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Penalizations of the Brownian motion with a functional of its local times

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Abstract

In this article, we study the family of probability measures (indexed by $t \in \mathbf{R}_+^*$), obtained by penalization of the Brownian motion with a given functional of its local times at time t.

We prove that this family tends to a limit measure when t goes to infinity if the functional satisfies some conditions of domination, and we check these conditions in several particular cases.

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0. Introduction

Brownian penalizations have been studied in several articles, in particular in [4–6]. The general principle of these penalizations is the following: let $\mathbf{W}$ be the Wiener measure on $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$, $(X_t)_{t\geq 0}$ the canonical process, and $(\Gamma_t)_{t\geq 0}$ a family of positive weights such that $0 < \mathbf{W}[\Gamma_t] < \infty$; we consider the family of probability measures $(\mathbf{W}_t)_{t\geq 0}$, obtained from $\mathbf{W}$, by "penalization" with the weight Γ:

$$\mathbf{W}_t = \frac{\Gamma_t}{\mathbf{W}[\Gamma_t]}.\mathbf{W}$$

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In many different particular cases, the family $(\mathbf{W}_t)_{t\geq 0}$ tends to a limit measure $\mathbf{W}_\infty$ as $t \to \infty$, in the following sense: for all $s \geq 0$, and for Λ_s measurable with respect to $\mathcal{F}_s = \sigma\{X_u, u \leq s\}$,

$$\mathbf{W}_t(\Lambda_s) \underset{t\to\infty}{\to} \mathbf{W}_\infty(\Lambda_s).$$

Up to now, there does not exist a general theorem which covers all the different cases for which convergence holds. On the other hand, we remark that in many of these cases, one has

$$\Gamma_t = F((l_t^y(X))_{y\in\mathbf{R}})$$

where $(l_t^y(X))_{y\in\mathbf{R}}$ is the family of the local times of $(X_s)_{s\leq t}$, and F is a measurable functional from $\mathcal{C}(\mathbf{R}, \mathbf{R}_+)$ to $\mathbf{R}_+$.

These two facts led us to prove that if Γ is of this form, the limit measure $\mathbf{W}_\infty$ exists for a "large" class of functionals F.

This proof is the main topic of our article, which is divided into six sections.

In the first one, we define and explain the notation that we need to prove our main theorem, which is stated at the end of the section.

In Section 2, we prove an equality satisfied by an approximation of a given functional of local times, and in Section 3, we majorize the error term corresponding to this approximation.

This allows us to obtain, in Section 4, the asymptotic behaviour of the expectation of functionals which satisfy some particular conditions, and finally we prove the main theorem in Section 5.

In Section 6, we study the four following examples, for which the theorem applies:

(1) $F((l^y)_{y\in\mathbf{R}}) = \phi(l^0)$ (which corresponds to $\Gamma_t = \phi(l_t^0(X))$), where ϕ is a function from $\mathbf{R}_+$ to $\mathbf{R}_+$, dominated by an integrable and decreasing function ψ.

(2) $F((l^y)_{y\in\mathbf{R}}) = \phi(\inf\{y \geq 0, l^y = 0\})$ (which corresponds to the weight $\Gamma_t = \phi(\sup\{X_s, s \leq t\})$), where ϕ is a function from $\mathbf{R}_+ \cup \{\infty\}$ to $\mathbf{R}_+$, dominated by a decreasing function ψ, which is integrable on $\mathbf{R}_+$.

(3) $F((l^y)_{y\in\mathbf{R}}) = \exp(-\int_{-\infty}^{\infty} V(y) l^y \mathrm{d}y)$, where V is a positive measurable function, not a.e. equal to zero, and integrable with respect to $(1 + y^2)\mathrm{d}y$.

(4) $F((l^y)_{y\in\mathbf{R}}) = \phi(l^{y_1}, l^{y_2})$, where $y_1 < y_2$ and $\phi(l_1, l_2) \leq h(l_1 \wedge l_2)$, for a decreasing and integrable function h.

The three first examples have been already studied by B. Roynette, P. Vallois and M. Yor.

As a help to the reader, we mention that Sections 2 and 3 are quite technical, but it is possible to read the details of these sections after Sections 4 and 5, which contain the principal steps of the proof of the theorem.

1. Notation and statement of the main theorem

In this article, $(B_t)_{t\geq 0}$ denotes a standard one-dimensional Brownian motion, $(L_t^y)_{t\geq 0, y\in\mathbf{R}}$ the bicontinuous version of its local times, and $(\tau_l^a)_{l\geq 0, a\in\mathbf{R}}$ the family of its inverse local times.

To simplify this notation, we put $T_a = \tau_0^a$ (first hitting time at a of B) and $\tau_l^0 = \tau_l$.

For every $l \in \mathbf{R}_+$, $(Y_{l,+}^y)_{y\in\mathbf{R}}$ denotes a random process defined on the whole real line, such that its "positive part" $(Y_{l,+}^y)_{y\geq 0}$ is a two-dimensional squared Bessel process (BESQ(2)), its "negative part" $(Y_{l,+}^{-y})_{y\geq 0}$ is an independent zero-dimensional squared Bessel process (BESQ(0)), and its value at zero $Y_{l,+}^0$ is equal to l. In particular, by classical properties of BESQ(0) and BESQ(2) processes, there exists a.s. $y_0 \leq 0$ such that $Y_{l,+}^y = 0$ iff $y \leq y_0$.

We define also $(Y^y_{l,-})_{y\in\mathbf{R}}$ as a process which has the same law as $(Y^{-y}_{l,+})_{y\in\mathbf{R}}$, the process obtained from $(Y^y_{l,+})_{y\in\mathbf{R}}$ by "reversing the time".

In one of the penalization results shown in [5], B. Roynette, P. Vallois and M. Yor obtain a limit process $(Z^l_t)_{t\geq 0}$, such that $Z^l_t = B_t$ for $t \leq \tau_l$, $(|Z^l_{\tau_l+u}|)_{u\geq 0}$ is a BES(3) process independent of B, and $\epsilon = \mathrm{sgn}(Z^l_{\tau_l+u})$ $(u > 0)$ is an independent variable such that $\mathbf{P}(\epsilon=1)=\mathbf{P}(\epsilon=-1)=1/2$. This process can be informally considered to be a Brownian motion conditioned to have a total local time equal to l at level zero. By applying Ray–Knight theorems for Brownian local times (see [7]) to $(Z^l_t)_{t\geq 0}$, it is possible to show that the law of the family of its total local times is the half-sum of the laws of $(Y^y_{l,+})_{y\in\mathbf{R}}$ and $(Y^y_{l,-})_{y\in\mathbf{R}}$ ($(Y^y_{l,+})_{y\in\mathbf{R}}$ corresponds to the paths of $(Z^l_t)_{t\geq 0}$ such that $\epsilon = 1$, and $(Y^y_{l,-})_{y\in\mathbf{R}}$ corresponds to the paths such that $\epsilon = -1$).

This explains why the processes $(Y^y_{l,+})_{y\in\mathbf{R}}$ and $(Y^y_{l,-})_{y\in\mathbf{R}}$ occur naturally in the description of the asymptotic behaviour of Brownian local times.

We also need to define some modifications of $(Y^y_{l,+})_{y\in\mathbf{R}}$ and $(Y^y_{l,-})_{y\in\mathbf{R}}$: for $l \geq 0$, $a \geq 0$, $(Y^y_{l,a})_{y\in\mathbf{R}}$ denotes a process such that $(Y^y_{l,a})_{y\geq 0}$ is markovian with the infinitesimal generator of BESQ(2) for $y \leq a$ and the infinitesimal generator of BESQ(0) for $y \geq a$, $(Y^{-y}_{l,a})_{y\geq 0}$ is an independent BESQ(0) process, and $Y^0_{l,a} = l$. For $a \leq 0$, $(Y^y_{l,a})_{y\in\mathbf{R}}$ has the same law as $(Y^{-y}_{l,-a})_{y\in\mathbf{R}}$.

Now, let F be a functional from $\mathcal{C}(\mathbf{R},\mathbf{R}_+)$ to $\mathbf{R}_+$, which is measurable with respect to the σ-field generated by the topology of uniform convergence on compact sets. We consider the following quantities, which will naturally appear in the asymptotics of $\mathbf{E}[F((L^y_t)_{y\in\mathbf{R}})]$:

$$I_+(F) = \int_0^\infty \mathrm{d}l\, \mathbf{E}[F((Y^y_{l,+})_{y\in\mathbf{R}})]$$

$$I_-(F) = \int_0^\infty \mathrm{d}l\, \mathbf{E}[F((Y^y_{l,-})_{y\in\mathbf{R}})]$$

$$I(F) = I_+(F) + I_-(F).$$

We observe that $I(F)$ is the integral of F with respect to the σ-finite measure I on $\mathcal{C}(\mathbf{R},\mathbf{R}_+)$, defined by

$$I = \int_0^\infty \mathrm{d}l\, P_{l,+} + \int_0^\infty \mathrm{d}l\, P_{l,-}$$

where $P_{l,+}$ is the law of $(Y^y_{l,+})_{y\in\mathbf{R}}$ and $P_{l,-}$ is the law of $(Y^y_{l,-})_{y\in\mathbf{R}}$.

At the end of this section, we give some conditions on F which turn out to be sufficient for obtaining our penalization result.

Unfortunately, these conditions are not very simple and we need three more definitions before stating the main theorem:

Definition 1.1 (*a condition of domination*). Let c and n be in $\mathbf{R}_+$ (generally n will be an integer). For every decreasing function h from $\mathbf{R}_+$ to $\mathbf{R}_+$, we say that a measurable function F from $\mathcal{C}(\mathbf{R},\mathbf{R}_+)$ to $\mathbf{R}_+$ satisfies the condition $C(c,n,h)$ iff the following holds for every continuous function l from $\mathbf{R}$ to $\mathbf{R}_+$:

(1) $F((l^y)_{y\in\mathbf{R}})$ depends only on $(l^y)_{y\in[-c,c]}$.

(2) $F((l^y)_{y\in\mathbf{R}}) \leq \left(\frac{\sup_{y\in[-c,c]} l^y+c}{\inf_{y\in[-c,c]} l^y+c}\right)^n h(\inf_{y\in[-c,c]} l^y)$.

Intuitively, a functional of the local times satisfies the above condition if it depends only on the local times on a compact set, and if it is small when these local times are large and do not vary too much.

Now, let us use the notation

$$N_c(h) = ch(0) + \int_0^\infty h(y)\mathrm{d}y.$$

If $N_c(h) < \infty$, it is possible to prove our main theorem for all functionals F which satisfy the condition $C(c, n, h)$, but this condition is restrictive, since the functional F must not depend on the local times outside of $[-c, c]$.
In order to relax this restriction, we need the following definition:

Definition 1.2 (*a less restrictive condition of domination*). Let n be in $\mathbf{R}_+$ and F be a positive and measurable function from $\mathcal{C}(\mathbf{R}, \mathbf{R}_+)$ to $\mathbf{R}$.
For all $M \geq 0$, let us say that F satisfies the condition $D(n, M)$ iff there exists a sequence $(c_k)_{k\geq 1}$ in $[1, \infty[$, a sequence $(h_k)_{k\geq 1}$ of decreasing functions from $\mathbf{R}_+$ to $\mathbf{R}_+$, and a sequence $(F_k)_{k\geq 0}$ of measurable functions from $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ to $\mathbf{R}_+$, such that:
(1) $F_0 = 0$ and $(F_k)_{k\geq 1}$ tends to F pointwise.
(2) For all $k \geq 1$, $|F_k - F_{k-1}|$ satisfies the condition $C(c_k, n, h_k)$.
(3) $\sum_{k\geq 1} N_{c_k}(h_k) \leq M$.
We define the quantity $N^{(n)}(F)$ as the infimum of $M \geq 0$ such that F satisfies the condition $D(n, M)$.

Intuitively, if $N^{(n)}(F) < \infty$, it means that F can be well approximated by functionals which satisfy conditions given in Definition 1.1.
In particular, if F satisfies the condition $C(c, n, h)$ for $c \geq 1$, one has $N^{(n)}(F) \leq N_c(h)$ (one can prove that F satisfies the condition $D(n, N_c(h))$, by taking in Definition 1.2 $c_k = c$, $h_k = h\mathbf{1}_{k=1}$, $F_0 = 0$ and $F_k = F$ if $k \geq 1$).
Now, for a given functional F, we need to define some other functionals, informally obtained from F by "shifting" the space and adding a given function to the local time family.
More precisely, let us consider the following definition:

Definition 1.3 (*local time and space shift*). Let x be a real number. If F is a measurable functional from $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ to $\mathbf{R}_+$, and if $(l_0^y)_{y\in\mathbf{R}}$ is a continuous function from $\mathbf{R}$ to $\mathbf{R}_+$, we denote by $F^{(l_0^y)_{y\in\mathbf{R}},x}$ the functional from $\mathcal{C}(\mathbf{R}, \mathbf{R}_+)$ to $\mathbf{R}_+$ which satisfies

$$F^{(l_0^y)_{y\in\mathbf{R}},x}((l^y)_{y\in\mathbf{R}}) = F((l_0^y + l^{y-x})_{y\in\mathbf{R}})$$

for every function $(l^y)_{y\in\mathbf{R}}$.

This notation and the functionals defined in this way appear naturally when we consider the conditional expectation $\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})|(B_u)_{u\leq s}]$, for $0 < s < t$, and apply the Markov property.
We are now able to state the main theorem of the article:

Theorem. *Let F be a functional from $\mathcal{C}(\mathbf{R}, \mathbf{R}_+)$ to $\mathbf{R}_+$ such that $I(F) > 0$ and $N^{(n)}(F) < \infty$ for some $n \geq 0$.*
If $\mathbf{W}$ denotes the standard Wiener measure on $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$, $(X_t)_{t\geq 0}$ the canonical process, and $(l_t^y(X))_{t\in\mathbf{R}_+, y\in\mathbf{R}}$ the continuous family of its local times ($\mathbf{W}$-a.s. well defined), the probability

measure:

$$\mathbf{W}_t^F = \frac{F\left(\left(l_t^y(X)\right)_{y\in\mathbf{R}}\right)}{\mathbf{W}\left[F\left(\left(l_t^y(X)\right)_{y\in\mathbf{R}}\right)\right]}.\mathbf{W}$$

is well defined for every t which is large enough, and there exists a probability measure $\mathbf{W}_\infty^F$ *such that*

$$\mathbf{W}_t^F(\Lambda_s) \underset{t\to\infty}{\to} \mathbf{W}_\infty^F(\Lambda_s)$$

for every $s \geq 0$ *and* $\Lambda_s \in \mathcal{F}_s = \sigma\{X_u, u \leq s\}$.
Moreover, this limit measure satisfies the following equality:

$$\mathbf{W}_\infty^F(\Lambda_s) = \mathbf{W}\left(\mathbf{1}_{\Lambda_s}.\frac{I\left(F^{(l_s^y(X))_{y\in\mathbf{R}},X_s}\right)}{I(F)}\right).$$

Remark 1.1. A consequence of the theorem is the fact that if $I(F) > 0$ and $N^{(n)}(F) < \infty$ for some $n \geq 0$, the process $\frac{(I(F^{(L_s^y)_{y\in\mathbf{R}},B_s}))_{s\geq 0}}{I(F)}$ is a martingale. In three of the four examples studied in Section 6, we compute explicitly this martingale, and in the two first ones, we check that this computation agrees with the results obtained by B. Roynette, P. Vallois and M. Yor.

Remark 1.2. We point out that our notation, $l_t^y(X)$, for the local times given in the theorem, differs from the notation L_t^y, which is used for the local times of $(B_s)_{s\leq t}$. This is because, in one case, we consider the canonical process $(X_t)_{t\geq 0}$ on a given probability space, and in the other case, we consider a Brownian motion on a space which is not made precise. Hence, the two mathematical objects deserve different writings, despite the fact that they are strongly related.

Remark 1.3. The domination conditions placed on F in the theorem are modelled on the case $F((l^y)_{y\in\mathbf{R}}) = \phi(l^0)$, for which convergence holds if ϕ is a positive and integrable function (see [6]).
In Definition 1.1, it is natural to replace $\phi(l^0)$ by $h(\inf_{y\in[-c,c]} l^y)$; the factor $(\frac{\sup_{y\in[-c,c]}+c}{\inf_{y\in[-c,c]}+c})^n$ is given in order to relax the condition $C(c, n, h)$ if the local time has large variations near level 0. Intuitively, the motivation for the hypotheses of the theorem is the fact that if they are satisfied, the mass of $\mathbf{W}_t^F$ arises from paths that do not spend large amounts of time in a neighbourhood of the origin, as in particular cases studied by B. Roynette, P. Vallois and M. Yor.

2. An approximation of the functionals of local times

In order to prove the theorem, we need to study the expectation of $F((L_t^y)_{y\in\mathbf{R}})$, where F is a function from $\mathcal{C}(\mathbf{R}, \mathbf{R}_+)$ to $\mathbf{R}_+$.
However, in general, it is difficult to do that directly, so in this section, we will replace $F((L_t^y)_{y\in\mathbf{R}})$ by an approximation.
For the study of this approximation, we need to consider the following quantities:

$$\mathcal{I}_{l,+}^c = \int_{-c}^c Y_{l,+}^y \mathrm{d}y, \qquad \mathcal{I}_{l,-}^c = \int_{-c}^c Y_{l,-}^y \mathrm{d}y, \qquad \mathcal{I}_{l,a}^c = \int_{-c}^c Y_{l,a}^y \mathrm{d}y$$

for $c \in \mathbf{R}_+$ or $c = \infty$, $a \in \mathbf{R}$;

$$\mathcal{Y}^c_{l,+} = \frac{1}{2}(Y^c_{l,+} + Y^{-c}_{l,+}), \qquad \mathcal{Y}^c_{l,-} = \frac{1}{2}(Y^c_{l,-} + Y^{-c}_{l,-}), \qquad \mathcal{Y}^c_{l,a} = \frac{1}{2}(Y^c_{l,a} + Y^{-c}_{l,a})$$

for $c \in \mathbf{R}_+$, $a \in \mathbf{R}$;

$$I_{c,t,+}(F) = \int_0^\infty \mathrm{d}l\, \mathbf{E}\left[F((Y^y_{l,+})_{y\in\mathbf{R}})\frac{\mathrm{e}^{-(\mathcal{Y}^c_{l,+})^2/2(t-\mathcal{I}^c_{l,+})}}{\sqrt{1-\mathcal{I}^c_{l,+}/t}}\phi\left(\frac{\mathcal{I}^c_{l,+}}{t}\right)\right]$$

$$I_{c,t,-}(F) = \int_0^\infty \mathrm{d}l\, \mathbf{E}\left[F((Y^y_{l,-})_{y\in\mathbf{R}})\frac{\mathrm{e}^{-(\mathcal{Y}^c_{l,-})^2/2(t-\mathcal{I}^c_{l,-})}}{\sqrt{1-\mathcal{I}^c_{l,-}/t}}\phi\left(\frac{\mathcal{I}^c_{l,-}}{t}\right)\right]$$

and

$$I_{c,t}(F) = I_{c,t,+}(F) + I_{c,t,-}(F)$$

for $c \in \mathbf{R}_+$, $t > 0$, where ϕ denotes the function from $\mathbf{R}_+$ to $\mathbf{R}_+$ such that $\phi(x) = 1$ in $x \le 1/3$, $\phi(x) = 2 - 3x$ if $1/3 \le x \le 2/3$ and $\phi(x) = 0$ if $x \ge 2/3$ (in particular, this function is continuous with compact support included in $[0, 1[$).

We observe that the expression $\frac{\mathrm{e}^{(\mathcal{Y}^c_{l,+})^2/2(t-\mathcal{I}^c_{l,+})}}{\sqrt{1-\mathcal{I}^c_{l,+}/t}}$ is not well defined if $\mathcal{I}^c_{l,+} \ge t$; but this is not important here, since $\phi(\mathcal{I}^c_{l,+}/t) = 0$ in that case.

Now, the main result of this section is the following proposition:

Proposition 2.1. *For all measurable functionals from $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ to $\mathbf{R}_+$, such that $F((l^y)_{y\in\mathbf{R}})$ depends only on $(l^y)_{y\in[-c,c]}$ for some $c \ge 0$, the following equality holds:*

$$\sqrt{2\pi t}\,\mathbf{E}\left[F((L^y_t)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\ge c}\phi\left(\frac{1}{t}\int_{-c}^c L^y_t\,\mathrm{d}y\right)\right] = I_{c,t}(F)$$

for all $t > 0$.

Proof. Let G_0 be a functional from $\mathcal{C}(\mathbf{R}_+, \mathbf{R}) \times \mathbf{R}_+$ to $\mathbf{R}_+$, such that the process $(G_0((X_s)_{s\ge0}, t))_{t\ge0}$, defined on the canonical space $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$, is progressively measurable.

If we denote by H_0 the functional defined by

$$H_0((\omega_s)_{s\ge0}, t) = G_0((\omega_t - \omega_{(t-s)_+})_{s\ge0}, t)$$

for every continuous function ω from $\mathbf{R}_+$ to $\mathbf{R}$ and all $t \ge 0$, H_0 satisfies the same conditions as G_0; moreover, $G_0((\omega_s)_{s\ge0}, t)$ and $H_0((\omega_s)_{s\ge0}, t)$ depend only on $(\omega_s)_{s\le t}$, therefore one can write

$$G((\omega_s)_{s\le t}) = G_0((\omega_s)_{s\ge0}, t)$$
$$H((\omega_s)_{s\le t}) = H_0((\omega_s)_{s\ge0}, t).$$

Now, by results by C. Leuridan (see [2]), P. Biane and M. Yor (see [1]), one has

$$\int_0^\infty \mathrm{d}t\, H((B_s)_{s\le t}) = \int_0^\infty \mathrm{d}l \int_{-\infty}^\infty \mathrm{d}a\, H((B_s)_{s\le\tau^a_l}).$$

Moreover, for all $t \geq 0$, $(B_s)_{s\leq t}$ and $(B_t - B_{t-s})_{s\leq t}$ have the same law; therefore, $G((B_s)_{s\leq t})$ and $H((B_s)_{s\leq t})$ have the same law and

$$\int_0^\infty \mathrm{d}t\, \mathbf{E}[G((B_s)_{s\leq t})] = \int_0^\infty \mathrm{d}l \int_{-\infty}^\infty \mathrm{d}a\, \mathbf{E}[H((B_s)_{s\leq \tau_l^a})].$$

Let us consider a continuous process $(Z_s^{l,a})_{s\geq 0}$ such that $Z_s^{l,a} = B_s$ for $s \leq \tau_l$ and $(Z^{l,a}_{\tau_l+u})_{u\leq T_{a\to 0}}$ is the time-reversed process of a Brownian motion starting from a, independent of B, and considered up to its first hitting time of zero (denoted by $T_{a\to 0}$); the trajectory of $Z^{l,a}$ after time $\tau_l + T_{a\to 0}$ is not important.

By using invariance properties of Brownian motion by time and space reversals, it is not difficult to check that $(Z_s^{l,a})_{s\leq \tau_l+T_{a\to 0}}$ has the same law as $(a - B_{\tau_l^a - s})_{s\leq \tau_l^a}$; therefore, $G((Z_s^{l,a})_{s\leq \tau_l+T_{a\to 0}})$ has the same law as $H((B_s)_{s\leq \tau_l^a})$ and

$$\int_0^\infty \mathrm{d}t\, \mathbf{E}[G((B_s)_{s\leq t})] = \int_0^\infty \mathrm{d}l \int_{-\infty}^\infty \mathrm{d}a\, \mathbf{E}[G((Z_s^{l,a})_{s\leq \tau_l+T_{a\to 0}})].$$

Hence, for all Borel sets U of $\mathbf{R}_+^*$, if we define $J_{c,U}(F)$ by

$$J_{c,U}(F) = \int_U \mathrm{d}t\, \mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\geq c}\phi\left(\frac{1}{t}\int_{-c}^c L_t^y \mathrm{d}y\right)\right]$$

we have, by taking G_0 and G such that $G((B_s)_{s\leq t}) = F((L_t^y)_{y\in\mathbf{R}})$,

$$\begin{aligned} J_{c,U}(F) &= \int_0^\infty \mathrm{d}t\, \mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\geq c}\phi\left(\frac{\int_{-c}^c L_t^y \mathrm{d}y}{\int_{-\infty}^\infty L_t^y \mathrm{d}y}\right)\mathbf{1}_{\int_{-\infty}^\infty L_t^y \mathrm{d}y \in U}\right] \\ &= \int_0^\infty \mathrm{d}l \int_{\mathbf{R}\setminus[-c,c]} \mathrm{d}a\, \mathbf{E}\left[F((L^{y,l,a})_{y\in\mathbf{R}})\phi\left(\frac{\int_{-c}^c L^{y,l,a}\mathrm{d}y}{\int_{-\infty}^\infty L^{y,l,a}\mathrm{d}y}\right)\mathbf{1}_{\int_{-\infty}^\infty L^{y,l,a}\mathrm{d}y\in U}\right] \end{aligned}$$

where $(L^{y,l,a})_{y\in\mathbf{R}}$ is the continuous family of the total local times of $Z^{l,a}$.

Hence, by the Ray–Knight theorem applied to the independent processes $(B_s = Z_s)_{s\leq \tau_l}$ and $(Z_{\tau_l+u})_{u\leq T_{a\to 0}}$, and classical additivity properties of squared Bessel processes,

$$\begin{aligned} J_{c,U}(F) &= \int_0^\infty \mathrm{d}l \int_{\mathbf{R}\setminus[-c,c]} \mathrm{d}a\, \mathbf{E}\left[F((Y_{l,a}^y)_{y\in\mathbf{R}})\phi\left(\frac{\mathcal{I}_{l,a}^c}{\mathcal{I}_{l,a}^\infty}\right)\mathbf{1}_{\mathcal{I}_{l,a}^\infty\in U}\right] \\ &= \int_0^\infty \mathrm{d}l \int_{\mathbf{R}\setminus[-c,c]} \mathrm{d}a\, \mathbf{E}\left[F((Y_{l,a}^y)_{y\in\mathbf{R}})\mathbf{E}\left[\phi\left(\frac{\mathcal{I}_{l,a}^c}{\mathcal{I}_{l,a}^\infty}\right)\mathbf{1}_{\mathcal{I}_{l,a}^\infty\in U}\,\middle|\, (Y_{l,a}^y)_{y\in[-c,c]}\right]\right] \end{aligned}$$

since $F((Y_{l,a}^y)_{y\in\mathbf{R}})$ depends only on $(Y_{l,a}^y)_{y\in[-c,c]}$.

Now, if θ is a given continuous function from $[-c, c]$ to $\mathbf{R}_+$, the integrals $\int_c^\infty Y_{l,a}^y \mathrm{d}y$ and $\int_{-\infty}^{-c} Y_{l,a}^y \mathrm{d}y$ are independent conditionally on $(Y_{l,a}^y = \theta^y)_{y\in[-c,c]}$ and their conditional laws are respectively equal to the laws of $\int_0^\infty Y^y_{\theta^c,(a-c)_+}\mathrm{d}y$ and $\int_0^\infty Y^y_{\theta^{-c},(-a-c)_+}\mathrm{d}y$.

Therefore, by additivity properties of BESQ processes, the conditional law of

$$\mathcal{I}_{l,a}^\infty - \mathcal{I}_{l,a}^c = \int_{-\infty}^{-c} Y_{l,a}^y \mathrm{d}y + \int_c^\infty Y_{l,a}^y \mathrm{d}y$$

given $(Y^y_{l,a} = \theta^y)_{y\in[-c,c]}$, is equal to the law of

$$\int_0^\infty Y^y_{\theta^c+\theta^{-c},0}\mathrm{d}y + \int_0^\infty Y^y_{0,(|a|-c)_+}\mathrm{d}y$$

where $(Y^y_{\theta^c+\theta^{-c},0})_{y\geq 0}$ and $(Y^y_{0,(|a|-c)_+})_{y\geq 0}$ are supposed to be independent.
By the Ray–Knight theorem, $\int_0^\infty Y^y_{\theta^c+\theta^{-c},0}\mathrm{d}y$ has the same law as the time spent in $\mathbf{R}_+$ by $(B_s)_{s\leq \tau_{\theta^c+\theta^{-c}}}$; therefore,

$$\int_0^\infty Y^y_{\theta^c+\theta^{-c},0}\mathrm{d}y \overset{(d)}{=} \tau_{(\theta^c+\theta^{-c})/2} \overset{(d)}{=} T_{(\theta^c+\theta^{-c})/2}.$$

Moreover,

$$\int_0^\infty Y^y_{0,(|a|-c)_+}\mathrm{d}y \overset{(d)}{=} T_{(|a|-c)_+}.$$

Hence, the conditional law of $\mathcal{I}^\infty_{l,a} - \mathcal{I}^c_{l,a}$, given $(Y^y_{l,a} = \theta^y)_{y\in[-c,c]}$, is equal to the law of $T_{(|a|-c)_+ + (\theta^c+\theta^{-c})/2}$. Consequently,

$$J_{c,U}(F) = \int_0^\infty \mathrm{d}l \int_{\mathbf{R}\setminus[-c,c]} \mathrm{d}a\, \mathbf{E}\left[F((Y^y_{l,a})_{y\in\mathbf{R}})\psi_a(\mathcal{I}^c_{l,a}, \mathcal{Y}^c_{l,a})\right]$$

where, for $|a| > c$,

$$\psi_a(\mathcal{I}, \theta) = \mathbf{E}\left[\phi\left(\frac{\mathcal{I}}{\mathcal{I} + T_{|a|-c+\theta}}\right)\mathbf{1}_{\mathcal{I}+T_{|a|-c+\theta}\in U}\right].$$

Now, if, for all $u > 0$, p_u denotes the density of the law of T_u, one has

$$\psi_a(\mathcal{I}, \theta) = \int_U \phi(\mathcal{I}/t)p_{|a|-c+\theta}(t-\mathcal{I})\mathrm{d}t$$

and

$$J_{c,U}(F) = \int_U \mathrm{d}t \int_0^\infty \mathrm{d}l \int_{\mathbf{R}\setminus[-c,c]} \mathrm{d}a\, \mathbf{E}\left[F((Y^y_{l,a})_{y\in\mathbf{R}})\phi\left(\frac{\mathcal{I}^c_{l,a}}{t}\right)p_{|a|-c+\mathcal{Y}^c_{l,a}}(t-\mathcal{I}^c_{l,a})\right].$$

By hypothesis, $F((Y^y_{l,a})_{y\in\mathbf{R}})$ depends only on $(Y^y_{l,a})_{y\in[-c,c]}$. Moreover, for $a \geq c$, $(Y^y_{l,a})_{y\in[-c,c]}$ has the same law as $(Y^y_{l,+})_{y\in[-c,c]}$, and for $a \leq -c$, $(Y^y_{l,a})_{y\in[-c,c]}$ has the same law as $(Y^y_{l,-})_{y\in[-c,c]}$.
Hence, we have

$$\begin{aligned} J_{c,U}(F) &= \int_U \mathrm{d}t \int_0^\infty \mathrm{d}l\, \mathbf{E}\left[F((Y^y_{l,+})_{y\in\mathbf{R}})\phi\left(\frac{\mathcal{I}^c_{l,+}}{t}\right)\int_c^\infty p_{a-c+\mathcal{Y}^c_{l,+}}(t-\mathcal{I}^c_{l,+})\mathrm{d}a\right] \\ &\quad + \int_U \mathrm{d}t \int_0^\infty \mathrm{d}l\, \mathbf{E}\left[F((Y^y_{l,-})_{y\in\mathbf{R}})\phi\left(\frac{\mathcal{I}^c_{l,-}}{t}\right)\int_{-\infty}^{-c} p_{|a|-c+\mathcal{Y}^c_{l,-}}(t-\mathcal{I}^c_{l,-})\mathrm{d}a\right]. \end{aligned}$$

Now, for $\theta \geq 0$, $u > 0$,

$$\begin{aligned} \int_{-\infty}^{-c} p_{|a|-c+\theta}(u)\mathrm{d}a &= \int_c^\infty p_{a-c+\theta}(u)\mathrm{d}a = \int_\theta^\infty p_b(u)\mathrm{d}b \\ &= \int_\theta^\infty \frac{b}{\sqrt{2\pi u^3}}\mathrm{e}^{-b^2/2u}\mathrm{d}b = \frac{1}{\sqrt{2\pi u}}\mathrm{e}^{-\theta^2/2u}. \end{aligned}$$

Therefore,

$$J_{c,U}(F) = \int_U \mathrm{d}t \frac{I_{c,t}(F)}{\sqrt{2\pi t}}.$$

This equality is satisfied for every Borel set U. Hence, by definition of $J_{c,U}(F)$, the equality given in Proposition 2.1 occurs for almost every $t > 0$.
In order to prove it for all $t > 0$, we begin to suppose that F is bounded and continuous.
In this case, for all $s, t > 0$,

$$\left| \mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}}) \mathbf{1}_{|X_t|\geq c} \phi\left(\frac{1}{t}\int_{-c}^{c} L_t^y \mathrm{d}y\right)\right] - \mathbf{E}\left[F((L_s^y)_{y\in\mathbf{R}}) \mathbf{1}_{|X_s|\geq c} \phi\left(\frac{1}{s}\int_{-c}^{c} L_s^y \mathrm{d}y\right)\right]\right|$$
$$\leq \mathbf{E}\left[\left| F((L_t^y)_{y\in\mathbf{R}}) \phi\left(\frac{1}{t}\int_{-c}^{c} L_t^y \mathrm{d}y\right) - F((L_s^y)_{y\in\mathbf{R}}) \phi\left(\frac{1}{s}\int_{-c}^{c} L_s^y \mathrm{d}y\right)\right|\right]$$
$$+ \|F\|_\infty \mathbf{P} \quad (\exists u \in [s,t], |X_u| = c).$$

If t is fixed, the first term of this sum tends to zero when s tends to t, by continuity of F, ϕ and dominated convergence.
The second term tends also to

$$\|F\|_\infty \mathbf{P}(|X_t| = c) = 0.$$

Therefore, the function

$$t \to \mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}}) \mathbf{1}_{|X_t|\geq c} \phi\left(\frac{1}{t}\int_{-c}^{c} L_t^y \mathrm{d}y\right)\right]$$

is continuous.
Now, let us prove that $I_{c,t}(F)$ is also continuous with respect to t.
For all $t > 0$,

$$F((Y_{l,+}^y)_{y\in\mathbf{R}}) \frac{\mathrm{e}^{-(\mathcal{Y}_{l,+}^c)^2/2(s-\mathcal{I}_{l,+}^c)}}{\sqrt{1-\mathcal{I}_{l,+}^c/s}} \phi\left(\frac{\mathcal{I}_{l,+}^c}{s}\right) \underset{s\to t}{\to} F((Y_{l,+}^y)_{y\in\mathbf{R}}) \frac{\mathrm{e}^{-(\mathcal{Y}_{l,+}^c)^2/2(t-\mathcal{I}_{l,+}^c)}}{\sqrt{1-\mathcal{I}_{l,+}^c/t}} \phi\left(\frac{\mathcal{I}_{l,+}^c}{t}\right)$$

by continuity of ϕ (if $\mathcal{I}_{l,+}^c < t$, it is clear, and if $\mathcal{I}_{l,+}^c \geq t$, the two expressions are equal to zero for $s \leq 3t/2$).
Moreover, for $s \leq 2t$,

$$F((Y_{l,+}^y)_{y\in\mathbf{R}}) \frac{\mathrm{e}^{-(\mathcal{Y}_{l,+}^c)^2/2(s-\mathcal{I}_{l,+}^c)}}{\sqrt{1-\mathcal{I}_{l,+}^c/s}} \phi\left(\frac{\mathcal{I}_{l,+}^c}{s}\right) \leq \sqrt{3}\|F\|_\infty \mathrm{e}^{-(\mathcal{Y}_{l,+}^c)^2/4t}$$
$$\leq \sqrt{3}\|F\|_\infty \mathrm{e}^{-(Y_{l,+}^c)^2/16t}.$$

Recalling that the Lebesgue measure is invariant for the BESQ(2) process $(Y_{l,+}^y)_{y\geq 0}$, we have

$$\int_0^\infty \mathrm{d}l\, \mathbf{E}\left[\mathrm{e}^{-(Y_{l,+}^c)^2/16t}\right] = \int_0^\infty \mathrm{d}l\, \mathrm{e}^{-l^2/16t} < \infty.$$

By dominated convergence, $t \to I_{c,t,+}(F)$ is continuous.
Similar computations imply the continuity of $t \to I_{c,t,-}(F)$, and finally $t \to I_{c,t}(F)$ is continuous.
Consequently, for F continuous and bounded, the equality given in Proposition 2.1, which was proven for a.e. $t > 0$, remains true for every $t > 0$.

Now, by monotone class theorem (see [8]), it is not difficult to extend this equality to every measurable and positive function, which completes the proof of Proposition 2.1. □

This proposition has the following consequence:

Corollary 2.1. *Let F be a functional which satisfies the condition of Proposition 2.1. The two following properties hold:*

(1) *For all $t > 0$,*

$$\sqrt{2\pi t}\mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\geq c}\phi\left(\frac{1}{t}\int_{-c}^{c}L_t^y\mathrm{d}y\right)\right] \leq \sqrt{3}I(F).$$

(2) *When t goes to infinity,*

$$\sqrt{2\pi t}\mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\geq c}\phi\left(\frac{1}{t}\int_{-c}^{c}L_t^y\mathrm{d}y\right)\right] \to I(F).$$

Proof. The first property is obvious, since $\phi(x)/\sqrt{1-x} \leq \sqrt{3}$ for all $x \geq 0$.
In order to prove the second property, we distinguish two cases:

(1) If $I(F) < \infty$, we observe that

$$F((Y_{l,+}^y)_{y\in\mathbf{R}})\frac{\mathrm{e}^{-(\mathcal{Y}_{l,+}^c)^2/2(t-\mathcal{I}_{l,+}^c)}}{\sqrt{1-\mathcal{I}_{l,+}^c/t}}\phi\left(\frac{\mathcal{I}_{l,+}^c}{t}\right)$$

is smaller than $\sqrt{3}F((Y_{l,+}^y)_{y\in\mathbf{R}})$ and tends to $F((Y_{l,+}^y)_{y\in\mathbf{R}})$ when t goes to infinity.
By dominated convergence, $I_{c,t,+}(F) \to I_+(F)$.
Similarly, $I_{c,t,-}(F) \to I_-(F)$ and finally

$$I_{c,t}(F) \to I(F).$$

(2) If $I(F) = \infty$, we can suppose for example $I_+(F) = \infty$.
In this case,

$$I_{c,t}(F) \geq I_{c,t,+}(F) \geq \int_0^\infty \mathrm{d}l\,\mathbf{E}\left[F((Y_{l,+}^y)_{y\in\mathbf{R}})\mathrm{e}^{-(\mathcal{Y}_{l,+}^c)^2/2(t-\mathcal{I}_{l,+}^c)}\phi\left(\frac{\mathcal{I}_{l,+}^c}{t}\right)\right]$$

which tends to $I_+(F) = \infty$ when $t \to \infty$, by monotone convergence. □

Now, the next step in this article is the majorization of the difference between the quantity $\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})]$ and the expression given in Proposition 2.1.

3. Majorization of the error term

For every positive and measurable functional F, we denote by $\Delta_{c,t}(F)$ the error term that we need to majorize:

$$\Delta_{c,t}(F) = \left|\sqrt{2\pi t}\mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\geq c}\phi\left(\frac{1}{t}\int_{-c}^{c}L_t^y\mathrm{d}y\right)\right] - \sqrt{2\pi t}\mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\right]\right|.$$

It is easy to check that

$$\Delta_{c,t}(F) \leq \Delta_{c,t}^{(1)}(F) + \Delta_{c,t}^{(2)}(F)$$

where

$$\Delta^{(1)}_{c,t}(F) = \sqrt{2\pi t}\mathbf{E}\left[F((L^y_t)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\le c}\right]$$

and

$$\Delta^{(2)}_{c,t}(F) = \sqrt{2\pi t}\mathbf{E}\left[F((L^y_t)_{y\in\mathbf{R}})\mathbf{1}_{\int_{-c}^{c} L^y_t \mathrm{d}y \ge t/3}\right].$$

The following proposition gives some precise majorizations of these quantities, when F satisfies the conditions of Definition 1.1.

Proposition 3.1. *Let F be a functional from $\mathcal{C}(\mathbf{R}, \mathbf{R}_+)$ to $\mathbf{R}_+$ which satisfies the condition $C(c, n, h)$ for a positive, decreasing function h and $c, n \ge 0$.*
For all $t \ge 0$, one has the following majorizations:

(1) $\Delta^{(1)}_{c,t}(F) \le A_n \frac{N_c(h)}{1+(t/c^2)^{1/3}}$,
(2) $\Delta^{(2)}_{c,t}(F) \le A_n \frac{ch(0)}{1+(t/c^2)} \le A_n \frac{N_c(h)}{1+(t/c^2)}$,
(3) $\Delta_{c,t}(F) \le A_n \frac{N_c(h)}{1+(t/c^2)^{1/3}}$,
(4) $I(F) \le A_n N_c(h)$,

where $A_n > 0$ depends only on n.

In order to prove Proposition 3.1, we will need some inequalities for the processes $(L^y_t)_{y\in[-c,c]}$ and $(Y^y_{l,+})_{y\in[-c,c]}$.
More precisely, if we put $\Sigma^c_t = \sup_{y\in[-c,c]} L^y_t$, $\sigma^c_t = \inf_{y\in[-c,c]} L^y_t$, $\Theta^c_{l,+} = \sup_{y\in[-c,c]} Y^y_{l,+}$, $\theta^c_{l,+} = \inf_{y\in[-c,c]} Y^y_{l,+}$, $\Theta^c_{l,-} = \sup_{y\in[-c,c]} Y^y_{l,-}$, $\theta^c_{l,-} = \inf_{y\in[-c,c]} Y^y_{l,-}$, the following statement holds:

Lemma 3.1. *For all c, $t > 0$:*

(1) *If $a \ge 0$,*

$$\mathbf{P}\left(\frac{\Sigma^c_t + c}{\sigma^c_t + c} \ge a\right) \le A\mathrm{e}^{-\lambda a}.$$

(2) *If $a \ge 4$,*

$$\mathbf{P}\left(\frac{\Theta^c_{l,+} + c}{\theta^c_{l,+} + c} \ge a\right) \le A\mathrm{e}^{-\lambda\left(a+\frac{l}{c}\right)}.$$

(3) *If $a \ge 4$,*

$$\mathbf{P}\left(\frac{\Theta^c_{l,-} + c}{\theta^c_{l,-} + c} \ge a\right) \le A\mathrm{e}^{-\lambda\left(a+\frac{l}{c}\right)}.$$

Here $A > 0$, $0 < \lambda < 1$ are universal constants.

Proof of Lemma 3.1. (1) Let us suppose $a \ge 8$, $c > 0$.
In that case,

$$\mathbf{P}\left(\frac{\Sigma^c_t + c}{\sigma^c_t + c} \ge a, L^0_t \ge \frac{ac}{4}\right) \le \mathbf{P}\left(\frac{\Sigma^c_t + c}{\sigma^c_t + c} \ge 8, L^0_t \ge \frac{ac}{4}\right)$$
$$\le \sum_{k\in\mathbf{N}} \mathbf{P}\left(\frac{\Sigma^c_t}{\sigma^c_t} \ge 8, L^0_t \in [2^{k-2}ac, 2^{k-1}ac]\right)$$

$$\begin{aligned}
&\leq \sum_{k\in\mathbf{N}} \mathbf{P}(\Sigma_t^c \geq 2^k ac, L_t^0 \in [2^{k-2}ac, 2^{k-1}ac]) \\
&\quad + \sum_{k\in\mathbf{N}} \mathbf{P}(\sigma_t^c \leq 2^{k-3}ac, \Sigma_t^c \leq 2^k ac, L_t^0 \in [2^{k-2}ac, 2^{k-1}ac]) \\
&\leq \sum_{k\in\mathbf{N}} \left[\mathbf{P}(\Sigma_{\tau_{2^{k-1}ac}}^c \geq 2^k ac) + \mathbf{P}(\sigma_{\tau_{2^{k-2}ac}}^c \leq 2^{k-3}ac, \Sigma_{\tau_{2^{k-2}ac}}^c \leq 2^k ac)\right] \\
&= \sum_{k\in\mathbf{N}} \left[\alpha_c(2^{k-1}ac) + \beta_c(2^{k-2}ac)\right]
\end{aligned}$$

where for $l \geq 0$, $\alpha_c(l) = \mathbf{P}(\Sigma_{\tau_l}^c \geq 2l)$ and $\beta_c(l) = \mathbf{P}(\sigma_{\tau_l}^c \leq l/2, \Sigma_{\tau_l}^c \leq 4l)$.
Now, by the Ray–Knight theorem, $\alpha_c(l) \leq 2\mathbf{P}(\sup_{y\in[0,c]} Y_{l,0}^y \geq 2l)$, and by the Dubins–Schwarz theorem, $Y_{l,0}^y = l + \beta_{\int_0^y 4Y_{l,0}^z \mathrm{d}z}$, where β is a Brownian motion.
Hence, if $S = \inf\{y \geq 0, Y_{l,0}^y \geq 2l\}$, one has $\sup_{u\leq\int_0^S 4Y_{l,0}^z\mathrm{d}z} \beta_u = l$, and if we suppose $\sup_{y\in[0,c]} Y_{l,0}^y \geq 2l$, we have $S \leq c$, $\int_0^S 4Y_{l,0}^z \mathrm{d}z \leq \int_0^S 8l\mathrm{d}z \leq 8lc$, and finally $\sup_{u\leq 8lc} \beta_u \geq l$.
Consequently,

$$\alpha_c(l) \leq 2\mathbf{P}\left(\sup_{u\leq 8lc} \beta_u \geq l\right) = 2\mathbf{P}(|\beta_{8lc}| \geq l) \leq 4\mathbf{P}(\beta_{8lc} \geq l) \leq 4\mathrm{e}^{-l/16c}.$$

By the same kind of argument, one obtains

$$\beta_c(l) \leq 4\mathrm{e}^{-l/128c}$$

and finally

$$\mathbf{P}\left(\frac{\Sigma_t^c + c}{\sigma_t^c + c} \geq a, L_t^0 \geq \frac{ac}{4}\right) \leq 4\sum_{k\in\mathbf{N}} \left(\mathrm{e}^{-2^{k-1}a/16} + \mathrm{e}^{-2^{k-2}a/128}\right) \leq 520\mathrm{e}^{-a/512}.$$

On the other hand,

$$\begin{aligned}
\mathbf{P}\left(\frac{\Sigma_t^c + c}{\sigma_t^c + c} \geq a, L_t^0 \leq \frac{ac}{4}\right) &\leq \mathbf{P}\left(\Sigma_t^c + c \geq ac, L_t^0 \leq \frac{ac}{4}\right) \leq \mathbf{P}\left(\Sigma_{\tau_{ac/4}}^c \geq (a-1)c\right) \\
&\leq \mathbf{P}\left(\Sigma_{\tau_{ac/4}}^c \geq \frac{7ac}{8}\right) \leq \alpha_c\left(\frac{ac}{4}\right) \leq 4\mathrm{e}^{-a/64}.
\end{aligned}$$

Consequently,

$$\mathbf{P}\left(\frac{\Sigma_t^c + c}{\sigma_t^c + c} \geq a\right) \leq 524\mathrm{e}^{-a/512}$$

for all $a \geq 8$.
This inequality remains obviously true for $a \leq 8$ or $c = 0$, so the first part of Lemma 3.1 is proven.

(2) Let a be greater than 4. If $l \geq ac/4$,

$$\mathbf{P}\left(\frac{\Theta_{l,+}^c + c}{\theta_{l,+}^c + c} \geq 4\right) \leq \mathbf{P}\left(\Theta_{l,+}^c \geq 2l\right) + \mathbf{P}\left(\Theta_{l,+}^c \leq 2l, \theta_{l,+}^c \leq l/2\right) \leq 2\tilde{\alpha}_c(l) + \tilde{\beta}_c(l)$$

where

$$\tilde{\alpha}_c(l) = \mathbf{P}\left(\sup_{y\in[0,c]} Y_{l,+}^y \geq 2l\right)$$

and

$$\tilde{\beta}_c(l) = \mathbf{P}\left(\sup_{y\in[-c,c]} Y^y_{l,0} \le 2l, \inf_{y\in[-c,c]} Y^y_{l,0} \le l/2\right).$$

Now, $(Y^y_{l,+})_{y\ge 0}$ is a BESQ(2) process; hence, if $(\beta_y = (\beta^{(1)}_y, \beta^{(2)}_y))_{y\ge 0}$ is a standard two-dimensional Brownian motion,

$$\begin{aligned}\tilde{\alpha}_c(l) &= \mathbf{P}\left(\sup_{y\in[0,c]} Y^y_{l,+} \ge 2l\right) = \mathbf{P}\left(\sup_{y\le c} \|\beta_y + (\sqrt{l},0)\| \ge \sqrt{2l}\right)\\ &\le \mathbf{P}\left(\sup_{y\le c}\|\beta_y\| \ge \sqrt{l}(\sqrt{2}-1)\right) \le 8\mathrm{e}^{-l/50c}.\end{aligned}$$

Moreover,

$$\tilde{\beta}_c(l) \le \mathbf{P}\left(\sup_{y\in[-c,c]} Y^y_{l,0} \le 4l, \inf_{y\in[-c,c]} Y^y_{l,0} \le l/2\right) = \beta_c(l) \le 4\mathrm{e}^{-l/128c}.$$

Therefore, if $l \ge ac/4$,

$$\mathbf{P}\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c} \ge a\right) \le 20\mathrm{e}^{-l/128c}.$$

Now, let us suppose $l \le ac/4$. In this case,

$$\mathbf{P}\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c} \ge a\right) \le \mathbf{P}\left(\Theta^c_{ac/4,+} \ge 3ac/4\right) \le 2\tilde{\alpha}_c(ac/4) \le 16\mathrm{e}^{-a/200}.$$

Hence, for every $l \ge 0$, $a \ge 4$,

$$\mathbf{P}\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c} \ge a\right) \le 20\mathrm{e}^{-(a+(l/c))/1024}$$

which proves the second inequality of the lemma.
The proof of the third inequality is exactly similar. □

Now, we are able to prove the main result of the section, which was presented in Proposition 3.1.

Proof of Proposition 3.1. (1) For $c = 0$, $\Delta^{(1)}_{c,t}(F) = 0$, so we can suppose $c > 0$.
The functional F satisfies the condition $C(c, n, h)$; hence, for all $a \ge 1$,

$$\begin{aligned}\frac{\Delta^{(1)}_{c,t}(F)}{\sqrt{2\pi t}} &= \mathbf{E}\left[F((L^y_t)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\le c}\right]\\ &\le \mathbf{E}\left[\left(\frac{\Sigma^c_t+c}{\sigma^c_t+c}\right)^n h(\sigma^c_t)\mathbf{1}_{|B_t|\le c}\right]\\ &\le \mathbf{E}\left[\left(\frac{\Sigma^c_t+c}{\sigma^c_t+c}\right)^n h(0)\mathbf{1}_{\frac{\Sigma^c_t+c}{\sigma^c_t+c}\ge a}\right] + a^n\mathbf{E}\left[h(\sigma^c_t)\mathbf{1}_{|B_t|\le c}\mathbf{1}_{\frac{\Sigma^c_t+c}{\sigma^c_t+c}\le a}\right].\end{aligned}$$

Now, if $\frac{\Sigma^c_t+c}{\sigma^c_t+c} \le a$, $\frac{L^0_t+c}{\sigma^c_t+c} \le a$ and $\sigma^c_t \ge \left(\frac{L^0_t}{a} - c\right)_+$.

Therefore,

$$\frac{\Delta^{(1)}_{c,t}(F)}{\sqrt{2\pi t}} \leq h(0)\mathbf{E}\left[\left(\frac{\Sigma^c_t + c}{\sigma^c_t + c}\right)^n \mathbf{1}_{\frac{\Sigma^c_t+c}{\sigma^c_t+c} \geq a}\right] + a^n \mathbf{E}\left[h\left(\left(\frac{L^0_t}{a} - c\right)_+\right)\mathbf{1}_{|B_t| \leq c}\right].$$

By Lemma 3.1,

$$\begin{aligned}\mathbf{E}\left[\left(\frac{\Sigma^c_t + c}{\sigma^c_t + c}\right)^n \mathbf{1}_{\frac{\Sigma^c_t+c}{\sigma^c_t+c} \geq a}\right] &= a^n \mathbf{P}\left(\frac{\Sigma^c_t + c}{\sigma^c_t + c} \geq a\right) + \int_a^\infty n b^{n-1} \mathbf{P}\left(\frac{\Sigma^c_t + c}{\sigma^c_t + c} \geq b\right) \mathrm{d}b \\ &\leq A\left(a^n \mathrm{e}^{-\lambda a} + \int_a^\infty n b^{n-1} \mathrm{e}^{-\lambda b} \mathrm{d}b\right) \leq A\left(\frac{6}{\lambda}\right)^{n+1} (n+1)! a^n \mathrm{e}^{-\lambda a}.\end{aligned}$$

On the other hand, by using the probability density of $(L^0_t, |B_t|)$ (given for example in [3], Lemma 2.4),

$$\begin{aligned}\mathbf{E}\left[h\left(\left(\frac{L^0_t}{a} - c\right)_+\right)\mathbf{1}_{|B_t| \leq c}\right] &= \sqrt{\frac{2}{\pi t^3}} \int_0^\infty \mathrm{d}l \int_0^c \mathrm{d}x\, h\left(\left(\frac{l}{a} - c\right)_+\right)(l + x)\mathrm{e}^{-(l+x)^2/2t} \\ &= \sqrt{\frac{2}{\pi t^3}} h(0) \int_0^{ac} \mathrm{d}l \int_0^c \mathrm{d}x (l + x) \mathrm{e}^{-(l+x)^2/2t} \\ &\quad + \sqrt{\frac{2}{\pi t^3}} \int_{ac}^\infty \mathrm{d}l \int_0^c \mathrm{d}x\, h\left(\frac{l}{a} - c\right)(l + x)\mathrm{e}^{-(l+x)^2/2t} \\ &= \sqrt{\frac{2}{\pi}} \frac{c^2}{t} h(0) \int_0^a \mathrm{d}l \int_0^1 \mathrm{d}x \frac{c(l + x)}{\sqrt{t}} \mathrm{e}^{-c^2(l+x)^2/2t} \\ &\quad + \sqrt{\frac{2}{\pi}} \frac{ac^2}{t} \int_0^\infty \mathrm{d}l \int_0^1 \mathrm{d}x\, h(cl) \frac{c(al + a + x)}{\sqrt{t}} \mathrm{e}^{-c^2(al+a+x)^2/2t}.\end{aligned}$$

For all $\theta \geq 0$, $\theta \mathrm{e}^{-\theta^2/2} \leq \mathrm{e}^{-1/2} \leq 1$. Hence,

$$\mathbf{E}\left[h\left(\left(\frac{L^0_t}{a} - c\right)_+\right)\mathbf{1}_{|B_t| \leq c}\right] \leq \sqrt{\frac{2}{\pi}} \frac{ac^2}{t}\left(h(0) + \int_0^\infty h(cl)\mathrm{d}l\right) = \sqrt{\frac{2}{\pi}} \frac{ac}{t} N_c(h).$$

Moreover, for $0 < t \leq c^2$,

$$\mathbf{E}\left[h\left(\left(\frac{L^0_t}{a} - c\right)_+\right)\mathbf{1}_{|B_t| \leq c}\right] \leq h(0) \leq \frac{N_c(h)}{c} \leq \frac{a N_c(h)}{\sqrt{t}}.$$

The majorizations given above imply

$$\Delta^{(1)}_{c,t}(F) \leq A\left(\frac{6}{\lambda}\right)^{n+1} (n+1)! a^n \mathrm{e}^{-\lambda a} \sqrt{2\pi t}\, h(0) + \sqrt{2\pi} a^{n+1} \left(\frac{c}{\sqrt{t}} \wedge 1\right) N_c(h).$$

Now, let us choose a as a function of t.
For $t \leq c^2$, we take $a = 1$ and obtain

$$\begin{aligned}\Delta^{(1)}_{c,t}(F) &\leq A\left(\frac{6}{\lambda}\right)^{n+1} (n+1)! \mathrm{e}^{-\lambda} \sqrt{2\pi}\, c h(0) + \sqrt{2\pi} N_c(h) \\ &\leq \sqrt{2\pi}\left(1 + A\left(\frac{6}{\lambda}\right)^{n+1} (n+1)! \mathrm{e}^{-\lambda}\right) N_c(h).\end{aligned}$$

For $t \geq c^2$, we take $a = (t/c^2)^{1/6(n+1)}$;

$$\begin{aligned}\Delta^{(1)}_{c,t}(F) &\leq A\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\left(\frac{t}{c^2}\right)^{1/6}\mathrm{e}^{-\lambda\left(\frac{t}{c^2}\right)^{1/6(n+1)}}\sqrt{2\pi t}\,h(0)\\ &\quad+\sqrt{2\pi}\left(\frac{t}{c^2}\right)^{1/6}\frac{c}{\sqrt{t}}N_c(h)\\ &\leq \sqrt{2\pi}\left(1+A\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\right)N_c(h)\left(\frac{t}{c^2}\right)^{-1/3}\left(1+\frac{t}{c^2}\mathrm{e}^{-\lambda\left(\frac{t}{c^2}\right)^{1/6(n+1)}}\right)\\ &\leq \sqrt{2\pi}\left(1+A\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\right)\left(1+\sup_{u\geq 1}u\mathrm{e}^{-\lambda u^{1/6(n+1)}}\right)\left(\frac{t}{c^2}\right)^{-1/3}N_c(h)\end{aligned}$$

where $\sup_{u\geq 1} u\mathrm{e}^{-\lambda u^{1/6(n+1)}}$ is finite and depends only on n (we recall the λ is a universal constant).

In the two cases, the first inequality of Proposition 3.1 is satisfied.

(2) For $c = 0$, $\Delta^{(2)}_{c,t}(F) = 0$, so we can again suppose $c > 0$.

For $a \geq 1$,

$$\begin{aligned}\frac{\Delta^{(2)}_{c,t}(F)}{\sqrt{2\pi t}} &= \mathbf{E}\left[F((L^y_t)_{y\in\mathbf{R}})\mathbf{1}_{\int_{-c}^{c}L^y_t\,\mathrm{d}y\geq t/3}\right] \leq \mathbf{E}\left[\left(\frac{\Sigma^c_t+c}{\sigma^c_t+c}\right)^n h(\sigma^c_t)\mathbf{1}_{\Sigma^c_t\geq t/6c}\right]\\ &\leq h(0)\left(\mathbf{E}\left[\left(\frac{\Sigma^c_t+c}{\sigma^c_t+c}\right)^n\mathbf{1}_{\frac{\Sigma^c_t+c}{\sigma^c_t+c}\geq a}\right]+a^n\mathbf{P}\left(L^0_t\geq\frac{t}{6ac}-c\right)\right)\\ &\leq A\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\,a^n\mathrm{e}^{-\lambda a}h(0)+2a^n h(0)\mathrm{e}^{-\frac{1}{2t}\left(\frac{t}{6ac}-c\right)_+^2}.\end{aligned}$$

If $t \leq 12c^2$, we take $a = 1$;

$$\Delta^{(2)}_{c,t}(F) \leq ch(0)\sqrt{24\pi}\left(2+A\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\mathrm{e}^{-\lambda}\right).$$

If $t \geq 12c^2$, we take $a = \left(\frac{t}{12c^2}\right)^{1/3}$;

$$\begin{aligned}\Delta^{(2)}_{c,t}(F) &\leq \sqrt{2\pi t}\,h(0)\left[A\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\left(\frac{t}{12c^2}\right)^{n/3}\mathrm{e}^{-\lambda\left(\frac{t}{12c^2}\right)^{1/3}}\cdots\right.\\ &\quad\left.+\;2\left(\frac{t}{12c^2}\right)^{n/3}\mathrm{e}^{-\frac{c^2}{2t}\left(2(t/12c^2)^{2/3}-1\right)^2}\right]\\ &\leq\left(\frac{c^2}{t}\right)ch(0)\sqrt{2\pi}\,12^{3/2}\left(2+A\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\right)\\ &\quad\times\left(\frac{t}{12c^2}\right)^{\frac{n}{3}+\frac{3}{2}}\left(\mathrm{e}^{-\lambda\left(\frac{t}{12c^2}\right)^{1/3}}+\mathrm{e}^{-\frac{1}{24}\left(\frac{t}{12c^2}\right)^{1/3}}\right).\end{aligned}$$

The second inequality of Proposition 3.1 holds, since $\sup_{u\geq 1} u^{\frac{n}{3}+\frac{3}{2}}(\mathrm{e}^{-\lambda u^{1/3}}+\mathrm{e}^{-\frac{1}{24}\lambda u^{1/3}})$ is finite and depends only on n.

(3) This inequality is an immediate consequence of (1) and (2).

(4) For every $l \geq 0$,

$$\mathbf{E}[F((Y^y_{l,+})_{y\in\mathbf{R}})] \leq \mathbf{E}\left[\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c}\right)^n h(\theta^c_{l,+})\right]$$
$$\leq h(0)\mathbf{E}\left[\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c}\right)^n \mathbf{1}_{\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c}\geq 4}\right] + 4^n h\left(\left(\frac{l}{4}-c\right)_+\right).$$

Now, by Lemma 3.1,

$$\mathbf{E}\left[\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c}\right)^n \mathbf{1}_{\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c}\geq 4}\right] = 4^n\mathbf{P}\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c}\geq 4\right)$$
$$+\int_4^\infty nb^{n-1}\mathbf{P}\left(\frac{\Theta^c_{l,+}+c}{\theta^c_{l,+}+c}\geq b\right)\mathrm{d}b \leq A\mathrm{e}^{-\lambda l/c}\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!4^n\mathrm{e}^{-4\lambda}.$$

Hence,

$$\mathbf{E}[F((Y^y_{l,+})_{y\in\mathbf{R}})] \leq Ah(0)\mathrm{e}^{-\lambda l/c}\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\,4^n\mathrm{e}^{-4\lambda}+4^n h\left(\left(\frac{l}{4}-c\right)_+\right)$$

and, by integrating with respect to l,

$$I_+(F) \leq \frac{A}{\lambda}\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!4^n\mathrm{e}^{-4\lambda}ch(0)+4^{n+1}ch(0)+4^{n+1}\int_0^\infty h(l)\mathrm{d}l$$
$$\leq 4^{n+1}\left(1+\frac{A}{\lambda}\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\right)N_c(h).$$

By symmetry, the same inequality holds for $I_-(F)$, and

$$I(F) \leq 2^{2n+3}\left(1+\frac{A}{\lambda}\left(\frac{6}{\lambda}\right)^{n+1}(n+1)!\right)N_c(h)$$

which completes the proof of Proposition 3.1. $\square$

4. An estimation of the quantity $\mathbf{E}[F((L^y_t)_{y\in\mathbf{R}})]$

In this section, we majorize $\mathbf{E}[F((L^y_t)_{y\in\mathbf{R}})]$ using an equivalent of this quantity when t goes to infinity. The following statement holds:

Proposition 4.1. *Let F be a functional from $\mathcal{C}(\mathbf{R},\mathbf{R}_+)$ to $\mathbf{R}_+$, which satisfies the condition $C(c,n,h)$, for a positive, decreasing function h, and c, $n \geq 0$.*
The following properties hold:

(1) *For all $t > 0$,*

$$\sqrt{2\pi t}\,\mathbf{E}[F((L^y_t)_{y\in\mathbf{R}})] \leq K_n N_c(h)$$

where $K_n > 0$ depends only on n.

(2) *If* $N_c(h) < \infty$,

$$\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] \underset{t\to\infty}{\to} I(F).$$

Proof. We suppose $N_c(h) < \infty$.
Proposition 3.1 implies the following:

$$\Delta_{c,t}(F) \le A_n N_c(h)$$
$$\Delta_{c,t}(F) \underset{t\to\infty}{\to} 0.$$

Moreover, by Corollary 2.1,

$$\sqrt{2\pi t}\mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\ge c}\phi\left(\frac{1}{t}\int_{-c}^{c} L_t^y \mathrm{d}y\right)\right] \underset{t\to\infty}{\to} I(F)$$
$$\sqrt{2\pi t}\mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\ge c}\phi\left(\frac{1}{t}\int_{-c}^{c} L_t^y \mathrm{d}y\right)\right] \le \sqrt{3}I(F) \le \sqrt{3}A_n N_c(h)$$

for all $t > 0$.
Now, by definition, one has

$$\left|\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] - \sqrt{2\pi t}\mathbf{E}\left[F((L_t^y)_{y\in\mathbf{R}})\mathbf{1}_{|B_t|\ge c}\phi\left(\frac{1}{t}\int_{-c}^{c} L_t^y \mathrm{d}y\right)\right]\right| = \Delta_{c,t}(F).$$

Therefore,

$$\sqrt{2\pi t}\,\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] \underset{t\to\infty}{\to} I(F)$$
$$\sqrt{2\pi t}\,\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] \le (1+\sqrt{3})A_n N_c(h)$$

which proves Proposition 4.1. □

The following result is an extension of Proposition 4.1 to a larger class of functionals F:

Proposition 4.2. *Let* $F : \mathcal{C}(\mathbf{R}, \mathbf{R}_+) \to \mathbf{R}_+$ *be a positive and measurable functional. The following properties hold for all* $n \ge 0$*:*

(1) *For all* $t > 0$,

$$\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] \le K_n N^{(n)}(F).$$

(2) *If* $N^{(n)}(F) < \infty$,

$$\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] \underset{t\to\infty}{\to} I(F).$$

Proof. We suppose $N^{(n)}(F) < \infty$.
(1) Let us take M such that $N^{(n)}(F) < M$.
By definition, F satisfies the condition $D(n, M)$, so there exists $(c_k)_{k\ge1}$,$(h_k)_{k\ge1}$, $(F_k)_{k\ge0}$ as in Definition 1.2.
One has $F = \sum_{k\ge1}(F_k - F_{k-1})$; hence,

$$\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] \le \sum_{k\ge1}\sqrt{2\pi t}\,\mathbf{E}[|F_k - F_{k-1}|((L_t^y)_{y\in\mathbf{R}})]$$
$$\le K_n \sum_{k\ge1} N_{c_k}(h_k) \le K_n M.$$

By taking $M \to N^{(n)}(F)$, one obtains the first part of Proposition 4.2.

(2) In order to prove the convergence, let us consider the equality

$$\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] = \sum_{k\geq 1}\sqrt{2\pi t}\,\mathbf{E}[(F_k - F_{k-1})_+((L_t^y)_{y\in\mathbf{R}})] - \sum_{k\geq 1}\sqrt{2\pi t}\mathbf{E}[(F_k - F_{k-1})_-((L_t^y)_{y\in\mathbf{R}})]$$

where the two sums are convergent.
By Proposition 4.1, the two terms indexed by k tend to $I((F_k - F_{k-1})_+)$ and $I((F_k - F_{k-1})_-)$ when t goes to infinity, and they are bounded by $K_n N_{c_k}(h_k)$.
Hence, by dominated convergence,

$$\sqrt{2\pi t}\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})] \underset{t\to\infty}{\to} \sum_{k\geq 1} I((F_k - F_{k-1})_+) - \sum_{k\geq 1} I((F_k - F_{k-1})_-).$$

Now, by the definition of I,

$$\sum_{k\geq 1} I((F_k - F_{k-1})_+) = I\left(\sum_{k\geq 1}(F_k - F_{k-1})_+\right)$$

$$\sum_{k\geq 1} I((F_k - F_{k-1})_-) = I\left(\sum_{k\geq 1}(F_k - F_{k-1})_-\right).$$

Therefore, if $G = \sum_{k\geq 1}(F_k - F_{k-1})_+$, and $H = \sum_{k\geq 1}(F_k - F_{k-1})_-$, one has

$$\sum_{k\geq 1} I((F_k - F_{k-1})_+) - \sum_{k\geq 1} I((F_k - F_{k-1})_-) = I(G) - I(H)$$

where

$$I(G) - I(H) = I(G - H) = I(F)$$

since $I(G) + I(H) \leq 2K_n \sum_{k\geq 1} N_{c_k}(h_k) < \infty$. □

Proposition 4.2 is proven, and we now have all we need for the proof of the main theorem, which is given in Section 5.

5. Proof of the main theorem

Our proof of the theorem starts with a general lemma (which does not involve the Wiener measure):

Lemma 5.1. *If $F : \mathcal{C}(\mathbf{R}, \mathbf{R}_+) \to \mathbf{R}_+$ is a measurable functional, $l_0 \in \mathcal{C}(\mathbf{R}, \mathbf{R}_+)$, $x \in \mathbf{R}$, and $n \geq 0$,*

$$N^{(n)}\left(F^{(l_0^y)_{y\in\mathbf{R}},x}\right) \leq 2^n \left(1 + \left(\sup_{z\in\mathbf{R}} l_0^z\right)^n\right)(1 + |x|)^{n+1} N^{(n)}(F).$$

Proof of Lemma 5.1. Let M be greater than $N^{(n)}(F)$.
There exists a sequence $(c_k)_{k\geq 1}$ in $[1, \infty[$, a sequence $(h_k)_{k\geq 1}$ of decreasing functions from $\mathbf{R}_+$ to $\mathbf{R}_+$, and a sequence $(F_k)_{k\geq 0}$ of measurable functions $\mathcal{C}(\mathbf{R}, \mathbf{R}_+) \to \mathbf{R}_+$, such that:
(1) $F_0 = 0$, and $F_k \underset{k\to\infty}{\to} F$.

(2) $(|F_k - F_{k-1}|)((l^y)_{y\in\mathbf{R}})$ depends only on $(l^y)_{|y|\le c_k}$, and

$$(|F_k - F_{k-1}|)((l^y)_{y\in\mathbf{R}}) \le \left(\frac{\sup_{|y|\le c} l^y + c_k}{\inf_{|y|\le c} l^y + c_k}\right)^n h_k\left(\inf_{|y|\le c} l^y\right)$$

(3) $\sum_{k\ge 1} N_{c_k}(h_k) \le M$.

These conditions imply the following ones for the sequence $(G_k = F_k^{(l_0^y)_{y\in\mathbf{R}},x})_{k\ge 1}$:

(1) $G_0 = 0$, and $G_k \underset{k\to\infty}{\to} F^{(l_0^y)_{y\in\mathbf{R}},x}$.

(2) $(|G_k - G_{k-1}|)((l^y)_{y\in\mathbf{R}})$ depends only on $(l^z)_{|z|\le c_k+|x|}$ and

$$\begin{aligned}
(|G_k - G_{k-1}|)((l^y)_{y\in\mathbf{R}}) &\le \left(\frac{\sup_{z\in[-c_k-x,c_k-x]} (l_0^{z+x} + l^z) + c_k}{\inf_{z\in[-c_k-x,c_k-x]} (l_0^{z+x} + l^z) + c_k}\right)^n \\
&\quad \times h_k\left(\inf_{z\in[-c_k-x,c_k-x]} (l_0^{z+x} + l^z)\right) \\
&\le \left(\frac{\sup_{z\in\mathbf{R}} l_0^z + \sup_{|z|\le c_k+|x|} l^z + c_k + |x|}{\inf_{|z|\le c_k+|x|} l^z + c_k}\right)^n h_k\left(\inf_{|z|\le c_k+|x|} l^z\right) \\
&\le 2^n\left[\left(\frac{\sup_{z\in\mathbf{R}} l_0^z}{c_k}\right)^n + \left(\frac{\sup_{|z|\le c_k+|x|} l^z + c_k + |x|}{\inf_{|z|\le c_k+|x|} l^z + c_k + |x|}\right)^n \left(1 + \frac{|x|}{\inf_{|z|\le c_k+|x|} l^z + c_k}\right)^n\right] \\
&\quad \times h_k\left(\inf_{|z|\le c_k+|x|} l^z\right) \\
&\le 2^n\left(\left(\sup_{z\in\mathbf{R}} l_0^z\right)^n + (1+|x|)^n\right)\left(\frac{\sup_{|z|\le c_k+|x|} l^z + c_k + |x|}{\inf_{|z|\le c_k+|x|} l^z + c_k + |x|}\right)^n h_k\left(\inf_{|z|\le c_k+|x|} l^z\right).
\end{aligned}$$

Therefore, $|G_k - G_{k-1}|$ satisfies the condition $C\left(c_k + |x|, n, 2^n((\sup_{z\in\mathbf{R}} l_0^z)^n + (1+|x|)^n)h_k\right)$.

(3) Now,

$$\begin{aligned}
N_{c_k+|x|}\left(2^n\left(\left(\sup_{z\in\mathbf{R}} l_0^z\right)^n + (1+|x|)^n\right)h_k\right) &\le 2^n\left(\left(\sup_{z\in\mathbf{R}} l_0^z\right)^n + (1+|x|)^n\right) \\
&\quad \times \frac{c_k + |x|}{c_k} N_{c_k}(h_k) \\
&\le 2^n\left(1 + \left(\sup_{z\in\mathbf{R}} l_0^z\right)^n\right)(1+|x|)^{n+1} N_{c_k}(h_k)
\end{aligned}$$

and $\sum_{k\ge 1} N_{c_k}(h_k) \le M$.

Therefore,

$$N^{(n)}\left(F^{(l_0^y)_{y\in\mathbf{R}},x}\right) \le 2^n\left(1 + \left(\sup_{z\in\mathbf{R}} l_0^z\right)^n\right)(1+|x|)^{n+1} M.$$

By taking $M \to N^{(n)}(F)$, we obtain the majorization stated in Lemma 5.1. □

Proof of the Theorem. $\sqrt{2\pi t}\mathbf{W}[F((l_t^y(X))_{y\in\mathbf{R}})]$ tends to $I(F) > 0$ when t goes to infinity, so it is strictly positive if t is large enough, and $\mathbf{W}_t^F$ is well defined.
If t is large enough, by the Markov property,

$$\mathbf{W}_t^F(\Lambda_s) = \mathbf{W}\left[\mathbf{1}_{\Lambda_s}\frac{\mathbf{W}\left[F\left(\left(l_t^y(X)\right)_{y\in\mathbf{R}}\right)|\sigma\{X_u, u\le s\}\right]}{\mathbf{W}\left[F\left(\left(l_t^y(X)\right)_{y\in\mathbf{R}}\right)\right]}\right]$$

$$= \mathbf{W}\left[\mathbf{1}_{\Lambda_s}\frac{\Psi_{t-s}\left((l_s^y(X))_{y\in\mathbf{R}}, X_s\right)}{\mathbf{W}\left[F\left(\left(l_t^y(X)\right)_{y\in\mathbf{R}}\right)\right]}\right]$$

where, for all continuous functions l from $\mathbf{R}$ to $\mathbf{R}_+$, and for all $x\in\mathbf{R}$, $u>0$,

$$\Psi_u\left((l^y)_{y\in\mathbf{R}}, x\right) = \mathbf{W}\left[F^{(l^y)_{y\in\mathbf{R}},x}\left((l_u^y(X))_{y\in\mathbf{R}}\right)\right].$$

By Proposition 4.2,

$$\frac{\Psi_{t-s}\left((l_s^y(X))_{y\in\mathbf{R}}, X_s\right)}{\mathbf{W}\left[F\left(\left(l_t^y(X)\right)_{y\in\mathbf{R}}\right)\right]} \underset{t\to\infty}{\to} \frac{I\left(F^{(l_s^y(X))_{y\in\mathbf{R}},X_s}\right)}{I(F)}.$$

Moreover, for $t\ge 2s$,

$$\sqrt{2\pi t}\,\Psi_{t-s}\left((l_s^y(X))_{y\in\mathbf{R}}, X_s\right) \le \sqrt{\frac{t}{t-s}}N^{(n)}\left(F^{(l_s^y(X))_{y\in\mathbf{R}},X_s}\right)$$
$$\le 2^{n+1/2}\left(1+\left(\sup_{z\in\mathbf{R}} l_s^z(X)\right)^n\right)(1+|X_s|)^{n+1}\,N^{(n)}(F)$$

and for t large enough,

$$\sqrt{2\pi t}\mathbf{W}\left[F\left((l_t^y(X))_{y\in\mathbf{R}}\right)\right] \ge I(F)/2.$$

Hence, for t large enough,

$$\frac{\Psi_{t-s}\left((l_s^y(X))_{y\in\mathbf{R}}, X_s\right)}{\mathbf{W}\left[F\left(\left(l_t^y(X)\right)_{y\in\mathbf{R}}\right)\right]} \le \frac{2^{n+3/2}\left(1+\left(\sup_{z\in\mathbf{R}} l_s^z(X)\right)^n\right)(1+|X_s|)^{n+1}\,N^{(n)}(F)}{I(F)}.$$

Now,

$$\mathbf{W}\left[\left(1+\left(\sup_{z\in\mathbf{R}} l_s^z(X)\right)^n\right)(1+|X_s|)^{n+1}\right]$$
$$\le \left(\mathbf{W}\left[\left(1+\left(\sup_{z\in\mathbf{R}} l_s^z(X)\right)^n\right)^2\right]\right)^{1/2}\left(\mathbf{W}\left[(1+|X_s|)^{2n+2}\right]\right)^{1/2} < \infty$$

since $\sup_{z\in\mathbf{R}} l_s^z(X)$ and $|X_s|$ have moments of any order.
By dominated convergence, we obtain the theorem. □

6. Examples

In this section, we check that the conditions of the theorem are satisfied in three examples studied by B. Roynette, P. Vallois and M. Yor, and one more particular case.

(I) *First example* (penalization with local time at level zero)
We take $F((l^y)_{y\in\mathbf{R}}) = \phi(l^0)$ where ϕ is bounded and dominated by a positive, decreasing and integrable function ψ.
F satisfies the condition $C(1, 0, \psi)$. Hence,

$$N^{(0)}(F) \leq N_1(\psi) = \psi(0) + \int_0^\infty \psi(y)\mathrm{d}y < \infty.$$

On the other hand,

$$I(F) = 2\int_0^\infty \phi(l)\mathrm{d}l$$
$$F^{(l_s^y(X))_{y\in\mathbf{R}},X_s}((l^y)_{y\in\mathbf{R}}) = l_s^0(X) + l^{-X_s}$$

and

$$I\left(F^{(l_s^y(X))_{y\in\mathbf{R}},X_s}\right) = \int_0^\infty \mathrm{d}l\left(\mathbf{E}\left[\phi(l_s^0(X) + Y_{l,+}^{-X_s})\right] + \mathbf{E}\left[\phi(l_s^0(X) + Y_{l,-}^{-X_s})\right]\right).$$

Now, by using the fact that the Lebesgue measure is invariant for a BESQ(2) process, we obtain

$$\int_0^\infty \mathrm{d}l\,\mathbf{E}\left[\phi(l_s^0(X) + Y_{l,-\mathrm{sgn}(X_s)}^{-X_s})\right] = \int_0^\infty \mathrm{d}l\,\phi(l_s^0(X) + l) = \int_{l_s^0(X)}^\infty \phi(l)\mathrm{d}l.$$

Moreover, the image of the Lebesgue measure under a BESQ(0) process taken at time $x \geq 0$ is the sum of the Lebesgue measure and $2x$ times the Dirac measure at 0; more precisely, for all measurable functions $f : \mathbf{R}_+ \to \mathbf{R}_+$, one has

$$\int_0^\infty \mathrm{d}l\,\mathbf{E}[f(Y_{l,-}^x)] = 2xf(0) + \int_0^\infty \mathrm{d}y f(y).$$

Therefore,

$$\int_0^\infty \mathrm{d}l\,\mathbf{E}\left[\phi(l_s^0(X) + Y_{l,\mathrm{sgn}(X_s)}^{-X_s})\right] = 2|X_s|\phi(l_s^0(X)) + \int_{l_s^0(X)}^\infty \phi(l)\mathrm{d}l$$

and finally

$$I\left(F^{(l_s^y(X))_{y\in\mathbf{R}},X_s}\right) = 2\left(|X_s|\phi(l_s^0(X)) + \int_{l_s^0(X)}^\infty \phi(l)\mathrm{d}l\right).$$

Consequently, if ϕ is not a.e. equal to zero, we can apply the theorem, and for $s \geq 0$, $\Lambda_s \in \mathcal{F}_s = \sigma\{X_u, u \leq s\}$,

$$\mathbf{W}_\infty^F(\Lambda_s) = \mathbf{W}\left(\mathbf{1}_{\Lambda_s}.\frac{|X_s|\phi(l_s^0(X)) + \Phi(l_s^0(X))}{\Phi(0)}\right)$$

where $\Phi(x) = \int_x^\infty \phi(l)\mathrm{d}l$.
This result is consistent with the limit measure obtained by Roynette, Vallois and Yor in [6].

(II) *Second example* (penalization with the supremum)
We take $F((l^y)_{y\in\mathbf{R}}) = \phi(\inf\{y \geq 0, l^y = 0\})$, where ϕ is dominated by a decreasing function $\psi : \mathbf{R}_+ \cup \{\infty\} \to \mathbf{R}_+$ such that $\int_0^\infty \psi(y)\mathrm{d}y < \infty$.
Let us recall that for this choice of F, $F\left((l_t^y(X))_{y\in\mathbf{R}}\right) = \phi(S_t)$, where S_t denotes the supremum of $(X_s)_{s\leq t}$.
Now, we take for $k \in \mathbf{N}$,

$$F_k((l^y)_{y\in\mathbf{R}}) = \phi_{2^k-1}(\inf\{y \geq 0, l^y = 0\})$$

where $\phi_{2^k-1} = \phi.\mathbf{1}_{]-\infty,2^k-1[}$.
(1) One has $F_0 = 0$ and $F_k \underset{k\to\infty}{\to} F$ pointwise.
(2) $(|F_k - F_{k-1}|)((l^y)_{y\in\mathbf{R}})$ depends only on $(l^y)_{|y|\leq 2^k-1}$ and

$$\begin{aligned}(|F_k - F_{k-1}|)((l^y)_{y\in\mathbf{R}}) &\leq \phi(\inf\{y \geq 0, l^y = 0\})\mathbf{1}_{\inf\{y\geq 0, l^y=0\}\in[2^{k-1}-1,2^k-1[} \\ &\leq \psi(2^{k-1}-1)\mathbf{1}_{\inf_{|y|\leq 2^k-1} l^y=0}.\end{aligned}$$

Hence, $|F_k - F_{k-1}|$ satisfies the condition $C(2^k-1, 0, \psi(2^{k-1}-1)\mathbf{1}_{\{0\}})$.
(3) Therefore,

$$N^{(0)}(F) \leq \sum_{k\geq 1}(2^k-1)\psi(2^{k-1}-1) \leq \psi(0) + 4\int_0^\infty \psi < \infty.$$

Moreover,

$$I(F) = \int_0^\infty \mathrm{d}l\, \mathbf{E}\left[\phi\left(\inf\{y \geq 0, Y_{l,+}^y = 0\}\right)\right] + \int_0^\infty \mathrm{d}l\, \mathbf{E}\left[\phi\left(\inf\{y \geq 0, Y_{l,-}^y = 0\}\right)\right].$$

The first integral is equal to zero and $\inf\{y \geq 0, Y_{l,-}^y = 0\}$ is the inverse of an exponential variable of parameter $l/2$.
Therefore,

$$\begin{aligned}I(F) &= \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}y \frac{l}{2y^2}\mathrm{e}^{-l/2y}\phi(y)\mathrm{d}y \\ &= \int_0^\infty \mathrm{d}y\phi(y)\int_0^\infty \mathrm{d}l \frac{l}{2y^2}\mathrm{e}^{-l/2y} = 2\int_0^\infty \phi(y)\mathrm{d}y.\end{aligned}$$

By similar computations, we obtain

$$\begin{aligned}I\left(F^{(l_s^y(X))_{y\in\mathbf{R}},X_s}\right) &= \int_0^\infty \mathrm{d}l\, \mathbf{E}[\phi(S_s \vee (X_s + \inf\{y \geq 0, Y_{l,-}^y = 0\}))] \\ &= 2\left((S_s - X_s)\phi(S_s) + \int_{S_s}^\infty \phi(y)\mathrm{d}y\right).\end{aligned}$$

Consequently, if ϕ is not a.e. equal to zero, the sequence $(\mathbf{W}_t^F)_{t\geq 0}$ satisfies for every $s \geq 0$, $\Lambda_s \in \mathcal{F}_s = \sigma\{X_u, u \leq s\}$,

$$\mathbf{W}_t^F(\Lambda_s) \underset{t\to\infty}{\to} \mathbf{W}_\infty^F(\Lambda_s)$$

where

$$\mathbf{W}_t^F = \frac{\phi(S_t)}{\mathbf{W}[\phi(S_t)]}.\mathbf{W}$$

and

$$\mathbf{W}_{\infty}^{F}(\Lambda_s) = \mathbf{W}\left[\mathbf{1}_{\Lambda_s} \frac{(S_s - X_s)\phi(S_s) + \Phi(S_s)}{\Phi(0)}\right].$$

This corresponds to B. Roynette, P. Vallois and M. Yor's penalization results for the supremum (see [6]).

(III) *Third example* (exponential penalization with an integral of the local times)
Let us take $F((l^y)_{y\in\mathbf{R}}) = \exp(-\int_{-\infty}^{\infty} V(y)l^y \mathrm{d}y)$, where V is a positive measurable function, not a.e. equal to zero, and integrable with respect to $(1+y^2)\mathrm{d}y$ (this condition is a little more restrictive than the condition obtained by Roynette, Vallois and Yor in [4]).
In that case, there exists $c \geq 1$ such that

$$\int_{-c}^{c} V(y)\mathrm{d}y > 0$$

We consider the following approximations of F: $F_0 = 0$, and for $k \geq 1$,
$F_k((l^y)_{y\in\mathbf{R}}) = \exp(-\int_{-2^k c}^{2^k c} V(y)l^y \mathrm{d}y)$.
The following holds:

(1) $F_0 = 0$ and $F_k \underset{k\to\infty}{\to} F$.

(2) $|F_k - F_{k-1}|((l^y)_{y\in\mathbf{R}})$ depends only on $(l^y)_{y\in[-2^k c, 2^k c]}$ and if $k \geq 2$,

$$\begin{aligned}
|F_k - F_{k-1}|((l^y)_{y\in\mathbf{R}}) &\leq \left(\int_{[-2^k c, 2^k c]\setminus[-2^{k-1}c, 2^{k-1}c]} V(y)\mathrm{d}y\right)\cdots \\
&\quad \cdots \left(\sup_{y\in[-2^k c, 2^k c]} l^y\right) \exp\left(-\int_{-2^{k-1}c}^{2^{k-1}c} V(y)l^y\mathrm{d}y\right) \\
&\leq \left(\int_{[-2^k c, 2^k c]\setminus[-2^{k-1}c, 2^{k-1}c]} V(y)\mathrm{d}y\right)\left(\frac{\sup_{y\in[-2^k c, 2^k c]} l^y + 2^k c}{\inf_{y\in[-2^k c, 2^k c]} l^y + 2^k c}\right) \\
&\quad \times \left(\inf_{y\in[-2^k c, 2^k c]} l^y + 2^k c\right)\cdots \exp\left[-\left(\int_{-2^{k-1}c}^{2^{k-1}c} V(y)\mathrm{d}y\right)\left(\inf_{y\in[-2^k c, 2^k c]} l^y\right)\right].
\end{aligned}$$

Moreover,

$$|F_1 - F_0|((l^y)_{y\in\mathbf{R}}) \leq \exp\left[-\left(\int_{2c}^{2c} V(y)\mathrm{d}y\right)\left(\inf_{y\in[-2c,2c]} l^y\right)\right].$$

Therefore, if we put $\rho = \int_{-c}^{c} V(y)\mathrm{d}y > 0$, for every $k \geq 1$, $|F_k - F_{k-1}|$ satisfies the condition $C(2^k c, 1, h_k)$ where the decreasing function h_k is defined by

$$h_k(l) = \left(\mathbf{1}_{k=1} + \int_{[-2^k c, 2^k c]\setminus[-2^{k-1}c, 2^{k-1}c]} V(y)\mathrm{d}y\right)(l + 2^k c + \rho^{-1})\mathrm{e}^{-\rho l}.$$

(3) One has

$$N_{2^k c}(h_k) \leq \left(\mathbf{1}_{k=1} + \int_{[-2^k c, 2^k c]\setminus[-2^{k-1}c, 2^{k-1}c]} V(y)\mathrm{d}y\right)(2^{2k}c^2 + 2^{k+1}c\rho^{-1} + 2\rho^{-2}).$$

Hence,

$$\sum_{k\geq 1} N^{(1)}_{2^k c}(h_k) \leq (1+\rho^{-1}+\rho^{-2})\left(4c^2 + \sum_{k\geq 1} 2^{2k}c^2 \int_{[-2^k c, 2^k c]\setminus[-2^{k-1}c, 2^{k-1}c]} V(y)\mathrm{d}y\right)$$
$$\leq 4(1+\rho^{-1}+\rho^{-2})\left(c^2 + \int_{\mathbf{R}} (1+y^2)V(y)\right) < \infty.$$

Moreover, by properties of BESQ processes, for all $l \geq 0$, $y \in \mathbf{R}$,

$$\mathbf{E}\left[Y^y_{l,+}\right] \leq l + 2|y|$$

and

$$\mathbf{E}\left[\int_{\mathbf{R}} Y^y_{l,+} V(y)\mathrm{d}y\right] \leq \int_{\mathbf{R}} (l+2|y|)V(y)\mathrm{d}y < \infty.$$

Therefore,

$$\mathbf{E}\left[\exp\left(-\int_{\mathbf{R}} Y^y_{l,+} V(y)\mathrm{d}y\right)\right] > 0$$

and $I(F) > 0$.
Consequently, the theorem applies in this case and B. Roynette, P. Vallois and M. Yor's penalization result holds (see [4]).

(IV) *Fourth example* (penalization with local times at two levels)
This example is a generalization of the first one.
Let us take, for $y_1 < y_2$, $F((l^y)_{y\in\mathbf{R}}) = \phi(l^{y_1}, l^{y_2})$ where $\phi(l_1, l_2) \leq h(l_1 \wedge l_2)$ for a positive, integrable and decreasing function h.
In that case, F satisfies the condition $C(|y_1| \vee |y_2|, 0, h)$, so the theorem applies if we have $I(F) > 0$.
For $y > 0$, $z, z' \geq 0$, let $q^{(0)}_y(z, z')$ be the density at z' of a BESQ(0) process starting from level z and taken at time y, $Q^{(0)}_y(z, 0)$ the probability that this process is equal to zero, and $q^{(2)}_y(z, z')$ the density at z' of a BESQ(2) process starting from z and taken at time y. If $0 < y_1 < y_2$, one has

$$\begin{aligned} I(F) = {} & \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, q^{(2)}_{y_1}(l, l_1) q^{(2)}_{y_2-y_1}(l_1, l_2)\phi(l_1, l_2) \\ & + \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, q^{(0)}_{y_1}(l, l_1) q^{(0)}_{y_2-y_1}(l_1, l_2)\phi(l_1, l_2) \\ & + \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}l_1\, q^{(0)}_{y_1}(l, l_1) Q^{(0)}_{y_2-y_1}(l_1, 0)\phi(l_1, 0) + \int_0^\infty \mathrm{d}l\, Q^{(0)}_{y_1}(l, 0)\phi(0, 0). \end{aligned}$$

Now, by properties of time-reversed BESQ processes, $q^{(0)}_y(z, z') = q^{(4)}_y(z', z)$ (where $q^{(4)}$ is the density of the BESQ(4) process) and $q^{(2)}_y(z, z') = q^{(2)}_y(z', z)$. Hence,

$$\int_0^\infty q^{(0)}_y(z, z')\mathrm{d}z = \int_0^\infty q^{(4)}_y(z', z)\mathrm{d}z = 1$$

and

$$\int_0^\infty q^{(2)}_y(z, z')\mathrm{d}z = \int_0^\infty q^{(2)}_y(z', z)\mathrm{d}z = 1$$

since $q^{(2)}$ and $q^{(4)}$ are probability densities with respect to the second variable.

Moreover,

$$\int_0^\infty Q_y^{(0)}(z,0)\mathrm{d}z = \int_0^\infty \mathrm{e}^{-z/2y}\mathrm{d}z = 2y.$$

Therefore,

$$\begin{aligned} I(F) = {} & \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2 (q_{y_2-y_1}^{(2)}(l_1,l_2) + q_{y_2-y_1}^{(0)}(l_1,l_2))\phi(l_1,l_2) \\ & + \int_0^\infty \mathrm{d}l_1 Q_{y_2-y_1}^{(0)}(l_1,0)\phi(l_1,0) + 2y_1\phi(0,0) \end{aligned}$$

for $0 \leq y_1 < y_2$.
Similar computations give for $y_1 < y_2 \leq 0$

$$\begin{aligned} I(F) = {} & \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2 (q_{y_2-y_1}^{(2)}(l_2,l_1) + q_{y_2-y_1}^{(0)}(l_2,l_1))\phi(l_1,l_2) \\ & + \int_0^\infty \mathrm{d}l_2 Q_{y_2-y_1}^{(0)}(l_2,0)\phi(0,l_2) + 2|y_2|\phi(0,0). \end{aligned}$$

For $y_1 < 0 < y_2$, we have

$$\begin{aligned} I(F) = {} & \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, q_{y_2}^{(2)}(l,l_2) q_{|y_1|}^{(0)}(l,l_1)\phi(l_1,l_2) \\ & + \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}l_2\, q_{y_2}^{(2)}(l,l_2) Q_{|y_1|}^{(0)}(l,0)\phi(0,l_2) \\ & + \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, q_{y_2}^{(0)}(l,l_2) q_{|y_1|}^{(2)}(l,l_1)\phi(l_1,l_2) \\ & + \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}l_1 Q_{y_2}^{(0)}(l,0) q_{|y_1|}^{(2)}(l,l_1)\phi(l_1,0). \end{aligned}$$

Now, for $y', y'' > 0$, and $z, z', z'' \leq 0$, the two following equalities hold:

$$\int_0^\infty q_{y'}^{(2)}(z,z') q_{y''}^{(0)}(z,z'')\mathrm{d}z = \frac{y' q_{y'+y''}^{(2)}(z',z'') + y'' q_{y'+y''}^{(0)}(z',z'')}{y'+y''}$$

$$\int_0^\infty q_{y'}^{(2)}(z,z') Q_{y''}^{(0)}(z,0)\mathrm{d}z = \frac{y''}{y'+y''} Q_{y'+y''}^{(0)}(z',0)$$

(the first one can be proven by using [10], Lemma 3, and the relation $q_y^{(0)}(z,z') = q_y^{(4)}(z',z)$; the second is a consequence of the equality $Q_{y''}^{(0)}(z,0) = \mathrm{e}^{-z/2y''} = 2y'' q_{y''}^{(2)}(0,z)$).
Therefore,

$$\begin{aligned} I(F) = {} & \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2 \left[q_{y_2-y_1}^{(2)}(l_1,l_2) + \frac{|y_1| q_{y_2-y_1}^{(0)}(l_2,l_1) + y_2 q_{y_2-y_1}^{(0)}(l_1,l_2)}{y_2-y_1} \right] \phi(l_1,l_2) \\ & + \int_0^\infty \mathrm{d}l_1 \frac{y_2}{y_2-y_1} Q_{y_2-y_1}^{(0)}(l_1,0)\phi(l_1,0) \\ & + \int_0^\infty \mathrm{d}l_2 \frac{|y_1|}{y_2-y_1} Q_{y_2-y_1}^{(0)}(l_2,0)\phi(0,l_2). \end{aligned}$$

This computation of $I(F)$ implies the following:

(1) For $0 < y_1 < y_2$, the theorem applies iff

$$\int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, \phi(l_1, l_2) + \int_0^\infty \mathrm{d}l_1\, \phi(l_1, 0) + \phi(0, 0) > 0.$$

(2) For $0 = y_1 < y_2$, it applies iff

$$\int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, \phi(l_1, l_2) + \int_0^\infty \mathrm{d}l_1\, \phi(l_1, 0) > 0.$$

(3) For $y_1 < 0 < y_2$, it applies iff

$$\int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, \phi(l_1, l_2) + \int_0^\infty \mathrm{d}l_1\, \phi(l_1, 0) + \int_0^\infty \mathrm{d}l_2\, \phi(0, l_2) > 0.$$

(4) For $y_1 < y_2 = 0$, it applies iff

$$\int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, \phi(l_1, l_2) + \int_0^\infty \mathrm{d}l_2\, \phi(0, l_2) > 0.$$

(5) For $y_1 < y_2 < 0$, it applies iff

$$\int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2\, \phi(l_1, l_2) + \int_0^\infty \mathrm{d}l_2\, \phi(0, l_2) + \phi(0, 0) > 0.$$

If the theorem holds, it is possible to compute $I(F^{(l_s^y(X))_{y\in\mathbf{R}}, X_s})$ in order to obtain the density, restricted to $\mathcal{F}_s$, of $\mathbf{W}_\infty^F$ with respect to $\mathbf{W}$.

For $X_s \leq y_1 < y_2$, we have

$$\begin{aligned} I(F^{(l_s^y(X))_{y\in\mathbf{R}}, X_s}) &= \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2 (q^{(2)}_{y_2-y_1}(l_1, l_2) \\ &\quad + q^{(0)}_{y_2-y_1}(l_1, l_2))\phi(l_s^{y_1}(X) + l_1, l_s^{y_2}(X) + l_2) \\ &\quad + \int_0^\infty \mathrm{d}l_1 Q^{(0)}_{y_2-y_1}(l_1, 0)\phi(l_s^{y_1}(X) + l_1, l_s^{y_2}(X)) + 2(y_1 - X_s)\phi(l_s^{y_1}(X), l_s^{y_2}(X)). \end{aligned}$$

For $y_1 < X_s < y_2$,

$$\begin{aligned} I(F^{(l_s^y(X))_{y\in\mathbf{R}}, X_s}) &= \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2 \Bigg[q^{(2)}_{y_2-y_1}(l_1, l_2) \cdots \\ &\quad + \frac{(X_s - y_1) q^{(0)}_{y_2-y_1}(l_2, l_1) + (y_2 - X_s) q^{(0)}_{y_2-y_1}(l_1, l_2)}{y_2 - y_1} \Bigg] \phi(l_s^{y_1}(X) + l_1, l_s^{y_2}(X) + l_2) \\ &\quad + \int_0^\infty \mathrm{d}l_1 \frac{y_2 - X_s}{y_2 - y_1} Q^{(0)}_{y_2-y_1}(l_1, 0)\phi(l_s^{y_1}(X) + l_1, l_s^{y_2}(X)) \\ &\quad + \int_0^\infty \mathrm{d}l_2 \frac{X_s - y_1}{y_2 - y_1} Q^{(0)}_{y_2-y_1}(l_2, 0)\phi(l_s^{y_1}(X), l_s^{y_2}(X) + l_2). \end{aligned}$$

For $y_1 < y_2 \leq X_s$,

$$\begin{aligned} I(F^{(l_s^y(X))_{y\in\mathbf{R}}, X_s}) &= \int_0^\infty \mathrm{d}l_1 \int_0^\infty \mathrm{d}l_2 (q^{(2)}_{y_2-y_1}(l_2, l_1) \\ &\quad + q^{(0)}_{y_2-y_1}(l_2, l_1))\phi(l_s^{y_1}(X) + l_1, l_s^{y_2}(X) + l_2) \end{aligned}$$

$$+\int_0^\infty \mathrm{d}l_2\; Q^{(0)}_{y_2-y_1}(l_2,0)\phi(l_s^{y_1}(X), l_s^{y_2}(X)+l_2) + 2(X_s - y_2)\phi(l_s^{y_1}(X), l_s^{y_2}(X)).$$

These formulæ give an explicit expression for the limit measure obtained in our last example.

Remark 6.1. It is not difficult to extend this example to a functional of a finite number of local times. We have only considered the case of two local times in order to avoid too complicated notation.

Remark 6.2. The main theorem cannot be extended to every functional F. For example, if we consider the functional

$$F((l^y)_{y\in\mathbf{R}}) = \exp\left(-\int_{-\infty}^{\infty}(l^y)^2\mathrm{d}y\right)$$

which corresponds to Edwards' model in dimension 1 (see [9]), the expectation $\mathbf{E}[F((L_t^y)_{y\in\mathbf{R}})]$ tends exponentially to zero, and $I(F) = 0$, since for all $l > 0$,

$$\int_{-\infty}^{\infty}(Y_{l,+}^y)^2\mathrm{d}y = \infty$$

almost surely.

Therefore, it is impossible to study this case like the examples given above.
Another case for which the theorem cannot apply is the functional

$$F((l^y)_{y\in\mathbf{R}}) = \phi(\sup(l^y)_{y\in\mathbf{R}})$$

where ϕ is a bounded function with compact support.
It would be interesting to find another way to study such penalizations.

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