l,0}, Y_{l,0}\rangle_y = 4\int_0^y Y_{l,0}^x\, dx.$$

Therefore:

$$Y_{l,0}^y = 2\, B_{\int_0^y Y_{l,0}^x\, dx}^{(l/2)}$$

where $B^{(l/2)}$ is a Brownian motion starting at $l/2$. Moreover, since $Y_{l,0}$ stays at zero when it hits 0, the hitting time of zero for $B^{(l/2)}$ is $S = \int_0^\infty Y_{l,0}^x\, dx$. Hence, the change of variable $s = \int_0^y Y_{l,0}^x\, dx$ gives:

$$\int_0^\infty (\mu - Y_{l,0}^y)\, Y_{l,0}^y\, dy = \int_0^S (\mu - 2 B_s^{(l/2)})\, ds.$$

15

Therefore, one has the equalities:

$$\mathbb{E}\left[e^{\int_0^\infty \left[-(Y_{l,0}^y)^2+\mu Y_{l,0}^y\right]dy}\, g\left(\int_0^\infty Y_{l,0}^y\, dy\right)\right] = \mathbb{E}\left[e^{\int_0^S (\mu-2B_s^{(l/2)})\, ds} g(S)\right]$$

$$= \int_0^\infty e^{\mu v}\, g(v)\mathbb{E}\left[e^{-\int_0^v 2B_s^{(l/2)}\, ds}\Big| S = v\right]\mathbb{P}[S \in dv].$$

Now, this formula implies (18), since the density at v of the law of S is equal to $\alpha_l(v)$ and the law of $(B_s^{(l/2)})_{s \le v}$, conditionally on $S = v$, is equal to the law of $V^{(l/2,v)}$ (see, for example, [5]).

It only remains to prove the integrability of $\bar{K}_l^{(\rho)}$. One easily checks that:

$$\bar{K}_l^{(\rho)} = \mathbb{E}\left[e^{\int_0^S (\rho-2B_s^{(l/2)})\, ds}\right].$$

Hence, if one sets:

$$f(x) = \text{Ai}(2^{-1/3}(2x-\rho)),$$

for the Airy function Ai (which is, up to a multiplicative constant, the unique bounded solution of the differential equation $\text{Ai}''(x) = x\,\text{Ai}(x)$), the process N defined by:

$$N_t = f(B_t^{(l/2)})\exp\left(\int_0^t (\rho - 2B_s^{(l/2)})\, ds\right)$$

is a local martingale.

Moreover, since ρ is smaller than $-2^{1/3}$ times the largest zero of Airy function, the function f is strictly positive on $\mathbb{R}_+$ and N is positive. By stopping N at time S, one gets a true martingale since $0 \le N_{t \wedge S} \le ||f||_\infty e^{\rho t}$, and by Doob's stopping theorem and Fatou's lemma, one has:

$$\bar{K}_l^{(\rho)} \le \frac{f(l/2)}{f(0)}.$$

Since Airy function decays faster than exponentially at infinity, the boundedness and the integrability of $\bar{K}_l^{(\rho)}$ are proven. $\qquad\square$

It is now easy to prove that Lemma 3.2 implies Proposition 1.3: by using this lemma, the definition of $Y_{l,a}$ and Markov property at level a, one can see that the left hand side of (17) is equal to:

$$\int_0^\infty dl \int_0^M da \, \bar{K}_l^{(\rho)} \mathbb{E}\left[e^{\int_0^a \left[-(Y_l^y)^2 + \rho Y_l^y\right]} \bar{K}_{Y_l^a}^{(\rho)}\right] \tag{19}$$

Now, $\bar{K}_{Y_l^a}^{\rho}$ is uniformly bounded and $-x^2 + \rho x \leq \frac{\rho^2}{4}$ for all $x \in \mathbb{R}$; hence, the quantity (19) is bounded by a constant times:

$$e^{M\rho^2/4} \int_0^M da \int_0^\infty dl \, \bar{K}_l^{(\rho)},$$

which is finite.

Hence, one has (15), and finally Proposition 1.3. $\qquad\square$

Remark: Proposition 1.3 remains true if one replaces ρ by any real ρ' which is strictly smaller than $-2^{1/3}$ times the largest zero of Airy function (for example, one can take $\rho' = 2.9$).

4 Proof of Proposition 1.4 (first part)

The purpose of this first part is to prove the following proposition, which, in particular, gives another expression for the left hand side of (10):

Proposition 4.1 *For $u, v, t, l > 0$, let us define the quantities:*

$$J_l(u, v) = \mathbb{E}\left[e^{-2\int_0^u R_w^{(l/2)} \, dw} \chi_v(2R_u^{(l/2)})\right], \tag{20}$$

where χ_v is given by (8) and $(R_w^{(l/2)})_{w\geq 0}$ is a Bessel process of dimension 2, starting at $l/2$;

$$J_l(t) := \int_0^t J_l(t-v,v)\,dv \tag{21}$$

and for all $t \in \mathbb{R}$,

$$J_l^{(\rho)}(t) = e^{\rho t}\mathbb{1}_{t>0}\,J_l(t).$$

Then, there exists a subset E of $\mathbb{R}_+$, such that the complement of E is Lebesgue-negligible, and for all $T \in E$:

$$e^{\rho T}\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\,\mathbb{1}_{X_T\geq M}\right]$$

$$= \int_0^\infty dl\,\mathbb{E}\left[e^{\int_{-\infty}^M\left(-[Y_l^y+f(y)]^2+\rho Y_l^y\right)dy}\,J_{Y_l^M}^{(\rho)}\left(T-\int_{-\infty}^M Y_l^y\,dy\right)\right]. \tag{22}$$

Moreover, for all measurable functions h from $(\mathbb{R}_+)^2$ to $\mathbb{R}_+$, and for all $l>0$:

$$\int_0^\infty db\,\mathbb{E}\left[e^{-\int_0^\infty(Y_{l,b}^y)^2\,dy}\,h\left(\int_0^b Y_{l,b}^y\,dy,\int_b^\infty Y_{l,b}^y\,dy\right)\right]$$

$$= \int_{(\mathbb{R}_+)^2} h(u,v)\,J_l(u,v)\,du\,dv. \tag{23}$$

and for all measurable functions g from $\mathbb{R}_+$ to $\mathbb{R}_+$:

$$\int_0^\infty db\,\mathbb{E}\left[e^{-\int_0^\infty(Y_{l,b}^y)^2\,dy}\,g\left(\int_0^\infty Y_{l,b}^y\,dy\right)\right] = \int_0^\infty g(t)\,J_l(t)\,dt. \tag{24}$$

Remark: In Proposition 4.1, it is natural to expect that E is empty, even if we don't need it to prove our main result.

Proof of Proposition 4.1: Equation (22) is a consequence of (23) and (24); therefore, we begin our proof by these two equalities. By monotone class theorem, it is sufficient to prove (23) for functions h of the form: $h(x,y) = h_1(x)h_2(y)$, where h_1 and h_2 are

measurable functions from $\mathbb{R}_+$ to $\mathbb{R}_+$.

By Lemma 3.2, for all $l > 0$:

$$\mathbb{E}\left[e^{-\int_0^\infty (Y_{l,0}^y)^2\, dy}\, h_2\left(\int_0^\infty Y_{l,0}^y\, dy\right)\right] = \int_0^\infty K_l(v)\, h_2(v)\, dv.$$

Hence, by applying Markov property to the process $Y_{l,b}$ at level b:

$$\int_0^\infty db\, \mathbb{E}\left[e^{-\int_0^\infty (Y_{l,b}^y)^2\, dy}\, h_1\left(\int_0^b Y_{l,b}^y\, dy\right) h_2\left(\int_b^\infty Y_{l,b}^y\, dy\right)\right]$$

$$= \int_0^\infty db\, \mathbb{E}\left[e^{-\int_0^b (Y_l^y)^2\, dy}\, h_1\left(\int_0^b Y_l^y\, dy\right) \int_0^\infty K_{Y_l^b}(v)\, h_2(v)\, dv\right]$$

$$= \int_0^\infty dv\, h_2(v)\, \mathbb{E}\left[\int_0^\infty db\, e^{-\int_0^b (Y_l^y)^2\, dy}\, h_1\left(\int_0^b Y_l^y\, dy\right) K_{Y_l^b}(v)\right]. \qquad (25)$$

Now, the function $\tilde{y}$ from $\mathbb{R}_+$ to $\mathbb{R}_+$, given by:

$$\tilde{y}(s) = \inf\{y \in \mathbb{R}_+, \int_0^y Y_l^{y'}\, dy' = s\}$$

is well-defined, continuous, strictly increasing and tending to infinity at infinity.

Hence, one can consider the process $(\tilde{Q}_s^{(l)})_{s \geq 0}$ such that:

$$\tilde{Q}_s^{(l)} = Y_l^{\tilde{y}(s)}.$$

One has:

$$d\tilde{y}(s) = \frac{ds}{Y_l^{\tilde{y}(s)}} = \frac{ds}{\tilde{Q}_s^{(l)}},$$

and the s.d.e.:

$$d\tilde{Q}_s^{(l)} = 2\sqrt{Y_l^{\tilde{y}(s)}}\, d\hat{B}_{\tilde{y}(s)} + 2\, d\tilde{y}(s) = 2\, dB_s + \frac{2\, ds}{\tilde{Q}_s^{(l)}},$$

where $\hat{B}$ and B are Brownian motions: the processes $\tilde{Q}^{(m)}$ and $2R^{(m/2)}$ have the same law.

By a change of variable in (25) ($b = \tilde{y}(s), y = \tilde{y}(u)$):

$$\int_0^\infty db\, \mathbb{E}\left[e^{-\int_0^\infty (Y_{l,b}^y)^2\, dy}\, h_1\left(\int_0^b Y_{l,b}^y\, dy\right) h_2\left(\int_b^\infty Y_{l,b}^y\, dy\right)\right]$$

$$= \int_0^\infty dv\, h_2(v)\, \mathbb{E}\left[\int_0^\infty d\tilde{y}(s)\, e^{-\int_0^s (Y_l^{\tilde{y}(u)})^2 \, d\tilde{y}(u)}\, h_1(s)\, K_{Y_l^{\tilde{y}(s)}}(v)\right]$$

$$= \int_0^\infty dv\, h_2(v)\, \mathbb{E}\left[\int_0^\infty ds\, h_1(s)\, e^{-2\int_0^s R_u^{(l/2)}\, du}\, \frac{K_{2R_s^{(l/2)}}(v)}{2R_s^{(l/2)}}\right],$$

which implies (23).

The equality (24) is easily obtained by applying (23) to the function $h : (u,v) \to g(u+v)$.

Now, it remains to deduce (22) from (23) and (24).

For all measurable and positive functions g, one has, by Lemma 3.1:

$$\int_0^\infty dT\, g(T)\, \mathbb{P}\left[e^{-\int_{-\infty}^\infty [L_T^y + f(y)]^2\, dy}\, \mathbb{1}_{X_T \geq M}\right]$$

$$= \int_0^\infty dl \int_M^\infty da\, \mathbb{E}\left[e^{-\int_{-\infty}^\infty [Y_{l,a}^y + f(y)]^2\, dy}\, g\left(\int_{-\infty}^\infty Y_{l,a}^y\, dy\right)\right].$$

On the other hand, by applying Markov property (for the process $Y_{l,a}$, at level M), and by using the fact that $f(y) = 0$ for $y \geq M$, one obtains, for all positive and measurable functions h_1 and h_2:

$$\int_0^\infty dl \int_M^\infty da\, \mathbb{E}\left[e^{-\int_{-\infty}^\infty [Y_{l,a}^y + f(y)]^2\, dy}\, h_1\left(\int_{-\infty}^M Y_{l,a}^y\, dy\right)\, h_2\left(\int_M^\infty Y_{l,a}^y\, dy\right)\right]$$

$$= \int_0^\infty dl \int_0^\infty db\, \mathbb{E}\left[e^{-\int_{-\infty}^M [Y_l^y + f(y)]^2\, dy}\, h_1\left(\int_{-\infty}^M Y_l^y\, dy\right) \ldots\right.$$

$$\left. \ldots \mathbb{E}\left[e^{-\int_0^\infty \left(\widehat{Y}_{Y_l^M,b}^y\right)^2\, dy}\, h_2\left(\int_0^\infty \widehat{Y}_{Y_l^M,b}^y\, dy\right)\, |Y_l^M\right]\right], \tag{26}$$

where $\widehat{Y}_{Y_l^M,b}$ is a process which has, conditionally on $Y_l^M = l'$, the same law as $Y_{l',b}$.

Now, by putting the integral with respect to db just before the second expectation in the right hand side of (26), and by applying (24) to $g = h_2$, one obtains that the left hand side of (26) is equal to:

$$= \int_0^\infty dl\, \mathbb{E}\left[e^{-\int_{-\infty}^M [Y_l^y + f(y)]^2\, dy}\, h_1\left(\int_{-\infty}^M Y_l^y\, dy\right) \int_0^\infty h_2(t)\, J_{Y_l^M}(t)\, dt\right].$$

Hence, by monotone class theorem, for all measurable functions h from $\mathbb{R}_+^2$ to $\mathbb{R}_+$:

$$\int_0^\infty dl \int_M^\infty da\, \mathbb{E}\left[e^{-\int_{-\infty}^\infty [Y_{l,a}^y + f(y)]^2\, dy}\, h\left(\int_{-\infty}^M Y_{l,a}^y\, dy, \int_M^\infty Y_{l,a}^y\, dy\right)\right]$$

$$= \int_0^\infty dl\, \mathbb{E}\left[e^{-\int_{-\infty}^M [Y_l^y + f(y)]^2\, dy} \int_0^\infty h\left(\int_{-\infty}^M Y_l^y\, dy, t\right) J_{Y_l^M}(t)\, dt\right].$$

By applying this equality to the function $h : (u, v) \to g(u + v)$, we obtain:

$$\int_0^\infty dT\, g(T)\, \mathbb{P}\left[e^{-\int_{-\infty}^\infty [L_T^y + f(y)]^2\, dy}\, \mathbb{1}_{X_T \geq M}\right]$$

$$= \int_0^\infty dT\, g(T) \int_0^\infty dl\, \mathbb{E}\left[e^{-\int_{-\infty}^M [Y_l^y + f(y)]^2\, dy} \dots\right.$$

$$\left.\dots J_{Y_l^M}\left(T - \int_{-\infty}^M Y_l^y\, dy\right) \mathbb{1}_{\int_{-\infty}^M Y_l^y\, dy < T}\right].$$

Since this equality is true for all g, there exists a subset E of $\mathbb{R}_+$, such that the complement

of E is Lebesgue-negligible, and for all $T \in E$:

$$\mathbb{P}\left[e^{-\int_{-\infty}^\infty [L_T^y + f(y)]^2\, dy}\, \mathbb{1}_{X_T \geq M}\right]$$

$$= \int_0^\infty dl\, \mathbb{E}\left[e^{-\int_{-\infty}^M [Y_l^y + f(y)]^2\, dy}\, J_{Y_l^M}\left(T - \int_{-\infty}^M Y_l^y\, dy\right) \mathbb{1}_{\int_{-\infty}^M Y_l^y\, dy < T}\right],$$

which implies (22). $\qquad\square$

5   Some estimates

In this section, we prove the following propositions, which give estimates for the different

quantities introduced earlier in this paper. In the sequel of this paper, C denotes a

universal and strictly positive constant, which may change from line to line.

Proposition 5.1 *For all $l, v > 0$, $\mu \in \mathbb{R}$, one has the majorization:*

$$K_l^{(\mu)}(v) \leq C\, lv^{-3/2} e^{(\mu - 2.9)v - l^2/8v}, \tag{27}$$

where $K_l^{(\mu)}(v)$ is defined by (7).

Proposition 5.2 *For all $M > 0$, the random variable $\int_0^M Y_0^y \, dy$ admits a density D_M with respect to Lebesgue measure, such that for all $u \geq 0$:*

$$(D_M * K_l^{(\rho)})(u) \leq C_M e^{-\nu_M l},$$

where $C_M, \nu_M > 0$ depend only on M.

Proposition 5.3 *For all $l, u, v > 0$:*

$$J_l(u, v) \leq \frac{C \, e^{-2.9v}}{(u + v)\sqrt{v}} e^{-l^2/8(u+v)}, \tag{28}$$

and

$$J_l(u, v) \leq \frac{C}{\sqrt{v}} \, e^{-2.8v - \rho u}, \tag{29}$$

if $u \geq 2$ (recall that $J_l(u, v)$ is defined by (20)).

Moreover, the function χ_v from $\mathbb{R}_+^*$ to $\mathbb{R}$ (recall that $\chi_v(l) = \frac{K_l(v)}{l}$), is in $L^2(\nu)$, and for fixed $l, v > 0$ and u going to infinity:

$$e^{\rho u} J_l(u, v) \longrightarrow \langle \chi_v | e_0 \rangle e_0(l). \tag{30}$$

Proposition 5.4 *For all $t > 0$:*

$$e^{\rho t} J_l(t) \leq C \left(1 + \frac{1}{\sqrt{t}} \right),$$

where $J_l(t)$ is defined by (21).

Moreover, for l fixed and t going to infinity:

$$e^{\rho t} J_l(t) \longrightarrow K \, e_0(l), \tag{31}$$

where K is the universal constant defined in Proposition 1.4.

Proof of Proposition 5.1: For all $l \geq 0$, the process $V^{(l/2,v)}$ is, by coupling, stochastically larger than $V^{(l/2,v)}$. Therefore, by scaling property:

$$\mathbb{E}\left[e^{-2\int_0^v V_u^{(l/2,v)}\,du}\right] \leq \mathbb{E}\left[e^{-2\int_0^v V_u^{(0,v)}\,du}\right] = \mathbb{E}\left[e^{-2v^{3/2}\int_0^1 V_u^{(0,1)}\,du}\right].$$

Now, the Laplace transform of $\int_0^1 V_u^{(0,1)}\,du$ (the area under a normalized Brownian excursion) is known (see, for example, [9]); one has, for $\lambda > 0$:

$$\mathbb{E}\left[e^{-\lambda\int_0^1 V_u^{(0,1)}\,du}\right] = \sqrt{2\pi}\,\lambda \sum_{n=1}^{\infty} e^{-u_n(\lambda^2/2)^{1/3}}$$

where $-u_1 > -u_2 > -u_3 > \dots$ are the (negative) zeros of the Airy function.

Therefore

$$\mathbb{E}\left[e^{-2\int_0^v V_u^{(l/2,v)}\,du}\right] \leq \sqrt{8\pi v^3} \sum_{n=1}^{\infty} e^{-2^{1/3}u_n v}$$

and:

$$\mathbb{E}\left[e^{-2\int_0^v V_u^{(l/2,v)}\,du}\right] \leq (C\,e^{-2.9v}) \sum_{n=1}^{\infty} e^{-(2^{1/3}u_n - 2.91)v}, \tag{32}$$

since $v^{3/2}$ is dominated by $e^{0.01\,v}$.

Now, $2^{1/3}u_1 > 2.91$; hence, for $v > 1$, the infinite sum in (32) is smaller than

$$\sum_{n=1}^{\infty} e^{-(2^{1/3}u_n - 2.91)}$$

which is finite, since u_n grows sufficiently fast with n (as $n^{2/3}$). Consequently, for $v \geq 1$:

$$\mathbb{E}\left[e^{-2\int_0^v V_u^{(l/2,v)}\,du}\right] \leq C\,e^{-2.9v}.$$

This majorization, which remains obviously true for $v \leq 1$ if we choose $C > e^{2.9}$, implies easily (27). $\qquad\square$

Proof of Proposition 5.2: In [2], the density of the law of $\int_0^1 Y_0^y \, dy$ is explicitly given:

$$D_1(x) = \pi \sum_{n=0}^{\infty} (-1)^n \left(n + \frac{1}{2}\right) e^{-\left(n+\frac{1}{2}\right)^2 \pi^2 x / 2}.$$

This formula proves that D_1 is continuous on $\mathbb{R}_+^*$ and that for $x \geq 1$:

$$D_1(x) \leq \pi e^{-\pi^2(x-1)/8} \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) e^{-\left(n+\frac{1}{2}\right)^2 \pi^2 / 2} \leq C e^{-x}. \tag{33}$$

Moreover, D_1 satisfies the functional equation:

$$D_1(x) = \left(\frac{2}{\pi x}\right)^{3/2} D_1\left(\frac{4}{\pi^2 x}\right),$$

which proves that, for $x \leq \frac{4}{\pi^2}$:

$$D_1(x) \leq C \, x^{-3/2} e^{-4/\pi^2 x} \leq C.$$

This inequality and the continuity of D_1 imply that (33) applies for all $x \in \mathbb{R}_+^*$.

By scaling property of squared Bessel processes, the density D_M exists and one has:

$$D_M(x) = \frac{1}{M^2} D_1\left(\frac{x}{M^2}\right),$$

which implies

$$D_M(x) \leq \frac{C}{M^2} e^{-\frac{x}{M^2}}.$$

Therefore, for all $u \geq 0$:

$$
\begin{aligned}
(D_M * K_l^{(\rho)})(u) &= \int_0^u D_M(u-v) K_l^{(\rho)}(v) \, dv \\
&\leq \frac{C}{M^2} \int_0^u e^{-\frac{u-v}{M^2}} \, l v^{-3/2} e^{(\rho-2.9)v - l^2/8v} \, dv \\
&\leq \frac{C}{M^2} e^{-\left(0.7 \wedge \frac{1}{M^2}\right)u} \int_0^u l v^{-3/2} e^{-l^2/8v} \, dv.
\end{aligned}
$$

The last inequality comes from the fact that $\rho - 2.9 \leq 0.7$, which implies, for $0 \leq u \leq v$:

$$-\frac{u-v}{M^2} + (\rho - 2.9)v \leq -\left(\frac{u-v}{M^2} + 0.7v\right) \leq -\left(0.7 \wedge \frac{1}{M^2}\right)u.$$

Now, the integral $\int_0^u lv^{-3/2}e^{-l^2/8v}\,dv$ is proportional to the probability that a Brownian motion starting at $l/2$ reaches zero before time u.

Hence:

$$\int_0^u lv^{-3/2}e^{-l^2/8v}\,dv \leq Ce^{-l^2/8u},$$

and finally:

$$
\begin{aligned}
(D_M * K_l^{(\rho)})(u) \quad &\leq \frac{C}{M^2}e^{-0.7u/(1+M^2)-l^2/8u} \\
&\leq \frac{C}{M^2}e^{-2\sqrt{(\frac{0.7u}{1+M^2})(l^2/8u)}} \\
&\leq \frac{C}{M^2}e^{-\frac{l}{2(1+M)}},
\end{aligned}
$$

which proves Proposition 5.2. $\hspace{1cm}\square$

Proof of Proposition 5.3: By definition of $J_l(u,v)$, one has:

$$J_l(u,v) \leq \mathbb{E}\left[\chi_v\left(2R_u^{(l/2)}\right)\right].$$

Now, the majorization (27) implies:

$$\chi_v\left(2R_u^{(l/2)}\right) \leq C\,v^{-3/2}e^{-2.9\,v}e^{-\left(R_u^{(l/2)}\right)^2/2v}.$$

By using the explicit expression of the Laplace transform of the squared bidimensional Bessel process (see, for example, [15]), one obtains:

$$J_l(u,v) \leq C\,v^{-3/2}e^{-2.9\,v}\,\frac{v}{u+v}\,e^{-l^2/8(u+v)}$$

which implies (28).

In order to prove (29), let us consider, on the set of measurable functions from $\mathbb{R}_+^*$

to $\mathbb{R}_+$, the semi-group of operators $(\Phi^s)_{s\geq 0}$ associated to the process $2R$ (twice a Bessel process of dimension 2), and the semi-group $(\widetilde{\Phi}^s)_{s\geq 0}$ associated to the same process, killed at rate x at level x.

For all positive and measurable functions ψ, and for all $l > 0$, one has:

$$[\Phi^s(\psi)](l) = \mathbb{E}\left[\psi(2R_s^{(l/2)})\right]. \tag{34}$$

$$[\widetilde{\Phi}^s(\psi)](l) = \mathbb{E}\left[e^{-2\int_0^s R_u^{(l/2)}\, du}\, \psi(2R_s^{(l/2)})\right]. \tag{35}$$

Now, let us observe that the measure ν on $\mathbb{R}_+^*$ is reversible, and hence invariant by the semigroup of $2R$. Since, for every measurable and positive function ψ,

$$\left(\widetilde{\Phi}^s(\psi)\right)^2 \leq (\Phi^s(\psi))^2 \leq \Phi^s(\psi^2),$$

one gets

$$||\widetilde{\Phi}^s(\psi)||_{L^2(\nu)}^2 \leq \int_{\mathbb{R}_+^*} \psi^2\, d\nu = ||\psi||_{L^2(\nu)}^2. \tag{36}$$

The inequality (36) proves that the semigroup $(\widetilde{\Phi}^s)_{s\geq 0}$ can be considered as a semigroup of continuous linear operators on $L^2(\nu)$.

Moreover, the infinitesimal generator of $2R$, killed at rate x at level x, is the operator $\mathcal{K}$ defined at the beginning of our paper. Hence, if $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\nu)$ such that e_n is an eigenvector of $\mathcal{K}$, corresponding to the eigenvalue $-\rho_n$ $(-\rho = -\rho_0 > -\rho_1 \geq -\rho_2 \geq -\rho_3 \geq ...)$, one has, for all n:

$$\widetilde{\Phi}^s(e_n) = e^{-\rho_n s}\, e_n.$$

Now, for all $\psi \in L^2(\nu)$, one has the representation:

$$\psi = \sum_{n\geq 0} \langle \psi | e_n \rangle e_n,$$

26

and, by linearity and continuity of $\widetilde{\Phi}^s$:

$$\widetilde{\Phi}^s(\psi) = \sum_{n \geq 0} e^{-\rho_n s} \langle \psi | e_n \rangle e_n. \tag{37}$$

In particular:

$$||\widetilde{\Phi}^s(\psi)||^2_{L^2(\nu)} = \sum_{n \geq 0} e^{-2\rho_n s} (\langle \psi | e_n \rangle)^2 \leq e^{-2\rho s} \sum_{n \geq 0} (\langle \psi | e_n \rangle)^2,$$

which implies:

$$||\widetilde{\Phi}^s(\psi)||_{L^2(\nu)} \leq e^{-\rho s} ||\psi||_{L^2(\nu)}. \tag{38}$$

Moreover, one has the equality:

$$e^{\rho s} \widetilde{\Phi}^s(\psi) - \langle \psi | e_0 \rangle e_0 = \sum_{n=1}^{\infty} e^{(\rho - \rho_n)s} \langle \psi | e_n \rangle e_n,$$

which implies that

$$e^{\rho s} \widetilde{\Phi}^s(\psi) \xrightarrow[s \to \infty]{} \langle \psi | e_0 \rangle e_0$$

in $L^2(\nu)$.

Now, by definition:

$$J_l(u, v) = \left(\widetilde{\Phi}^u(\chi_v) \right)(l) \tag{39}$$

where $\chi_v \in L^2(\nu)$, by the majorization (27).

Hence, in $L^2(\nu)$,

$$e^{\rho u} J_\bullet(u, v) \xrightarrow[s \to \infty]{} \langle \chi_v | e_0 \rangle e_0$$

where $J_\bullet(u, v)$ is the function defined by:

$$\left(J_\bullet(u, v) \right)(l) = J_l(u, v).$$

In order to prove the corresponding pointwise convergence (which is (30)), let us observe that for all $\psi \in L^2(\nu)$, $l > 0$:

$$|[\widetilde{\Phi}^1(\psi)](l)| \leq \mathbb{E}[|\psi(2R_1^{(l/2)})|] \quad \leq \left(\mathbb{E}[(\psi(2R_1^{(l/2)}))^2]\right)^{1/2}$$
$$\leq \left[\int_0^\infty p_1^{(2)}\left(\frac{l}{2}, x\right)(\psi(2x))^2\, dx\right]^{1/2}$$
$$\leq \left[\int_0^\infty x\, (\psi(2x))^2\, dx\right]^{1/2}$$
$$\leq ||\psi||_{L^2(\nu)}. \tag{40}$$

Here, we use the majorization $p_1^{(2)}(x, y) \leq y$, which comes from the fact that the transition densities of a bidimensional Brownian motion are uniformly bounded by $1/2\pi$ at time 1. By (40), one has for $s > 1$, $l > 0$, $\psi \in L^2(\nu)$:

$$|e^{\rho s}\left(\widetilde{\Phi}^s(\psi)\right)(l) - \langle \psi|e_0\rangle e_0(l)| \quad = |\left(e^{\rho s}\widetilde{\Phi}^s(\psi) - \langle \psi|e_0\rangle e_0\right)(l)|$$
$$= e^\rho|\left(\widetilde{\Phi}^1\left(e^{\rho(s-1)}\widetilde{\Phi}^{s-1}(\psi) - \langle \psi|e_0\rangle e_0\right)\right)(l)|$$
$$\leq e^\rho\, ||e^{\rho(s-1)}\widetilde{\Phi}^{s-1}(\psi) - \langle \psi|e_0\rangle e_0||_{L^2(\nu)}$$
$$\xrightarrow[s\to\infty]{} 0.$$

By applying this convergence to χ_v, one obtains the pointwise (and in fact uniform) convergence (30).

Now, it remains to prove (29).

For $s \geq 1$, $l > 0$, by (38) and (40),

$$|[\widetilde{\Phi}^s(\psi)](l)| = |[\widetilde{\Phi}^1(\widetilde{\Phi}^{s-1}(\psi))](l)| \leq ||\Phi^{s-1}(\psi)||_{L^2(\nu)} \leq e^{-\rho(s-1)}||\psi||_{L^2(\nu)}. \tag{41}$$

By (39), and by semigroup property of $\widetilde{\Phi}$, one has for $u > 1$, $l, v > 0$:

$$J_l(u, v) = [\widetilde{\Phi}^{u-1}(J_\bullet(1, v))](l). \tag{42}$$

Moreover, by (28):

$$J_l(1, v) \leq Ce^{-2.9v}v^{-1/2}e^{-l^2/8(1+v)},$$

which implies:

$$\begin{aligned}
||J_\bullet(1, v)||^2_{L^2(\nu)} \quad &\leq Ce^{-5.8v}\int_0^\infty \frac{l}{v}e^{-l^2/4(1+v)}\,dl \\
&\leq Ce^{-5.8v}\frac{1+v}{v} \\
&\leq \frac{C}{v}e^{-5.6v},
\end{aligned}$$

and

$$||J_\bullet(1, v)||_{L^2(\nu)} \leq \frac{C}{\sqrt{v}}e^{-2.8v}.$$

For $u \geq 2$, we can combine (41) and (42), and we obtain:

$$J_l(u, v) \leq Ce^{-\rho(u-2)}||J_\bullet(1, v)||_{L^2(\nu)} \leq \frac{C}{\sqrt{v}}e^{-\rho u - 2.8v},$$

which is (29).

The proof of Proposition 5.3 is now complete. □

Proof of Proposition 5.4: Let us split the integral corresponding to $J_l(t)$ into two parts:

$$A(t) := \int_0^{(t-2)_+} J_l(t - v, v)\,dv,$$

$$B(t) := \int_{(t-2)_+}^t J_l(t - v, v)\,dv.$$

One has:

$$e^{\rho t}A(t) = \int_0^\infty \mathbb{1}_{t-v \geq 2}J_l(t - v, v)\,e^{\rho t}\,dv;$$

where, by (30):

$$\mathbb{1}_{t-v \geq 2}J_l(t - v, v)\,e^{\rho t} \longrightarrow e^{\rho v}\langle \chi_v|e_0\rangle\,e_0(l),$$

29

for l, v fixed and t tending to infinity. Moreover, by (29) and the fact that $\rho - 2.8 < -0.6$:

$$\mathbb{1}_{t-v \geq 2} J_l(t-v, v) \, e^{\rho t} \leq \frac{C}{\sqrt{v}} e^{-2.8v - \rho(t-v) + \rho t} \leq \frac{C}{\sqrt{v}} e^{-0.6v},$$

which is integrable on $\mathbb{R}_+$.

Hence:

$$e^{\rho t} A(t) \leq C, \tag{43}$$

and by dominated convergence:

$$e^{\rho t} A(t) \xrightarrow[t \to \infty]{} K \, e_0(l), \tag{44}$$

where l is fixed and

$$K = \int_0^\infty e^{\rho v} \, \langle \, \chi_v | e_0 \, \rangle \, dv$$

is the constant defined in Proposition 1.4. The majorization (43) implies that K is necessarily finite.

The integral $B(t)$ can be estimated in the following way: by (28),

$$
\begin{aligned}
e^{\rho t} B(t) \quad &\leq e^{\rho t} \int_{(t-2)_+}^t \frac{C \, e^{-2.9v}}{t \sqrt{v}} \, e^{-l^2/8t} \, dv \\
&\leq \frac{C \, e^{-0.7t}}{t} e^{-l^2/8t} \int_0^t \frac{dv}{\sqrt{v}} \\
&\leq \frac{C \, e^{-0.7t}}{\sqrt{t}} e^{-l^2/8t} \\
&\leq \frac{C}{\sqrt{t}}.
\end{aligned}
\tag{45}
$$

Proposition 5.4 is a consequence of (43), (44) and (45). $\qquad \square$

The estimates given in this section are used in the second part of the proof of Proposition 1.4.

6 Proof of Proposition 1.4 (second part)

In this section, we estimate the right hand side of (22), which is, by Proposition 4.1, equal to the left hand side of (10) in Proposition 1.4.

We need the following lemma:

Lemma 6.1 *For all $M > 0$, and all functions f, g from $\mathbb{R}_+$ to $\mathbb{R}_+$:*

$$\int_0^\infty dl \, \mathbb{E}\left[e^{\int_{-\infty}^M \left(-\left[Y_l^y + f(y)\right]^2 + \rho Y_l^y\right) dy} \, g\left(\int_{-\infty}^M Y_l^y \, dy\right)\right] \leq C_M' \int_0^\infty g,$$

where $C_M' > 0$ is finite and depends only on M.

Proof: One has the following majorization:

$$\int_0^\infty dl \, \mathbb{E}\left[e^{\int_{-\infty}^M \left(-\left[Y_l^y + f(y)\right]^2 + \rho Y_l^y\right) dy} \, g\left(\int_{-\infty}^M Y_l^y \, dy\right)\right]$$

$$\leq e^{M\rho^2/4} \int_0^\infty dl \, \mathbb{E}\left[e^{\int_{-\infty}^0 \left[-\left(Y_l^y\right)^2 + \rho Y_l^y\right] dy} \, g\left(\int_{-\infty}^M Y_l^y \, dy\right)\right] \qquad (46)$$

since f is nonnegative and $-x^2 + \rho x \leq \rho^2/4$ for all $x \in \mathbb{R}$.

Now, for all positive and measurable functions h_1 and h_2:

$$\int_0^\infty dl \, \mathbb{E}\left[e^{\int_{-\infty}^0 \left[-\left(Y_l^y\right)^2 + \rho Y_l^y\right] dy} \, h_1\left(\int_{-\infty}^0 Y_l^y \, dy\right) h_2\left(\int_0^M Y_l^y \, dy\right)\right]$$

$$= \int_0^\infty dl \, \mathbb{E}\left[e^{\int_0^\infty \left[-\left(Y_{l,0}^y\right)^2 + \rho Y_{l,0}^y\right] dy} \, h_1\left(\int_0^\infty Y_{l,0}^y \, dy\right)\right] \mathbb{E}\left[h_2\left(\int_0^M Y_l^y \, dy\right)\right]$$

$$= \int_0^\infty dl \int_0^\infty K_l^{(\rho)}(v) \, h_1(v) \, dv \, \mathbb{E}\left[h_2\left(\int_0^M Y_l^y \, dy\right)\right],$$

by Lemma 3.2.

By additivity properties of squared Bessel processes, the law of $\int_0^M Y_l^y \, dy$ is the convolution of the law σ_1 of $\int_0^M Y_{l,0}^y dy$ and the law σ_2 of $\int_0^M Y_0^y \, dy$.

Since by Proposition 5.2, σ_2 has the density D_M with respect to Lebesgue measure, we have the equality:

$$\mathbb{E}\left[h_2\left(\int_0^M Y_l^y\,dy\right)\right] = \int_0^\infty dt\,h_2(t)\int_0^t \sigma_1(du)\,D_M(t-u),$$

which implies:

$$\int_0^\infty dl\,\mathbb{E}\left[e^{\int_{-\infty}^0 \left[-\left(Y_l^y\right)^2+\rho Y_l^y\right]dy}\,h_1\left(\int_{-\infty}^0 Y_l^y\,dy\right)h_2\left(\int_0^M Y_l^y\,dy\right)\right]$$

$$= \int_0^\infty dl\int_0^\infty K_l^{(\rho)}(v)\,h_1(v)\,dv\int_0^\infty dt\,h_2(t)\int_0^t \sigma_1(du)\,D_M(t-u).$$

By monotone class theorem and easy computations, for all positive and measurable functions g:

$$\int_0^\infty dl\,\mathbb{E}\left[e^{\int_{-\infty}^0 \left[-\left(Y_l^y\right)^2+\rho Y_l^y\right]dy}\,g\left(\int_{-\infty}^M Y_l^y\,dy\right)\right]$$

$$= \int_0^\infty dt\,g(t)\int_0^\infty dl\int_0^t \sigma_1(du)\,(K_l^{(\rho)} * D_M)(t-u).$$

Now, by Proposition 5.2:

$$(K_l^{(\rho)} * D_M)(t-u) \leq C_M e^{-\nu_M l},$$

and

$$\int_0^t \sigma_1(du) \leq 1,$$

since σ_1 is a probability measure.

Hence:

$$\int_0^\infty dl\,\mathbb{E}\left[e^{\int_{-\infty}^0 \left[-\left(Y_l^y\right)^2+\rho Y_l^y\right]dy}\,g\left(\int_{-\infty}^M Y_l^y\,dy\right)\right] \leq \frac{C_M}{\nu_M}\int_0^\infty g, \qquad (47)$$

The majorizations (46) and (47) imply Lemma 6.1. $\qquad\square$

After proving this lemma, let us take $T \in E$ and $\epsilon > 0$; by splitting the right hand side of (22) into two parts, we obtain:

$$e^{\rho T} \mathbb{P} \left[e^{-\int_{-\infty}^{\infty} [L_T^y + f(y)]^2 \, dy} \mathbb{1}_{X_T \geq M} \right] = I_{1,\epsilon} + I_{2,\epsilon},$$

where:

$$I_{1,\epsilon} = \int_0^{\infty} dl \, \mathbb{E} \left[e^{\int_{-\infty}^{M} \left(-[Y_l^y + f(y)]^2 + \rho Y_l^y \right) dy} \cdots \right.$$

$$\left. \cdots J_{Y_l^M}^{(\rho)} \left(T - \int_{-\infty}^{M} Y_l^y \, dy \right) \mathbb{1}_{\int_{-\infty}^{M} Y_l^y \, dy \notin [T - \epsilon, T]} \right],$$

and

$$I_{2,\epsilon} = \int_0^{\infty} dl \, \mathbb{E} \left[e^{\int_{-\infty}^{M} \left(-[Y_l^y + f(y)]^2 + \rho Y_l^y \right) dy} \cdots \right.$$

$$\left. \cdots J_{Y_l^M}^{(\rho)} \left(T - \int_{-\infty}^{M} Y_l^y \, dy \right) \mathbb{1}_{\int_{-\infty}^{M} Y_l^y \, dy \in [T - \epsilon, T]} \right].$$

By Proposition 5.4:

$$J_{Y_l^M}^{(\rho)} \left(T - \int_{-\infty}^{M} Y_l^y \, dy \right) \mathbb{1}_{\int_{-\infty}^{M} Y_l^y \, dy \notin [T - \epsilon, T]} \xrightarrow[T \to \infty]{} K \, e_0(Y_l^M),$$

and

$$J_{Y_l^M}^{(\rho)} \left(T - \int_{-\infty}^{M} Y_l^y \, dy \right) \mathbb{1}_{\int_{-\infty}^{M} Y_l^y \, dy \notin [T - \epsilon, T]} \leq C \left(1 + \frac{1}{\sqrt{\epsilon}} \right).$$

Since:

$$\int_0^{\infty} dl \, \mathbb{E} \left[e^{\int_{-\infty}^{M} \left(-[Y_l^y + f(y)]^2 + \rho Y_l^y \right) dy} \right]$$

$$\leq e^{M \rho^2 / 4} \int_0^{\infty} dl \, \mathbb{E} \left[e^{\int_0^{\infty} \left[-(Y_{l,0}^y)^2 + \rho Y_{l,0}^y \right] dy} \right]$$

$$= e^{M \rho^2 / 4} \int_0^{\infty} \bar{K}_l^{(\rho)} \, dl < \infty,$$

one obtains:

$$I_{1,\epsilon} \xrightarrow[T \to \infty]{} K \int_0^{\infty} dl \, \mathbb{E} \left[e^{\int_{-\infty}^{M} \left(-[Y_l^y + f(y)]^2 + \rho Y_l^y \right) dy} e_0(Y_l^M) \right] = K \, A_+^{1,M}(f) < \infty, \qquad (48)$$

by dominated convergence.

On the other hand, by Proposition 5.4:

$$I_{2,\epsilon} \leq C \int_0^\infty dl\, \mathbb{E}\left[e^{\int_{-\infty}^M \left(-\left[Y_l^y + f(y)\right]^2 + \rho Y_l^y\right) dy} \dots \right.$$

$$\left. \dots \left(1 + \left(T - \int_{-\infty}^M Y_l^y\, dy\right)^{-1/2}\right) \mathbb{1}_{\int_{-\infty}^M Y_l^y\, dy \in [T-\epsilon, T]} \right],$$

and by applying Lemma 6.1 to the function $g : t \to (1 + (T-t)^{-1/2})\mathbb{1}_{t \in [T-\epsilon, T]}$:

$$I_{2,\epsilon} \leq C\, C_M'(\epsilon + \sqrt{\epsilon}). \tag{49}$$

Therefore, by combining (48) and (49):

$$\limsup_{T \in E, T \to \infty} \left| e^{\rho T} \mathbb{P}\left[e^{-\int_{-\infty}^\infty [L_T^y + f(y)]^2\, dy}\, \mathbb{1}_{X_T \geq M} \right] - K\, A_+^{1,M}(f) \right| \leq C\, C_M'(\epsilon + \sqrt{\epsilon}),$$

and by taking $\epsilon \to 0$:

$$e^{\rho T} \mathbb{P}\left[e^{-\int_{-\infty}^\infty [L_T^y + f(y)]^2\, dy}\, \mathbb{1}_{X_T \geq M} \right] \xrightarrow[T \in E, T \to \infty]{} K\, A_+^{1,M}(f). \tag{50}$$

Now, let us prove the continuity, with respect to T, of the left hand side of (50).

If $T_0 \in \mathbb{R}_+$ and $T \leq T_0 + 1$ tends to T_0, then $\mathbb{P}$-almost surely, L_T^y tends to $L_{T_0}^y$ and $L_T^y \leq L_{T_0+1}^y$ for all $y \in \mathbb{R}$.

Since $y \to L_{T_0+1}^y + f(y)$ is square-integrable, by dominated convergence:

$$\int_{-\infty}^\infty [L_T^y + f(y)]^2\, dy \xrightarrow[T \to T_0]{} \int_{-\infty}^\infty [L_{T_0}^y + f(y)]^2\, dy.$$

Another application of dominated convergence gives:

$$\left| \mathbb{P}\left[e^{-\int_{-\infty}^\infty [L_T^y + f(y)]^2\, dy}\, \mathbb{1}_{X_{T_0} \geq M} \right] - \mathbb{P}\left[e^{-\int_{-\infty}^\infty [L_{T_0}^y + f(y)]^2\, dy}\, \mathbb{1}_{X_{T_0} \geq M} \right] \right| \xrightarrow[T \to T_0]{} 0. \tag{51}$$

Moreover:

$$\mathbb{P}\left[\left| \mathbb{1}_{X_{T_0} \geq M} - \mathbb{1}_{X_T \geq M} \right| \right] \leq \mathbb{P}\left[\exists t \in [T_0, T], X_t = M \right] \xrightarrow[T \to T_0]{} \mathbb{P}\left[X_{T_0} = M \right] = 0,$$

which implies:

$$\left| \mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\, \mathbb{1}_{X_T \geq M} \right] - \mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\, \mathbb{1}_{X_{T_0} \geq M} \right] \right| \xrightarrow[T \to T_0]{} 0. \tag{52}$$

The convergences (51) and (52) imply the continuity of:

$$T \longrightarrow e^{\rho T}\mathbb{P}\left[e^{-\int_{-\infty}^{\infty}[L_T^y+f(y)]^2\,dy}\, \mathbb{1}_{X_T \geq M} \right].$$

Since E is dense in $\mathbb{R}_+$, we can remove the condition $T \in E$ in (50), which completes the proof of Proposition 1.4. $\qquad\square$

7   A conjecture about the behaviour of $\mathbb{Q}^\beta$

In this paper, we have proven that one can construct a probability measure corresponding to the one-dimensional Edwards' model, for polymers of infinite length.

Moreover, there is an explicit expression for this probability $\mathbb{Q}^\beta$.

Now, the most natural question one can ask is the following: what is the behaviour of the canonical process X under $\mathbb{Q}^\beta$?

At this moment, we are not able to answer this question, which seems to be very difficult, because of the complicated form of the density D_s^β of $\mathbb{Q}^\beta_{|\mathcal{F}_s}$, with respect to $\mathbb{P}_{|\mathcal{F}_s}$.

However, it seems to be reasonable to expect that X_T has a ballistic behaviour, as in the case of Edwards' model on $[0, T]$; one can also expect a central-limit theorem.

Therefore, we can state the following conjecture:

Conjecture: Under $\mathbb{Q}^\beta$, the process X is transient, and:

$$\mathbb{Q}^\beta\left(X_t \xrightarrow[t \to \infty]{} +\infty \right) = \mathbb{Q}^\beta\left(X_t \xrightarrow[t \to \infty]{} -\infty \right) = 1/2.$$

Moreover, there exist universal positive constants a and σ such that:

$$\frac{|X_t|}{t} \xrightarrow[t\to\infty]{} a\beta^{1/3}$$

a.s., and such that the random variable

$$\frac{|X_t| - a\beta^{1/3}t}{\sqrt{t}}$$

converges in law to a centered gaussian variable of variance σ^2 (the factor $\beta^{1/3}$ comes from Brownian scaling).

It is possible that the constants in these convergences are the same as in [18], despite the fact that we don't have any argument to support this. It can also be interesting to study some large deviation results for the canonical process under $\mathbb{Q}^\beta$, and to compare them with the results given in [19]. On the other hand, if the proof of the conjecture above is too hard to obtain, it is perhaps less difficult to prove, by using Ray-Knight theorems, some properties of the total local times $(L_\infty^y)_{y\in\mathbb{R}}$ of X, which are expected to be finite because of the transience of X.

References

[1] E. Bolthausen, On the construction of the three-dimensional polymer measure. Probab. Theory Related Fields **97** (1993), 81–101.

[2] P. Biane, J. Pitman and M. Yor, Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. Bull. Amer. Math. Soc. (N.S.) **38** (2001), 435–465.

[3] P. Biane and M. Yor, Sur la loi des temps locaux browniens pris en un temps exponentiel, in *Séminaire de Probabilités, XXII*, 455–466. Lecture Notes in Math. **1321**, Springer, 1988.

[4] S.-F. Edwards, The statistical mechanics of polymers with excluded volume. Proc. Phys. Soc. **85** (1965), 613–624.

[5] R.-K. Getoor and M.-J. Sharpe, Excursions of Brownian motion and Bessel processes. Z. Wahrsch. Verw. Gebiete **47** (1979), 83–106.

[6] F. Hirsch and G. Lacombe, *Elements of functional analysis*. Graduate Texts in Mathematics **192**, Springer-Verlag, 1999.

[7] C. Leuridan, Le théorème de Ray-Knight à temps fixe, in *Séminaire de Probabilités, XXXII*, 376–396. Lecture Notes in Math. **1686**, Springer, 1998.

[8] J.-F. Le Gall, Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan, in *Séminaire de Probabilités, XIX*, 314–331. Lecture Notes in Math. **1123**, Springer, 1985.

[9] G. Louchard, The Brownian excursion area: a numerical analysis. Comput. Math. Appl. **10** (1984), 413–417.

[10] J. Rosen, Tanaka's formula and renormalization for intersections of planar Brownian motion. Ann. Probab. **14** (1986), 1245–1251.

[11] B. Roynette, P. Vallois and M. Yor, Limiting laws for long Brownian bridges perturbed by their one-sided maximum. III. Period. Math. Hungar. **50** (2005), 247–280.

[12] B. Roynette, P. Vallois and M. Yor, Limiting laws associated with Brownian motion perturbed by its maximum, minimum and local time. II. Studia Sci. Math. Hungar. **43** (2006), 295–360.

[13] B. Roynette, P. Vallois and M. Yor, Limiting laws associated with Brownian motion perturbed by normalized exponential weights. I. Studia Sci. Math. Hungar. **43** (2006), 171–246.

[14] B. Roynette, P. Vallois and M. Yor, Some penalisations of the Wiener measure. Jpn. J. Math. **1** (2006).

[15] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Chap. XI, 411, Springer-Verlag, 1991.

[16] S. Varadhan, Appendix to: Euclidian quantum field theory, by K. Symanzik. Local Quantum Fields Theory, Academic Press (1969).

[17] R. van der Hofstad, *One-dimensional random polymers*. CWI Tract **123**, Stichting Mathematisch Centrum Centrum voor Wiskunde en Informatica, 1998.

[18] R. van der Hofstad, F. den Hollander and W. König, Central limit theorem for the Edwards model. Ann. Probab. **25** (1997), 573–597.

[19] R. van der Hofstad, F. den Hollander and W. König, Large deviations for the one-dimensional Edwards model. Ann. Probab. **31** (2003), 2003–2039.

[20] M.-J. Westwater, On Edwards' model for long polymer chains. Comm. Math. Phys. **72** (1980), 131–174.

[21] M.-J. Westwater, On Edwards' model for polymer chains. III. Borel summability. Comm. Math. Phys. **84** (1982), 459–470.

[22] M.-J. Westwater, On Edwards' model for polymer chains, in *Trends and developments in the eighties*, 384–404, World Sci. Publishing, 1985.