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On some properties of a universal sigma-finite measure associated with a remarkable class of submartingales

By

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Abstract

In a previous work, we associated with any submartingale X of class (Σ), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfying some technical conditions, a σ-finite measure $\mathcal{Q}$ on $(\Omega, \mathcal{F})$, such that for all $t \geq 0$, and for all events $\Lambda_t \in \mathcal{F}_t$:

$$\mathcal{Q}[\Lambda_t, g \leq t] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_t} X_t],$$

where g is the last hitting time of zero by the process X. The measure $\mathcal{Q}$, which was previously studied in particular cases related with Brownian penalisations and problems in mathematical finance, enjoys some remarkable properties which are detailed in this paper. Most of these properties are related to a certain class of nonnegative martingales, defined as the local densities (with respect to $\mathbb{P}$) of the finite measures which are absolutely continuous with respect to $\mathcal{Q}$. In particular, we obtain a decomposition of any nonnegative supermartingale into three parts, one of them being a martingale in the class described above. If the initial supermartingale is a martingale, this decomposition corresponds to the decomposition of finite measures on $(\Omega, \mathcal{F})$ as sums of three measures, such that the first one is absolutely continuous with respect to $\mathbb{P}$, the second one is absolutely continuous with respect to $\mathcal{Q}$ and the third one is singular with respect to $\mathbb{P}$ and $\mathcal{Q}$. From the properties of the measure $\mathcal{Q}$, we also deduce a universal class of penalisation results of the probability measure $\mathbb{P}$ with a large class of functionals: the measure $\mathcal{Q}$ appears to be the unifying object in these problems.

Notation

In this paper, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ will denote a filtered probability space. $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ is the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ is the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}$. If Y is a random variable, we denote indifferently by $\mathbb{P}[Y]$ or by $\mathbb{E}_{\mathbb{P}}[Y]$ the expectation of X with respect to $\mathbb{P}$. If $(A_t)_{t\geq 0}$ is an increasing process, as usual, the increasing limit of A_t, when $t \to \infty$, is denoted A_∞.

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§1. Introduction

In a paper by Madan, Roynette and Yor [7], and a set of lectures by Yor [3], the authors prove that if $(M_t)_{t\geq 0}$ is a continuous nonnegative local martingale defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions, and such that $\lim_{t\to\infty} M_t = 0$, then for any $K \geq 0$ and any bounded $\mathcal{F}_t$-measurable variable F_t:

$$K\,\mathbb{P}\left[F_t\, \mathbb{1}_{g_K \leq t}\right] = \mathbb{P}\left[F_t (K - M_t)_+\right], \tag{1.1}$$

where $g_K = \sup\{t \geq 0 : M_t = K\}$. The formula (1.1), which represents the price of a European put option in terms of the probability distribution of some last passage time gives, in a particular case, a positive answer to the following problem, also stated in [3] and [7]: for which submartingales X can we find a σ-finite measure $\mathcal{Q}$ and the end of an optional set g such that

$$\mathcal{Q}\left[F_t\, \mathbb{1}_{g\leq t}\right] = \mathbb{P}\left[F_t X_t\right], \tag{1.2}$$

for any bounded $\mathcal{F}_t$-measurable variable F_t.? This problem was previously encountered in the literature in different situations. In [2], Azéma and Yor prove that for any continuous and uniformly integrable martingale M, (1.2) holds for $X_t = |M_t|$, $\mathcal{Q} = |M_\infty|.\mathbb{P}$ and $g = \sup\{t \geq 0 : M_t = 0\}$, or equivalently

$$|M_t| = \mathbb{P}[|M_\infty| \mathbb{1}_{g\leq t} | \mathcal{F}_t].$$

Here again the measure $\mathcal{Q}$ is finite. A particular case where the measure $\mathcal{Q}$ is not finite was obtained by Najnudel, Roynette and Yor in their study of Brownian penalisations (see [11]). For example, they prove the existence of the measure $\mathcal{Q}$ when $X_t = |W_t|$ is the absolute value of the standard Brownian Motion. In this case, the measure $\mathcal{Q}$ is not finite but only σ-finite and is singular with respect to the Wiener measure: it satisfies $\mathcal{Q}(g = \infty) = 0$, where $g = \sup\{t \geq 0 : \ W_t = 0\}$. Now, the existence of $\mathcal{Q}$ in all the examples cited above is a consequence of a general result proved by the authors of the present paper in [10]. The relevant class of submartingales is called (Σ), it was first introduced by Yor in [17] and some of its main properties were further studied in [12]. Let us recall its definition.

Definition 1.1 [12, 17]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A nonnegative submartingale (resp. local submartingale) $(X_t)_{t\geq 0}$ is of class (Σ), iff it can be decomposed as $X_t = N_t + A_t$ where $(N_t)_{t\geq 0}$ and $(A_t)_{t\geq 0}$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted processes satisfying the following assumptions:

- $(N_t)_{t\geq 0}$ is a càdlàg martingale (resp. local martingale);

- $(A_t)_{t\geq 0}$ is a continuous increasing process, with $A_0 = 0$;
- The measure (dA_t) is carried by the set $\{t \geq 0, X_t = 0\}$.

One notes that a process of class (Σ) is "almost" a martingale: outside the zeros of X, the process A does not increase. In fact many processes one often encounters fall into this class, e.g. $X_t = |M_t|$ where $(M_t)_{t\geq 0}$ is a continuous local martingale, $X_t = (M_t - K)_+$ where $(M_t)_{t\geq 0}$ is a càdlàg local martingale with only positive jumps and $K \in \mathbb{R}$ is a constant, $X_t = S_t - M_t$ where $(M_t)_{t\geq 0}$ is a local martingale with only negative jumps and $S_t = \sup_{u\leq t} M_u$. Other remarkable families of examples consist of a large class of recurrent diffusions on natural scale (such as some powers of Bessel processes of dimension $\delta \in (0, 2)$, see [10]) or of a function of a symmetric Lévy process; in these cases, A_t is the local time of the diffusion process or of the Lévy process.

Note that in the case where $A_\infty = \infty$, $\mathbb{P}$-almost surely (this condition holds if $(X_t)_{t\geq 0}$ is a reflected Brownian motion), and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual conditions, the measure $\mathcal{Q}$ cannot exist: otherwise, we would have for all $t \geq 0$,

$$\mathbb{P}[X_t] = \mathbb{P}[X_t \mathbb{1}_{g>t}] = \mathcal{Q}[g \leq t, g > t] = 0,$$

since the event $\{g > t\}$ is $\mathbb{P}$-almost sure, and then in $\mathcal{F}_t$. Hence, X would be indistinguishable from zero, which contradicts the fact that $A_\infty = \infty$. This issue explains why usual conditions are not assumed in the sequel of this paper. On the other hand, we also encounter some problems if we do not complete the probability spaces: for example, if $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F}_t$ is the σ-algebra generated by the canonical process X up to time t, and $\mathbb{P}$ is Wiener measure, then there does not exist a càdlàg and $(\mathcal{F}_t)_{t\geq 0}$-adapted version of the local time which is well-defined everywhere (and not only $\mathbb{P}$-almost surely). In order to avoid also this technical problem, we assume that the filtration satisfies some particular conditions, intermediate between the right-continuity and the usual conditions. These assumptions, called "natural conditions", were first introduced by Bichteler in [5], and then rediscovered in [8] (there they are also called N-usual conditions) where it is proved that most of the properties which generally hold under the usual conditions remain valid under the natural conditions (for example, existence of càdlàg versions of martingales, the Doob-Meyer decomposition, the début theorem, etc.). Let us recall here the definition.

Definition 1.2. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, satisfies the natural conditions iff the two following assumptions hold:

- The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous;

- For all $t \geq 0$, and for every $\mathbb{P}$-negligible set $A \in \mathcal{F}_t$, all the subsets of A are contained in $\mathcal{F}_0$.

This definition is slightly different from the definitions given in [5] and [8] but one can easily check that it is equivalent. The natural enlargement of a filtered probability space can be defined by using the following proposition:

Proposition 1.1 [8]. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. There exists a unique filtered probability space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ (with the same set Ω), such that:*

- *For all $t \geq 0$, $\widetilde{\mathcal{F}}_t$ contains $\mathcal{F}_t$, $\widetilde{\mathcal{F}}$ contains $\mathcal{F}$ and $\widetilde{\mathbb{P}}$ is an extension of $\mathbb{P}$;*
- *The space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ satisfies the natural conditions;*
- *For any filtered probability space $(\Omega, \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$ satisfying the two items above, $\mathcal{F}'_t$ contains $\widetilde{\mathcal{F}}_t$ for all $t \geq 0$, $\mathcal{F}'$ contains $\widetilde{\mathcal{F}}$ and $\mathbb{P}'$ is an extension of $\widetilde{\mathbb{P}}$.*

The space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ is called the natural enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Intuitively, the natural enlargement of a filtered probability space is its smallest extension which satisfies the natural conditions. We also introduce a class of filtered measurable spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ such that any compatible family $(\mathbb{Q}_t)_{t\geq 0}$ of probability measures, $\mathbb{Q}_t$ defined on $\mathcal{F}_t$, can be extended to a probability measure $\mathbb{Q}$ defined on $\mathcal{F}$.

Definition 1.3. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered measurable space, such that $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_t$, $t \geq 0$: $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. We say that the property[1] (P) holds if and only if $(\mathcal{F}_t)_{t\geq 0}$ enjoys the following properties:

- for all $t \geq 0$, $\mathcal{F}_t$ is generated by a countable number of sets;
- for all $t \geq 0$, there exists a Polish space Ω_t, and a surjective map π_t from Ω to Ω_t, such that $\mathcal{F}_t$ is the σ-algebra of the inverse images, by π_t, of Borel sets in Ω_t, and such that for all $B \in \mathcal{F}_t$, $\omega \in \Omega$, $\pi_t(\omega) \in \pi_t(B)$ implies $\omega \in B$;
- if $(\omega_n)_{n\geq 0}$ is a sequence of elements of Ω, such that for all $N \geq 0$,
$$\bigcap_{n=0}^{N} A_n(\omega_n) \neq \emptyset,$$
where $A_n(\omega_n)$ is the intersection of the sets in $\mathcal{F}_n$ containing ω_n, then:
$$\bigcap_{n=0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

[1](P) stands for Parthasarathy since such conditions where introduced by him in [13].

A fundamental example of a filtered measurable space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ satisfying the property (P) can be constructed as follows: we take Ω to be equal to $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, or $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$, the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ (for some $d \geq 1$), and for $t \geq 0$, we define $(\mathcal{F}_t)_{t\geq 0}$ as the natural filtration of the canonical process, and we set

$$\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t.$$

The combination of the property (P) and the natural conditions gives the following definition:

Definition 1.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. We say that it satisfies the property (NP) if and only if it is the natural enlargement of a filtered probability space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\geq 0}, \mathbb{P}^0)$ such that the filtered measurable space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\geq 0})$ enjoys property (P).

In [8] the following result on extension of probability measures is proved (in a slightly more general form):

Proposition 1.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, satisfying property (NP). Then, the σ-algebra $\mathcal{F}$ is the σ-algebra generated by $(\mathcal{F}_t)_{t\geq 0}$, and for all coherent families of probability measures $(\mathbb{Q}_t)_{t\geq 0}$ such that $\mathbb{Q}_t$ is defined on $\mathcal{F}_t$, and is absolutely continuous with respect to the restriction of $\mathbb{P}$ to $\mathcal{F}_t$, there exists a unique probability measure $\mathbb{Q}$ on $\mathcal{F}$ which coincides with $\mathbb{Q}_t$ on $\mathcal{F}_t$ for all $t \geq 0$.*

By using all the results and definitions above, one can state rigorously the main result of [10] in its most general form:

Theorem 1.1. *Let $(X_t)_{t\geq 0}$ be a submartingale of the class (Σ) (in particular X_t is integrable for all $t \geq 0$), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ which satisfies the property (NP). In particular, $(\mathcal{F}_t)_{t\geq 0}$ satisfies the natural conditions and $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_t$, $t \geq 0$. Then, there exists a unique σ-finite measure $\mathcal{Q}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that for $g := \sup\{t \geq 0, X_t = 0\}$:*

- $\mathcal{Q}[g = \infty] = 0$;
- *For all $t \geq 0$, and for all $\mathcal{F}_t$-measurable, bounded random variables F_t,*

$$\mathcal{Q}\left[F_t \, \mathbb{1}_{g\leq t}\right] = \mathbb{P}\left[F_t X_t\right]. \tag{1.3}$$

Note that before being proved in its general form in [10], Theorem 1.1 was shown (under usual assumptions) by Cheridito, Nikeghbali and Platen in [6], in the particular case where the submartingale X is of class (D) (in fact, as mentioned in [6], the solution is essentially contained and somehow hidden in [1]). In this case, the measure $\mathcal{Q}$ is finite an satisfies:

$$\mathcal{Q} = X_\infty.\mathbb{P}.$$

Moreover, Theorem 1.1 has already been obtained in some special cases (but not under the most rigorous formulation with the correct assumption on the underlying filtered probability space) such as the case where X_t is the absolute value of the canonical process on the Wiener space or when $X_t = |Y_t|^{\alpha-1}$ where Y is a symmetric stable Lévy process of index $\alpha \in (1,2)$, although in this latter case the property (1.3) was not noticed ([16]). In fact, almost all our results will apply to a large class of symmetric Lévy processes including the symmetric stable Lévy processes of index $\alpha \in (1,2)$. We shall now detail a little more this last example since it provides natural examples of processes with jumps, living on the Skorokhod space. Let us define, one the space $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}$, $(\mathcal{F}_t)_{t\geq 0}$ as the natural filtration of the canonical process $(Y_t)_{t\geq 0}$, and let us set

$$\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t.$$

We consider on $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ the probability $\mathbb{P}$ under which $(Y_t)_{t\geq 0}$ is a symmetric Lévy process with exponent Ψ:

$$\mathbb{P}[\exp(i\xi Y_t)] = \exp(-t\Psi(\xi)).$$

Moreover, we assume that 0 is regular for itself and that $(Y_t)_{t\geq 0}$ is recurrent, or equivalently (see Bertoin [4]):

$$\int_{-\infty}^{\infty} \frac{d\xi}{1+\Psi(\xi)} < \infty \text{ and } \int_0 \frac{d\xi}{\Psi(\xi)} = \infty.$$

In the case where $(Y_t)_{t\geq 0}$ is a symmetric α-stable Lévy process of index $\alpha \in (1,2)$, $\Psi(\xi) = |\xi|^\alpha$ and the above conditions on Ψ are satisfied (see [4]). Salminen and Yor [15] have proved that if for some $x \in \mathbb{R}$:

$$v(x) = \frac{1}{\pi} \int_0^\infty \frac{1-\cos(\xi x)}{\Psi(\xi)} d\xi,$$

then

$$v(Y_t - x) = v(x) + N_t^x + L_t^x,$$

where N_t^x is a martingale and where $(L_t^x)_{t\geq 0}$ is the local time at level x of the Lévy process $(Y_t)_{t\geq 0}$. Since $(L_t^x)_{t\geq 0}$ is continuous, increasing, adapted and only increases when $Y_t = x$ (see Bertoin [4] Chapter V), the process $(v(Y_t - x))_{t\geq 0}$ is of class (Σ), moreover, $(Y_t)_{t\geq 0}$ is recurrent and 0 is regular for itself, which implies that $\lim_{t\to\infty} L_t^x = \infty$, $\mathbb{P}$-almost surely. Hence Theorem 1.1 applies, and for any $x \in \mathbb{R}$, there exists a σ-finite measure $\mathcal{Q}_x$, singular to $\mathbb{P}$ and such that all the properties of Theorem 1.1 are satisfied with $X_t = v(Y_t - x)$ and $g \equiv g_x = \sup\{t : Y_t = x\}$. In the special case of symmetric α-stable Lévy processes of index $\alpha \in (1,2)$, $v(x) = c(\alpha)|x|^{\alpha-1}$ for some explicit constant $c(\alpha)$ (see [15]). In the sequel, all our results which do not require any assumptions on the sign of the jumps will apply to this family of examples as well.

Let us now shortly recall the general construction of $\mathcal{Q}$ given in [10]. For a Borel, integrable, strictly positive and bounded function f from $\mathbb{R}$ to $\mathbb{R}$, one defines the function G as

$$G(x) = \int_x^\infty f(y)\,dy,$$

and then one proves that the process

$$\left(M_t^f := G(A_t) - \mathbb{P}[G(A_\infty)|\mathcal{F}_t] + f(A_t)X_t\right)_{t\geq 0}, \tag{1.4}$$

is a martingale with respect to $\mathbb{P}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$. Since $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the natural conditions and since $G(A_t) \geq G(A_\infty)$, one can suppose that this martingale is nonnegative and càdlàg, by choosing carefully the version of $\mathbb{P}[G(A_\infty)|\mathcal{F}_t]$. In this case, since $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the property (NP), there exists a unique finite measure $\mathcal{M}^f$ such that for all $t \geq 0$, and for all bounded, $\mathcal{F}_t$-measurable functionals Γ_t:

$$\mathcal{M}^f[\Gamma_t] = \mathbb{P}[\Gamma_t M_t^f].$$

Now, since f is strictly positive, one can define a σ-finite measure $\mathcal{Q}^f$ by:

$$\mathcal{Q}^f := \frac{1}{f(A_\infty)}\,.\mathcal{M}^f.$$

It is proved in [10] that if the function G/f is unformly bounded (this condition is, for example, satisfied for $f(x) = e^{-x}$), then $\mathcal{Q}^f$ satisfies the conditions defining $\mathcal{Q}$ in Theorem 1.1, which implies the existence part of this result. The uniqueness part is proved just after in a very easy way: one remarkable consequence of the uniqueness is the fact that $\mathcal{Q}^f$ does not depend on the choice of f.

One of the remarkable features of the measure $\mathcal{Q}$, in the special case of the Wiener space and when $(X_t)_{t\geq 0}$ is the absolute value of the Wiener process, is that it allows a unified view of some penalisation problems related with Wiener measure. More precisely, Roynette, Vallois and Yor ([14]) consider $\mathbb{W}$, the Wiener measure on the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ endowed with its canonical filtration $(\mathcal{F}_s)_{s\geq 0}$ (not completed), and then they define the σ-algebra $\mathcal{F}$ by

$$\mathcal{F} := \bigvee_{s\geq 0} \mathcal{F}_s.$$

They consider $(\Gamma_t)_{t\geq 0}$, a family of nonnegative random variables on the same space, such that

$$0 < \mathbb{W}[\Gamma_t] < \infty,$$

and for $t \geq 0$, they define the probability measure

$$\mathbb{Q}_t := \frac{\Gamma_t}{\mathbb{W}[\Gamma_t]} .\mathbb{W}.$$

Then they are able to prove that for many examples of families of functionals $(\Gamma_t)_{t\geq 0}$, there exists a probability measure $\mathbb{Q}_\infty$ which can be considered as the weak limit of $(\mathbb{Q}_t)_{t\geq 0}$ when t goes to infinity, in the following sense: for all $s \geq 0$ and for all events $\Lambda_s \in \mathcal{F}_s$,

$$\mathbb{Q}_t[\Lambda_s] \underset{t\to\infty}{\longrightarrow} \mathbb{Q}_\infty[\Lambda_s].$$

Finding the measure $\mathbb{Q}_\infty$ amounts to solving the penalisation problem associated with the functional $(\Gamma_t)_{t\geq 0}$. The functional Γ_t is typically some function of the local time or the running supremum of the Wiener process, or some Feynman-Kac functional of the Wiener process. In the monograph [11], Najnudel, Roynette and Yor have proved that the measure $\mathcal{Q}$ associated with the absolute value of the Wiener process allows a unified approach to many of the examples dealt with separately in the literature: under some technical conditions on the functionals $(\Gamma_t)_{t\geq 0}$, they show that the measure $\mathbb{Q}_\infty$ is absolutely continuous with respect to $\mathcal{Q}$ with an explicit density. In this paper, we shall completely solve the penalisation problem (under the assumptions of Theorem 1.1) with functionals of the form $\Gamma_t = F_t X_t$, where F_t is some functional satisfying some not so restrictive condition. In particular we need no assumption on the continuity of the paths of (X_t), nor any Markov or scaling properties.

More precisely, throughout this paper, we establish some of the fundamental properties of the measure $\mathcal{Q}$ (which also prepare the ground for a forthcoming work on penalisation of diffusion paths). A remarkable class of martingales defined as local densities (with respect to $\mathbb{P}$) of finite measures, absolutely continuous with respect to $\mathcal{Q}$, is involved in a crucial way. The precise definition of these martingales is given in Section 2, and they are explicitly computed in some particular cases. In Section 3, we study their behaviour when t goes to infinity, in the most interesting case where $A_\infty = \infty$, $\mathbb{P}$-almost surely, and we deduce some information about the behaviour of

$(X_t)_{t\geq 0}$ under the measure $\mathcal{Q}$. We shall then naturally deduce the announced universal penalisation results from our study of the asymptotic behaviour of these martingales and of $(X_t)_{t\geq 0}$ under $\mathcal{Q}$. In Section 4, we give a new decomposition of any nonnegative supermartingale into the sum of three nonnegative terms, such that the first one is a uniformly integrable martingale, and the second is a martingale in the class described above. If the initial supermartingale is a martingale, this decomposition can be interpreted as the decomposition of a finite measure on $(\Omega, \mathcal{F})$ as the sum of a three measures, one of them being absolutely continuous with respect to $\mathbb{P}$, the second one being absolutely continuous with respect to $\mathcal{Q}$, and the last one being singular with respect to $\mathbb{P}$ and $\mathcal{Q}$. On our way we shall also establish the following remarkable fact: if X is of class (Σ), with only positive jumps, and if $A_\infty = \infty$, then for any $a \in \mathbb{R}$, $(X_t - a)_+$ is of class (Σ) and the measure $\mathcal{Q}_a$ associated with it is the same as the measure $\mathcal{Q}$ associated with X. This invariance property was observed in [11] in the special case where X is the absolute value of the Wiener process.

§2. A remarkable class of martingales related to the measure $\mathcal{Q}$

Let us first remark that since $\mathcal{Q}[g = \infty] = 0$, one has

$$\mathcal{Q}[A_\infty = \infty] = 0,$$

i.e. A_∞ is finite $\mathcal{Q}$-almost everywhere. Let us state a useful result which was proved in [10]:

Proposition 2.1. *Let f be an integrable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then under the assumptions of Theorem 1.1, the measure*

$$\mathcal{M}^f := f(A_\infty).\mathcal{Q}$$

is the unique positive, finite measure such that for all $t \geq 0$, and for all bounded, $\mathcal{F}_t$-measurable functionals Γ_t:

$$\mathcal{M}^f[\Gamma_t] = \mathbb{P}[\Gamma_t M_t^f], \tag{2.1}$$

where the process $(M_t^f)_{t\geq 0}$ is given by:

$$M_t^f := G(A_t) - \mathbb{P}[G(A_\infty)|\mathcal{F}_t] + f(A_t)X_t$$

for

$$G(x) := \int_x^\infty f(y)dy.$$

In particular, $(M_t^f)_{t\geq 0}$ *is a martingale, càdlàg if one chooses a suitable version of the conditional expectation of* $G(A_\infty)$ *given* $\mathcal{F}_t$. *Moreover,* $(M_t^f)_{t\geq 0}$ *is uniquely determined by* f *in the following sense: two càdlàg martingales satisfying (2.1) are necessarily indistinguishable.*

Proposition 2.1 gives a relation between a finite measure which is absolutely continuous with respect to $\mathcal{Q}$ ($\mathcal{M}^f$), and a càdlàg martingale $(M_t^f)_{t\geq 0}$. This relation can be generalized as follows:

Proposition 2.2. *We suppose that the assumptions of Theorem 1.1 hold and we take the same notation. Let* F *be a* $\mathcal{Q}$*-integrable, nonnegative functional defined on* $(\Omega, \mathcal{F})$. *Then, there exists a càdlàg* $\mathbb{P}$*-martingale* $(M_t(F))_{t\geq 0}$ *such that the measure* $\mathcal{M}^F := F.\mathcal{Q}$ *is the unique finite measure satisfying, for all* $t \geq 0$, *and for all bounded,* $\mathcal{F}_t$*-measurable functionals* Γ_t:

$$\mathcal{M}^F[\Gamma_t] = \mathbb{P}[\Gamma_t M_t(F)].$$

The martingale $(M_t(F))_{t\geq 0}$ *is unique up to indistinguishability.*

Proof. Let $t \geq 0$, Γ_t be a nonnegative, $\mathcal{F}_t$-measurable functional such that:

$$\mathbb{P}[\Gamma_t] = 0,$$

and let f be an integrable, strictly positive function from $\mathbb{R}_+$ to $\mathbb{R}_+$. One has:

$$\mathbb{P}[M_t^f \, \Gamma_t] = 0,$$

and by Proposition 2.1,

$$\mathcal{Q}[f(A_\infty)\, \Gamma_t] = 0.$$

Since f is supposed to be strictly positive, one deduces that

$$\mathcal{Q}[\Gamma_t] = 0,$$

and finally,

$$\mathcal{Q}[F\, \Gamma_t] = 0.$$

Therefore, the restriction of the finite measure $\mathcal{M}^F$ to $\mathcal{F}_t$ is absolutely continuous with respect to $\mathbb{P}$, and there exists a nonnegative, $\mathcal{F}_t$-measurable random variable $M_t^{(0)}$ such that for all $\mathcal{F}_t$-measurable, bounded variables Γ_t:

$$\mathcal{M}^F[\Gamma_t] = \mathbb{P}[M_t^{(0)}\, \Gamma_t].$$

This equality, available for all $t \geq 0$, implies that $(M_t^{(0)})_{t\geq 0}$ is a $\mathbb{P}$-martingale. Since the underlying probability space satisfies the natural conditions, $(M_t^{(0)})_{t\geq 0}$ admits a càdlàg modification $(M_t)_{t\geq 0}$, and one has the equality:

$$\mathcal{M}^F[\Gamma_t] = \mathbb{P}[M_t \, \Gamma_t].$$

By the monotone class theorem this determines uniquely the measure $\mathcal{M}^F$. Moreover, if $(M'_t)_{t\geq 0}$ is a càdlàg martingale satisfying:

$$\mathcal{M}^F[\Gamma_t] = \mathbb{P}[M'_t \, \Gamma_t],$$

then for all $t \geq 0$, $M_t = M'_t$ almost surely, and since M and M' are càdlàg, they are indistinguishable. □

By Proposition 2.2, one can define a particular family of nonnegative, càdlàg $\mathbb{P}$-martingales: the martingales of the form $(M_t(F))_{t\geq 0}$, where F is a nonnegative, $\mathcal{Q}$-integrable functional F. By construction, these martingales correspond to the local densities, with respect to $\mathbb{P}$, of the finite measures which are absolutely continuous with respect to $\mathcal{Q}$. The situation is similar to the case of nonnegative, uniformly integrable martingales, which are the local densities of the finite measures, absolutely continuous with respect to $\mathbb{P}$. Proposition 2.2 does not give any explicit formula for the martingale $M_t(F)$. However, from Proposition 2.1, one deduces immediately the following result:

Corollary 2.1. *Under the assumptions of Theorem 1.1, for all integrable functions f from $\mathbb{R}_+$ to $\mathbb{R}_+$, $f(A_\infty)$ is integrable with respect to $\mathcal{Q}$, and the martingale $\big(M_t(f(A_\infty))\big)_{t\geq 0}$ is indistinguishable with the martingle $(M_t^f)_{t\geq 0}$ defined in Proposition 2.1.*

Remark. Let f be an integrable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. The martingale

$$(\mathbb{P}[G(A_\infty)|\mathcal{F}_t])_{t\geq 0}$$

admits a càdlàg version. If it is denoted by $(G_t)_{t\geq 0}$, one has:

$$M_t(F) = G(A_t) - G_t + Y_t,$$

where $(Y_t)_{t\geq 0}$ is a càdlàg modification of $(f(A_t)X_t)_{t\geq 0}$, which then exists for any choice of f (recall that $G(A_t)$ is continuous with respect to t). If f is bounded, one easily proves that $f(A_t)X_t$ is càdlàg with respect to t: in this case, $(Y_t)_{t\geq 0}$ is indistinguishable with $(f(A_t)X_t)_{t\geq 0}$. However, for unbounded f, the existence of Y is not trivial.

Another case for which one can give a simple expression for the martingale $(M_t(F))_{t\geq 0}$ is the case where $(X_t)_{t\geq 0}$ is of class (D). More precisely, one has the following result:

Proposition 2.3. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that the process $(X_t)_{t\geq 0}$ is of class (D). Then X_t tends a.s. to a limit X_∞ when t goes to infinity, and the measure $\mathcal{Q}$ is absolutely continuous with respect to $\mathbb{P}$, with density X_∞. Moreover, a nonnegative measurable functional F is integrable with respect to $\mathcal{Q}$ if and only if FX_∞ is integrable with respect to $\mathbb{P}$, and in this case, $(M_t(F))_{t\geq 0}$ is a càdlàg version (unique up to indistinguishability) of the conditional expectation $(\mathbb{P}[FX_\infty|\mathcal{F}_t])_{t\geq 0}$. In particular, it is uniformly integrable, and it converges a.s. and in L^1 to FX_∞ when t goes to infinity.*

Proof. The equality $\mathcal{Q} = X_\infty.\mathbb{P}$ is contained in [10], [1] and [6]. Let us shortly reprove it here. Since $(X_t)_{t\geq 0}$ is of class (D), the expectation of A_t is bounded, and then A_∞ is integrable, which implies that $(N_t)_{t\geq 0}$ is a uniformly integrable, càdlàg martingale. It admits an a.s. limit N_∞ for t going to infinity, and then X_∞ is well-defined. Moreover, if d_t is the infimum of $u > t$, such that $X_u = 0$, by the version of the début theorem given in [8], d_t is a stopping time. Moreover, $d_t = \infty$ if and only if $g \leq t$, and by right-continuity of X, $X_{d_t} = 0$ for $d_t < \infty$. One deduces:

$$\mathbb{P}[X_\infty \mathbb{1}_{g\leq t}|\mathcal{F}_t] = \mathbb{P}[X_\infty \mathbb{1}_{d_t=\infty}|\mathcal{F}_t] = \mathbb{P}[X_{d_t}|\mathcal{F}_t].$$

Now, since $(N_t)_{t\geq 0}$ is a uniformly integrable, càdlàg martingale,

$$\mathbb{P}[X_\infty \mathbb{1}_{g\leq t}|\mathcal{F}_t] = \mathbb{P}[N_{d_t} + A_{d_t}|\mathcal{F}_t] = N_t + A_t = X_t,$$

or equivalently, for all bounded, $\mathcal{F}_t$-measurable functionals Γ_t:

$$\mathbb{P}[\Gamma_t \, X_\infty \, \mathbb{1}_{g\leq t}] = \mathbb{P}[\Gamma_t \, X_t].$$

Moreover, under $X_\infty.\mathbb{P}$, $X_\infty > 0$ almost everywhere and then $g < \infty$. One deduces that $X_\infty.\mathbb{P}$ is equal to $\mathcal{Q}$. For any nonnegative functional F, it is then trivial that F is integrable with respect to $\mathcal{Q}$ iff FX_∞ is integrable with respect to $\mathbb{P}$, in this case, the finite measure $\mathcal{M}^F$ has density FX_∞ with respect to $\mathbb{P}$. By taking the restriction to $\mathcal{F}_t$, one deduces that the martingale $(M_t(F))_{t\geq 0}$ is a càdlàg version of the conditional expectation of FX_∞. $\square$

It is also possible to describe explicitly $\mathcal{Q}$ and $(M_t(F))_{t\geq 0}$ if $(X_t)_{t\geq 0}$ is a strictly positive martingale:

Proposition 2.4. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that $\mathbb{P}$-almost surely, $(X_t)_{t\geq 0}$ does not vanish: in particular, $(A_t)_{t\geq 0}$ is* *indistinguishable from zero, and* $(X_t)_{t\geq 0}$ *is a martingale. Then,* $\mathcal{Q}$ *is finite and it is the unique measure, such that for* $t \geq 0$*, the restriction of* $\mathcal{Q}$ *to* $\mathcal{F}_t$ *has density* X_t *with respect to the restriction of* $\mathbb{P}$ *to* $\mathcal{F}_t$*. Moreover, for any nonnegative,* $\mathcal{Q}$*-integrable functional* F*, the martingale* $(M_t(F))_{t\geq 0}$ *can be given by:*

$$M_t(F) = X_t\, \widetilde{\mathcal{Q}}[F\,|\mathcal{F}_t],$$

where $\widetilde{\mathcal{Q}}[F\,|\mathcal{F}_t]$ *is a càdlàg version of the conditional expectation of* F *given* $\mathcal{F}_t$*, with respect to the probability measure* $\widetilde{\mathcal{Q}}$ *obtained by dividing* $\mathcal{Q}$ *by its total mass (different from zero). In particular, the functional identically equal to one is* $\mathcal{Q}$*-integrable and* $(M_t(1))_{t\geq 0}$ *is indistinguishable from* $(X_t)_{t\geq 0}$*.*

Proof. Let $T_0 := \inf\{t \geq 0, X_t = 0\}$. By the début theorem (under natural conditions), T_0 is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. By assumption, for all $t \geq 0$, the event $\{T_0 > t\}$, which is in $\mathcal{F}_t$, holds $\mathbb{P}$-almost surely. Now, by the construction of $\mathcal{Q}$ given in [10] and described above, $\mathcal{Q}$ is absolutely continuous with respect to a finite measure, which is locally absolutely continuous with respect to $\mathbb{P}$. One deduces that for all $t \geq 0$, the event $\{T_0 > t\}$ holds $\mathcal{Q}$-almost everywhere. Hence, $\mathcal{Q}$-almost everywhere, T_0 is infinite and $(X_t)_{t\geq 0}$ does not vanish, which implies

$$\mathcal{Q}[\Gamma_t] = \mathbb{P}[\Gamma_t X_t]. \tag{2.2}$$

By the monotone class theorem, $\mathcal{Q}$ is the unique measure satisfying (2.2): it is finite since X_0 is integrable, its total mass is different from zero since $X_0 > 0$. Hence, $\widetilde{\mathcal{Q}}$ is well-defined. Moreover, if F is integrable with respect to $\mathcal{Q}$, it is also integrable with respect to $\widetilde{\mathcal{Q}}$, and the $\widetilde{\mathcal{Q}}$-martingale:

$$\left(\widetilde{\mathcal{Q}}[F\,|\mathcal{F}_t]\right)_{t\geq 0}$$

is well-defined and admits a càdlàg version $(Y_t)_{t\geq 0}$. Indeed, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the natural conditions, and then it is also the case for $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \widetilde{\mathcal{Q}})$, since for all $t \geq 0$, the restriction of $\widetilde{\mathcal{Q}}$ to $\mathcal{F}_t$ is equivalent to the restriction of $\mathbb{P}$ (recall that $X_t > 0$, $\mathbb{P}$-almost surely). Therefore, for all bounded, $\mathcal{F}_t$-measurable functionals Γ_t:

$$\begin{aligned}
\mathcal{Q}[F\,\Gamma_t] &= \mathcal{Q}[1]\,\widetilde{\mathcal{Q}}[F\,\Gamma_t] \\
&= \mathcal{Q}[1]\,\widetilde{\mathcal{Q}}\left[\Gamma_t\,\widetilde{\mathcal{Q}}[F\,|\mathcal{F}_t]\right] \\
&= \mathcal{Q}\left[\Gamma_t Y_t\right] \\
&= \mathbb{P}\left[\Gamma_t X_t Y_t\right]
\end{aligned}$$

Now, one has:

$$\mathcal{Q}[F\,\Gamma_t] = \mathcal{M}^F[\Gamma_t] = \mathbb{P}[\Gamma_t M_t(F)],$$

with the notation of Proposition 2.2. Hence, $(M_t(F))_{t\geq 0}$ is a modification of $(X_tY_t)_{t\geq 0}$, and since these two processes are càdlàg, they are indistinguishable. Moreover, the functional equal to one is $\mathcal{Q}$-integrable, since $\mathcal{Q}$ is finite. In this case, one can take $Y_t = 1$ for all $t \geq 0$, and $(M_t(F))_{t\geq 0}$ is indistinguishable from $(X_t)_{t\geq 0}$. □

After giving these simple examples for which one can explicitly compute $\mathcal{Q}$ and $M_t(F)$, it is natural to ask what happens in a more general situation. In Section 3, we study the case where $A_\infty = \infty$, $\mathbb{P}$-almost surely (this case occurs, in particular, when $(X_t)_{t\geq 0}$ is a reflected Brownian motion). Unfortunately, we are not able to give explicit expressions for the martingales of the form $(M_t(F))_{t\geq 0}$ in this case, but we obtain some information about their behaviour when t goes to infinity.

§3. The case $A_\infty = \infty$

As it was proved in [10], the measure $\mathcal{Q}$ is infinite if one supposes that $A_\infty = \infty$, $\mathbb{P}$-almost surely. More precisely, the image of $\mathcal{Q}$ by the functional A_∞ is the infinite measure: $\mathbb{P}[X_0].\delta_0 + \lambda$, where λ is Lebesgue measure on $\mathbb{R}_+$. Moreover, again for $A_\infty = \infty$, the martingale $(M_t(F))_{t\geq 0}$ tends $\mathbb{P}$-almost surely to zero for any $\mathcal{Q}$-integrable functional F. In particular, it cannot be uniformly integrable, except for $F = 0$, $\mathcal{Q}$-almost everywhere. More precisely, one has the slightly more general result:

Proposition 3.1. *Let us suppose that the assumptions of Theorem 1.1 are satisfied. Then on the set $\{A_\infty = \infty\}$, the martingale $M_t(F)$ tends $\mathbb{P}$-almost surely to zero when t goes to infinity.*

Proof. Let us use the notation of Proposition 2.2. For all $u > 0$, $v \geq t > 0$:

$$\mathcal{M}^F[A_t > u] = \mathbb{P}[M_v(F)\,\mathbb{1}_{A_t > u}].$$

Moreover, $\mathbb{P}$-almost surely:

$$M_v(F)\,\mathbb{1}_{A_t>u} \underset{v\to\infty}{\longrightarrow} M_\infty(F)\,\mathbb{1}_{A_t>u},$$

where $M_\infty(F)$ is the a.s. limit of $M_t(F)$ for t going to infinity. By Fatou's lemma, one deduces:

$$\mathbb{P}[M_\infty(F)\,\mathbb{1}_{A_t>u}] \leq \mathcal{M}^F[A_t > u] \leq \mathcal{M}^F[A_\infty > u].$$

Now, $M_\infty(F)\,\mathbb{1}_{A_\infty>u}$ is the almost sure limit of $M_\infty(F)\,\mathbb{1}_{A_t>u}$. Since $M_\infty(F)$ is integrable by Fatou's lemma, one has, by dominated convergence:

$$\mathbb{P}[M_\infty(F)\,\mathbb{1}_{A_\infty>u}] \leq \mathcal{M}^F[A_\infty > u].$$

By taking u going to infinity, we are done, since A_∞ is finite $\mathcal{M}^F$-almost everywhere. $\square$

Once the behaviour of $(M_t(F))_{t\geq 0}$ under $\mathbb{P}$ is known, it is natural to ask what happens under $\mathcal{Q}$. The following result implies that the behaviour of $(M_t(F))_{t\geq 0}$ is not the same. Moreover, it gives some information about the behaviour of $(X_t)_{t\geq 0}$ under $\mathcal{Q}$:

Proposition 3.2. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that $A_\infty = \infty$, $\mathbb{P}$-almost surely. Then $\mathcal{Q}$-almost everywhere, X_t tends to infinity with t, and*

$$\frac{M_t(F)}{X_t} \underset{t\to\infty}{\longrightarrow} F$$

for all nonnegative, $\mathcal{Q}$-integrable functionals F.

Remark. As we have seen in Proposition 2.2, two versions of $(M_t(F))_{t\geq 0}$ are indistinguishable with respect to $\mathbb{P}$. Since $\mathcal{Q}$ is absolutely continuous with respect to a finite measure which is locally absolutely continuous with respect to $\mathbb{P}$, the two versions are also indistinguishable with respect to $\mathcal{Q}$. Hence, $(M_t(F))_{t\geq 0}$ can be considered to be well-defined for all the problems concerning its behaviour under the measure $\mathcal{Q}$.

Proof. The functional $H := e^{-A_\infty}$ is $\mathcal{Q}$-integrable and one has:

$$M_t(H) = e^{-A_t}(1 + X_t)$$

(recall that $\mathbb{P}$-almost surely, $e^{-A_\infty} = 0$, since $A_\infty = \infty$). One deduces, for all bounded, $\mathcal{F}_t$-measurable random variables Γ_t:

$$\mathcal{M}^H[\Gamma_t] = \mathbb{P}\left[e^{-A_t}(1 + X_t)\,\Gamma_t\right].$$

This implies:

$$\mathbb{P}[\Gamma_t] = \mathcal{M}^H[Y_t\Gamma_t],$$

where

$$Y_t = \frac{e^{A_t}}{1 + X_t}.$$

Now, $(Y_t)_{t\geq 0}$ is a nonnegative, càdlàg martingale, with respect to the probability measure $\widetilde{\mathcal{M}}^H := \mathcal{M}^H/\mathcal{M}^H(1)$, and then, converges $\mathcal{M}^H$-almost everywhere to a limit random variable Y_∞. Now, since for all $u > 0$, $v \geq t > 0$,

$$\mathbb{P}[A_t \leq u] = \mathcal{M}^H[Y_v\, \mathbb{1}_{A_t\leq u}],$$

one has, by taking $v \to \infty$ and by using Fatou's lemma:

$$\mathcal{M}^H[Y_\infty \, \mathbb{1}_{A_t \leq u}] \leq \mathbb{P}[A_t \leq u],$$

which implies

$$\mathcal{M}^H[Y_\infty \, \mathbb{1}_{A_\infty \leq u}] \leq \mathbb{P}[A_t \leq u].$$

Now, since $A_\infty = \infty$, $\mathbb{P}$-almost surely, one has

$$\mathbb{P}[A_t \leq u] \underset{t \to \infty}{\longrightarrow} 0.$$

Hence,

$$\mathcal{M}^H[Y_\infty \, \mathbb{1}_{A_\infty \leq u}] = 0,$$

and finally (by taking u going to infinity):

$$\mathcal{M}^H[Y_\infty \, \mathbb{1}_{A_\infty < \infty}] = 0.$$

Since $A_\infty < \infty$, $\mathcal{Q}$-almost everywhere, $Y_\infty = 0$, $\mathcal{M}^H$-almost everywhere, which implies that X_t tends to infinity with t. On the other hand, for all nonnegative, integrable functionals F, and for all bounded, $\mathcal{F}_t$-measurable functionals Γ_t, one has:

$$\begin{aligned}\mathcal{M}^H\left[\Gamma_t \frac{M_t(F)}{M_t(H)}\right] = \mathcal{Q}\left[\Gamma_t \, H \frac{M_t(F)}{M_t(H)}\right] &= \mathbb{P}\left[\Gamma_t \, M_t(H) \frac{M_t(F)}{M_t(H)}\right] = \mathbb{P}[\Gamma_t \, M_t(F)] \\ &= \mathcal{Q}[\Gamma_t \, F] \\ &= \mathcal{M}^H\left[\Gamma_t \frac{F}{H}\right] = \mathcal{M}^H\left[\Gamma_t \, \widetilde{\mathcal{M}}^H\left[\frac{F}{H} \,|\mathcal{F}_t\right]\right].\end{aligned}$$

Note that all the equalities above are meaningful since $M_t(H)$ and H never vanish. Therefore, for all $t \geq 0$, one has almost surely:

$$\frac{M_t(F)}{M_t(H)} = \widetilde{\mathcal{M}}^H\left[\frac{F}{H} \,|\mathcal{F}_t\right],$$

which implies that

$$\frac{M_t(F)}{M_t(H)} \underset{t \to \infty}{\longrightarrow} \frac{F}{H},$$

$\widetilde{\mathcal{M}}^H$-almost surely, and then, $\mathcal{Q}$-almost everywhere. Now, since $X_t \to \infty$, $\mathcal{Q}$-almost everywhere, $X_t > 0$ for t large enough and:

$$\frac{M_t(H)}{X_t} = e^{-A_t}\left(1 + \frac{1}{X_t}\right) \underset{t \to \infty}{\longrightarrow} e^{-A_\infty}.$$

One deduces:

$$\frac{M_t(F)}{X_t} = \frac{M_t(F)}{M_t(H)} \frac{M_t(H)}{X_t} \underset{t \to \infty}{\longrightarrow} \frac{F}{H} e^{-A_\infty} = F.$$

□

In the case where $(X_t)_{t\geq 0}$ is a reflected Brownian motion, Proposition 3.2 is essentially proved in [11] and when X_t is a symmetric α-stable process of index $\alpha \in (1,2)$, it is proved in [16]. In the particular case of the reflected Brownian motion, the measure $\mathcal{Q}$ is strongly related to the last passage time at any level and not only at zero. This relation can be generalized as follows:

Proposition 3.3. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that the submartingale $(X_t)_{t\geq 0}$ has only positive jumps and that $A_\infty = \infty$ almost surely under $\mathbb{P}$. For $a \geq 0$, let $g^{[a]}$ be the last hitting time of the interval $[0,a]$:*

$$g^{[a]} = \sup\{t \geq 0, X_t \leq a\}.$$

Then for all $t \geq 0$, and for all $\mathcal{F}_t$-measurable, bounded variables Γ_t, the measure $\mathcal{Q}$ satisfies

$$\mathcal{Q}\left[\Gamma_t \, \mathbb{1}_{g^{[a]}\leq t}\right] = \mathbb{P}\left[\Gamma_t (X_t - a)_+\right]. \tag{3.1}$$

Moreover, $((X_t - a)_+)_{t\geq 0}$ is a submartingale of class (Σ) and the σ-finite measure obtained by applying Theorem 1.1 to it is equal to $\mathcal{Q}$.

Proof. Let:

$$d_t^{[a]} := \inf\{v > t, X_v \leq a\}.$$

By the début theorem (for natural conditions), $d_t^{[a]}$ is a stopping time. Now, for all $u > t$:

$$\mathcal{Q}\left[\Gamma_t \, \mathbb{1}_{g\leq u, d_t^{[a]} > u}\right] = \mathbb{P}\left[\Gamma_t \, \mathbb{1}_{d_t^{[a]} > u} X_u\right].$$

One deduces, by using the decomposition of the submartingale $(X_t)_{t\geq 0}$, and by applying martingale property to $(N_t)_{t\geq 0}$:

$$\begin{aligned}\mathcal{Q}\left[\Gamma_t \mathbb{1}_{g\leq u, d_t^{[a]}>u}\right] &= \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}>u}(N_u + A_u)\right] \\ &= \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}>u}(N_u + A_t)\right] \\ &= \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}>u} A_t\right] \\ &\quad + \mathbb{P}\left[\Gamma_t N_u\right] - \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}\leq u} N_u\right] \\ &= \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}>u} A_t\right] \\ &\quad + \mathbb{P}\left[\Gamma_t N_t\right] - \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}\leq u} N_{d_t^{[a]}}\right] \\ &= \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}>u} A_t\right] + \mathbb{P}\left[\Gamma_t N_t\right] \\ &\quad - \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}\leq u} X_{d_t^{[a]}}\right] + \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}\leq u} A_t\right] \\ &= \mathbb{P}\left[\Gamma_t X_t\right] - \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}\leq u} X_{d_t^{[a]}}\right].\end{aligned}$$

Now, by right continuity, $d_t^{[a]} = t$ if $X_t < a$, and since X has only positive jumps, for $X_t \geq a$ and $d_t^{[a]} < \infty$, $X_{d_t^{[a]}} = a$. One deduces that

$$\mathcal{Q}\left[\Gamma_t \mathbb{1}_{g\leq u, d_t^{[a]}>u}\right] = \mathbb{P}\left[\Gamma_t X_t\right] - \mathbb{P}\left[\Gamma_t \mathbb{1}_{d_t^{[a]}\leq u}(X_t \wedge a)\right].$$

When u tends to infinity, the event $\{g \leq u, d_t^{[a]} > u\}$ tends to the event $\{g^{[a]} \leq t\}$. Moreover, the event $\{d_t^{[a]} \leq u\}$ tends to $\{d_t^{[a]} < \infty\}$, which is almost sure under $\mathbb{P}$, since $A_\infty = \infty$. One deduces:

$$\mathcal{Q}\left[\Gamma_t \mathbb{1}_{g^{[a]}\leq t}\right] = \mathbb{P}\left[\Gamma_t X_t\right] - \mathbb{P}\left[\Gamma_t (X_t \wedge a)\right] = \mathbb{P}\left[\Gamma_t (X_t - a)_+\right].$$

Now, from Lemma 2.1 in [6], $((X_t - a)_+)_{t\geq 0}$ is also nonnegative submartingale of class (Σ). The supremum of its hitting times of zero is $g^{[a]}$. The formula (3.1) and the fact that $\{g^{[a]} < \infty\}$ holds $\mathcal{Q}$-almost everywhere (recall that $X_t \to \infty$ when $t \to \infty$, since $A_\infty = \infty$, $\mathbb{P}$-almost surely), imply that $\mathcal{Q}$ is the σ-finite measure obtained from the submartingale $((X_t - a)_+)_{t\geq 0}$. □

In their study of Brownian penalisations, Najnudel, Roynette and Yor ([11]) introduce a particular class of nonnegative processes which converge $\mathcal{Q}$-almost everywhere to a $\mathcal{Q}$-integrable functional. Let us state a similar definition in our general framework:

Definition 3.1. Let us suppose that the assumptions of Theorem 1.1 are satisfied. We say that a process $(F_t)_{t\geq 0}$ belongs to the class (C) if it is nonnegative, uniformly bounded, nonincreasing, càdlàg and adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$, if there exists $a > 0$ such that for all $t \geq 0$, $F_t = F_{g^{[a]}}$ on the set $\{t \geq g^{[a]}\}$, and if its decreasing limit at infinity, denoted F_∞, is $\mathcal{Q}$-integrable.

For example, the process $(F_t)_{t\geq 0}$ given by

$$F_t = \varphi(A_t),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is integrable and decreasing, is in the class (C), as well as

$$F_t := \exp\left(-\lambda A_t - \int_0^t q(X_s)ds\right),$$

where $\lambda > 0$ and where q is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$, with compact support. When a process $(F_t)_{t\geq 0}$ is in the class (C), the following proposition gives the behaviour of $\mathbb{P}[F_t X_t]$ when t goes to infinity.

Proposition 3.4. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that $A_\infty = \infty$, $\mathbb{P}$-almost surely. Let $(F_t)_{t\geq 0}$ be a process in the class (C). Then, if F_g is integrable with respect to $\mathcal{Q}$, one has:*

$$\mathbb{P}[F_t X_t] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[F_\infty].$$

Proof. It is sufficient to prove:

$$\mathcal{Q}[F_t \mathbb{1}_{g\leq t}] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[F_\infty].$$

Now, since the set $\{g^{[a]} \leq t\}$ is included in the set $\{g \leq t\}$, one can write:

$$\mathcal{Q}[F_t \mathbb{1}_{g\leq t}] = \mathcal{Q}[F_t \mathbb{1}_{g^{[a]}\leq t}] + \mathcal{Q}[F_t \mathbb{1}_{g\leq t<g^{[a]}}].$$

Moreover:

$$\mathcal{Q}[F_t \mathbb{1}_{g^{[a]}\leq t}] = \mathcal{Q}[F_\infty \mathbb{1}_{g^{[a]}\leq t}] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[F_\infty \mathbb{1}_{g^{[a]}<\infty}] = \mathcal{Q}[F_\infty].$$

The last equality is due to the fact that in the case where $A_\infty = \infty$, $\mathbb{P}$-almost surely, the process $(X_t)_{t\geq 0}$ tends $\mathcal{Q}$-almost everywhere to infinity with t. Hence, it is sufficient to prove that

$$\mathcal{Q}[F_t \mathbb{1}_{g\leq t<g^{[a]}}] \underset{t\to\infty}{\longrightarrow} 0.$$

Now, $F_t \mathbb{1}_{g\leq t<g^{[a]}}$ is dominated by F_g, integrable with respect to $\mathcal{Q}$, and tends to zero, $\mathcal{Q}$-almost surely, for t going to infinity. By dominated convergence, we are done. $\square$

Remark. Let $(X_t)_{t\geq 0}$ be the absolute value of the Wiener process and let $F_t := \exp(-\lambda L_t)$, where L_t is the local time of $(X_t)_{t\geq 0}$ at level 0. The process $(F_t)_{t\geq 0}$ is in the class (C) and it is known (see [10]) that L_∞ follows the Lebesgue measure on $\mathbb{R}_+$ under $\mathcal{Q}$. Consequently,

$$\mathbb{P}[\exp(-\lambda L_t)X_t] \underset{t\to\infty}{\longrightarrow} 1/\lambda,$$

although

$$\exp(-\lambda L_t)X_t \underset{t\to\infty}{\longrightarrow} 0,$$

$\mathbb{P}$-almost surely. Of course, due to the general feature of our results, the same result holds if one replaces X_t by $|Y_t|^{\alpha-1}$, where Y is a symmetric α-stable Lévy process with index $\alpha \in (1,2)$, and L_t would then stand for the local time of Y.

Here is another version of the same result (which does not involve the class (C)) and which is in fact more powerful and useful:

Proposition 3.5. *Let us suppose that the assumptions of Theorem 1.1 are satisfied and that $A_\infty = \infty$, $\mathbb{P}$-almost surely. Let $(F_t)_{t\geq 0}$ be a càdlàg, adapted, nonnegative process such that its limit F_∞ exists $\mathcal{Q}$-almost everywhere. We suppose that there exists a $\mathcal{Q}$-integrable, nonnegative functional H, such that for all $t \geq 0$, one has:*

$$F_t X_t \leq M_t(H)$$

$\mathbb{P}$-almost surely. Then, F_∞ is $\mathcal{Q}$-integrable and:

$$\mathbb{P}[F_t X_t] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[F_\infty].$$

Proof. For all $t \geq 0$, one has $\mathcal{Q}$-almost everywhere

$$F_t X_t \leq M_t(H). \tag{3.2}$$

Indeed, the event $\{F_t X_t > M_t(H)\}$ is $\mathcal{F}_t$-measurable and $\mathbb{P}$-negligible, and then, $\mathcal{Q}$-negligible. One deduces that $\mathbb{P}$-almost surely and $\mathcal{Q}$-almost everywhere, (3.2) is satisfied for all rationals $t \geq 0$, and then for all $t \geq 0$, since $(F_t)_{t\geq 0}$, $(M_t(H))_{t\geq 0}$ and $(X_t)_{t\geq 0}$ are càdlàg. By adding e^{-A_∞} to H, one can now suppose that $H > 0$ and $M_t(H) > 0$ for all t. Hence:

$$\mathbb{P}[F_t X_t] = \mathcal{M}^H\left[\frac{F_t X_t}{M_t(H)}\right].$$

Now, one has, uniformly in t:

$$\frac{F_t X_t}{M_t(H)} \leq 1 + \infty.\mathbb{1}_{\exists t\geq 0, F_t X_t > M_t(H)},$$

which is $\mathcal{M}^H$-integrable, since $\mathcal{M}^H$ is a finite measure and the event $\{\exists t \geq 0, F_t X_t > M_t(H)\}$ is $\mathcal{Q}$, and then $\mathcal{M}^H$-negligible. In particular:

$$\mathbb{P}[F_t X_t] \leq \mathcal{M}^H \left[1 + \infty . \mathbb{1}_{\exists t \geq 0, F_t X_t > M_t(H)}\right] = \mathcal{Q}[H] < \infty.$$

Moreover, $\mathcal{Q}$-almost everywhere:

$$\frac{F_t X_t}{M_t(H)} \underset{t \to \infty}{\longrightarrow} \frac{F_\infty}{H}.$$

By dominated convergence:

$$\mathbb{P}[F_t X_t] \underset{t \to \infty}{\longrightarrow} \mathcal{M}^H \left[\frac{F_\infty}{H}\right] = \mathcal{Q}[F_\infty].$$

Since for all $t \geq 0$,

$$\mathbb{P}[F_t X_t] \leq \mathcal{Q}[H],$$

one deduces that

$$\mathcal{Q}[F_\infty] \leq \mathcal{Q}[H] < \infty.$$

□

We now illustrate how the above result can be used. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable function. From the identity (1.4) defining the martingale $(M_t^f)_{t \geq 0}$, and using the fact that $A_\infty = \infty$, $\mathbb{P}$-almost surely, we have that

$$f(A_t) X_t \leq M_t^f.$$

Consequently the above Proposition applies to the case $F_t = f(A_t)$, with $f : \mathbb{R}_+ \to \mathbb{R}_+$ an integrable function. It also obviously applies to any function F_t which satisfying $F_t \leq f(A_t)$, for some integrable $f : \mathbb{R}_+ \to \mathbb{R}_+$. For instance, the result would apply to any $F_t = G_t f(A_t)$ where G_t is a bounded càdlàg $\mathcal{F}_t$-measurable process and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is integrable; in particular if $F_t = f(A_t) \exp\left(-\int_0^t q(X_s) ds\right)$, where q is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$, then the above proposition applies.

We are now able to state two universal penalisation results:

Proposition 3.6. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that $A_\infty = \infty$, $\mathbb{P}$-almost surely. Let $(F_t)_{t \geq 0}$ be a process in the class (C) such that F_g is integrable with respect to $\mathcal{Q}$ and F_∞ is not $\mathcal{Q}$-almost everywhere equal to zero. Then, for t sufficiently large, $0 < \mathbb{P}[F_t X_t] < \infty$, and one can define a measure $\mathbb{Q}_t$ by*

$$\mathbb{Q}_t = \frac{F_t X_t}{\mathbb{P}[F_t X_t]} . \mathbb{P}.$$

Moreover, there exists a probability measure $\mathbb{Q}_\infty$ which can be considered as the weak limit of $\mathbb{Q}_t$ when t goes to infinity, in the following sense: for all $s \geq 0$ and for all events $\Lambda_s \in \mathcal{F}_s$,

$$\mathbb{Q}_t[\Lambda_s] \underset{t\to\infty}{\longrightarrow} \mathbb{Q}_\infty[\Lambda_s].$$

The measure $\mathbb{Q}_\infty$ is absolutely continuous with respect to $\mathcal{Q}$:

$$\mathbb{Q}_\infty = \frac{F_\infty}{\mathcal{Q}[F_\infty]}.\mathcal{Q},$$

where $0 < \mathcal{Q}[F_\infty] < \infty$.

Proof. The integrability of F_∞ under $\mathcal{Q}$ is an immediate consequence of the integrability of F_g, and $\mathcal{Q}[F_\infty] > 0$ because F_∞ is not $\mathcal{Q}$-almost everywhere equal to zero. Moreover, for all $t \geq 0$, F_t is uniformly bounded and X_t is $\mathbb{P}$-integrable, which implies that $\mathbb{P}[F_t X_t]$ is finite. On the other hand, by Proposition 3.4,

$$\mathbb{P}[F_t X_t] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[F_\infty] > 0, \tag{3.3}$$

and then $\mathbb{P}[F_t X_t] > 0$ for t large enough. Now, one has, for $t > s$:

$$\mathbb{P}[F_t \mathbb{1}_{\Lambda_s} X_t] = \mathcal{Q}[F_t \mathbb{1}_{\Lambda_s} \mathbb{1}_{g \leq t}],$$

where, by the arguments in the proof of Proposition 3.4,

$$\mathcal{Q}[F_t \mathbb{1}_{\Lambda_s} \mathbb{1}_{g \leq t}] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[F_\infty \mathbb{1}_{\Lambda_s}].$$

Combining this with (3.3) completes the proof of the proposition. □

Proposition 3.7. *Let us suppose that the assumptions of Theorem 1.1 are satisfied and that $A_\infty = \infty$, $\mathbb{P}$-almost surely. Let $(F_t)_{t\geq 0}$ be a càdlàg, adapted, nonnegative process such that its limit F_∞ exists $\mathcal{Q}$-almost everywhere and is not $\mathcal{Q}$-almost everywhere equal to zero. We suppose that there exists a $\mathcal{Q}$-integrable, nonnegative functional H, such that for all $t \geq 0$, one has:*

$$F_t X_t \leq M_t(H)$$

$\mathbb{P}$-almost surely. Then, for t sufficiently large, $0 < \mathbb{P}[F_t X_t] < \infty$ and one can define a measure $\mathbb{Q}_t$ by

$$\mathbb{Q}_t = \frac{F_t X_t}{\mathbb{P}[F_t X_t]}.\mathbb{P}.$$

Moreover, there exists a probability measure $\mathbb{Q}_\infty$ which can be considered as the weak limit of $\mathbb{Q}_t$ when t goes to infinity, in the following sense: for all $s \geq 0$ and for all events $\Lambda_s \in \mathcal{F}_s$,

$$\mathbb{Q}_t[\Lambda_s] \underset{t\to\infty}{\longrightarrow} \mathbb{Q}_\infty[\Lambda_s].$$

The measure $\mathbb{Q}_\infty$ is absolutely continuous with $\mathcal{Q}$:

$$\mathbb{Q}_\infty = \frac{F_\infty}{\mathcal{Q}[F_\infty]}.\mathcal{Q},$$

where $0 < \mathcal{Q}[F_\infty] < \infty$.

Proof. Since for all $t \geq 0$, $F_t X_t \leq M_t(H)$, $\mathbb{P}$-almost surely, one has

$$\mathbb{P}[F_t X_t] \leq \mathbb{P}[M_t(H)] = \mathcal{Q}[H] < \infty.$$

On the other hand, by Proposition 3.5,

$$\mathbb{P}[F_t X_t] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[F_\infty] \in (0, \infty), \tag{3.4}$$

which implies that $\mathbb{P}[F_t X_t] > 0$ for t large enough. Moreover, by applying Proposition 3.5 to the family of functionals $(F_t \mathbb{1}_{\Lambda_s} \mathbb{1}_{t \geq s})_{t \geq 0}$, one deduces:

$$\mathbb{P}[F_t \mathbb{1}_{\Lambda_s} X_t] \underset{t\to\infty}{\longrightarrow} \mathcal{Q}[\mathbb{1}_{\Lambda_s} F_\infty].$$

Combining this with (3.4) completes the proof of the proposition. □

Remark. The results above give the behaviour of the quantity $\mathbb{P}[F_t X_t]$. In order to obtain penalisation results which do not necessarily involve X_t, e.g. of the form $F_t = f(A_t)$, we need to find an equivalent for $\mathbb{P}[F_t]$. Unfortunately, we are not able to give such an estimate in the general case, however, if $(X_t)_{t \geq 0}$ is a diffusion satisfying some technical conditions, this problem is solved in the companion paper [9], and we deduce a penalisation theorem, generalizing results given in [11].

§4. A new decomposition of nonnegative supermartingales

The following proposition gives a general decomposition of any nonnegative, càdlàg supermartingale, involving a uniformly martingale and a martingale of the form $(M_t(F))_{t \geq 0}$. This decomposition generalizes a result obtained in [11] (Theorem 1.2.5).

Proposition 4.1. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that $A_\infty = \infty$, $\mathbb{P}$-almost surely. Let Z be a nonnegative, càdlàg $\mathbb{P}$-supermartingale. We denote by Z_∞ the $\mathbb{P}$-almost sure limit of Z_t when t goes to infinity. Then, $\mathcal{Q}$-almost everywhere, the quotient Z_t/X_t is well-defined for t large enough and converges, when t goes to infinity, to a limit z_∞, integrable with respect to $\mathcal{Q}$, and $(Z_t)_{t \geq 0}$ decomposes as*

$$\left(Z_t = M_t(z_\infty) + \mathbb{P}[Z_\infty | \mathcal{F}_t] + \xi_t\right)_{t \geq 0},$$

where $(\mathbb{P}[Z_\infty | \mathcal{F}_t])_{t \geq 0}$ denotes a càdlàg version of the conditional expectation of Z_∞ with respect to $\mathcal{F}_t$, and $(\xi_t)_{t \geq 0}$ is a nonnegative, càdlàg $\mathbb{P}$-supermartingale, such that:

- *$Z_\infty \in L^1_+(\mathcal{F}, \mathbb{P})$, hence $\mathbb{P}[Z_\infty|\mathcal{F}_t]$ converges $\mathbb{P}$-almost surely and in $L^1(\mathcal{F}, \mathbb{P})$ towards Z_∞;*
- *$\frac{\mathbb{P}[Z_\infty|\mathcal{F}_t]+\xi_t}{X_t} \underset{t\to\infty}{\longrightarrow} 0$, $\mathcal{Q}$-almost everywhere;*
- *$M_t(z_\infty) + \xi_t \underset{t\to\infty}{\longrightarrow} 0$, $\mathbb{P}$-almost surely.*

Moreover, the decomposition is unique in the following sense: let z'_∞ be a $\mathcal{Q}$-integrable, nonnegative functional, Z'_∞ a $\mathbb{P}$-integrable, nonnegative random variable, $(\xi'_t)_{t\geq 0}$ a càdlàg, nonnegative $\mathbb{P}$-supermartingale, and let us suppose that for all $t \geq 0$,

$$Z_t = M_t(z'_\infty) + \mathbb{P}[Z'_\infty|\mathcal{F}_t] + \xi'_t.$$

Under these assumptions, if for t going to infinity, ξ'_t tends $\mathbb{P}$-almost surely to zero and ξ'_t/X_t tends $\mathcal{Q}$-almost everywhere to zero, then $z'_\infty = z_\infty$, $\mathcal{Q}$-almost everywhere, $Z'_\infty = Z_\infty$, $\mathbb{P}$-almost surely, and ξ' is $\mathbb{P}$-indistinguishable with ξ.

Proof. Let $H := e^{-A_\infty}$. Since Z is a càdlàg $\mathbb{P}$-supermartingale, it is easy to deduce that

$$\left(\frac{Z_t}{M_t(H)}\right)_{t\geq 0}$$

is a càdlàg supermartingale with respect to $\widetilde{\mathcal{M}}^H := \mathcal{M}^H/\mathcal{M}^H(1)$. Hence, it converges $\widetilde{\mathcal{M}}^H$-almost surely to a limit ζ. Since $M_t(H)/X_t$ converges $\widetilde{\mathcal{M}}^H$-a.s. to H, Z_t/X_t converges $\widetilde{\mathcal{M}}^H$-a.s., and then $\mathcal{Q}$-almost everywhere, to $z_\infty = \zeta\, H$. Moreover, since ζ is the $\widetilde{\mathcal{M}}^H$-a.s. limit of the $\widetilde{\mathcal{M}}^H$-supermartingale $(Z_t/M_t(H))_{t\geq 0}$, one has:

$$\mathcal{Q}[z_\infty] = \mathcal{M}^H[\zeta] \leq \mathcal{M}^H[Z_0/M_0(H)] < \infty.$$

Since z_∞ is $\mathcal{Q}$-integrable, $(M_t(z_\infty))_{t\geq 0}$ is well-defined. Now, for all nonnegative, $\mathcal{F}_t$-measurable functionals Γ_t:

$$\begin{aligned}
\mathcal{Q}[\Gamma_t z_\infty] &= \mathcal{Q}\left[\Gamma_t \lim_{u\to\infty} \frac{Z_u}{X_u}\right] \\
&= \mathcal{Q}\left[\Gamma_t \lim_{u\to\infty} \frac{Z_u}{X_u} \mathbb{1}_{g\leq u}\right] \\
&\leq \liminf_{u\to\infty} \mathcal{Q}\left[\Gamma_t \frac{Z_u}{X_u} \mathbb{1}_{g\leq u}\right] \\
&= \liminf_{u\to\infty} \mathbb{P}\left[\Gamma_t \frac{Z_u}{X_u} X_u\right] \\
&\leq \liminf_{u\to\infty} \mathbb{P}[\Gamma_t Z_u] \leq \mathbb{P}[\Gamma_t Z_t].
\end{aligned}$$

One deduces that for all $t \geq 0$, $M_t(z_\infty) \leq Z_t$, $\mathbb{P}$-a.s., which implies that $(M_t(z_\infty) \wedge Z_t)_{t\geq 0}$ is a càdlàg and adapted modification of $(M_t(z_\infty))_{t\geq 0}$. Since $(M_t(z_\infty))_{t\geq 0}$ is only defined up to càdlàg modifications (which are indistinguishable from each other), one can replace $(M_t(z_\infty))_{t\geq 0}$ by $(M_t(z_\infty) \wedge Z_t)_{t\geq 0}$, and then suppose that for all $t \geq 0$, $M_t(z_\infty) \leq Z_t$ everywhere. Note that if $(Z_t)_{t\geq 0}$ is supposed to be uniformly integrable, $(M_t(z_\infty))_{t\geq 0}$ is also uniformly integrable, and since it tends $\mathbb{P}$-almost surely to zero, it is $\mathbb{P}$-almost surely identically zero. This implies that $z_\infty = 0$, $\mathcal{Q}$-almost everywhere. Now, going back to the general case, let us define, for all $t \geq 0$:

$$\widetilde{Z}_t := Z_t - M_t(z_\infty).$$

Since $(M_t(z_\infty))_{t\geq 0}$ is a càdlàg $\mathbb{P}$-martingale, the process $(\widetilde{Z}_t)_{t\geq 0}$ is a càdlàg, nonnegative $\mathbb{P}$-supermartingale. Moreover, $M_t(z_\infty)$ tends $\mathbb{P}$-almost surely to zero when t goes to infinity, hence:

$$\widetilde{Z}_t \underset{t\to\infty}{\longrightarrow} Z_\infty,$$

$\mathbb{P}$-almost surely. Now, since $(\widetilde{Z}_t)_{t\geq 0}$ is a nonnegative supermartingale and $Z_\infty \geq 0$, $\mathbb{P}$-almost surely, we obtain, for all $t \geq 0$:

$$0 \leq \mathbb{P}[Z_\infty | \mathcal{F}_t] \leq \widetilde{Z}_t, \tag{4.1}$$

$\mathbb{P}$-almost surely. Hence, $\Big((\mathbb{P}[Z_\infty|\mathcal{F}_t])_+ \wedge \widetilde{Z}_t\Big)_{t\geq 0}$ is a càdlàg version of $(\mathbb{P}[Z_\infty|\mathcal{F}_t])_{t\geq 0}$ and one can suppose that (4.1) holds everywhere. Now, let us write, for all $t \geq 0$:

$$\xi_t := \widetilde{Z}_t - \mathbb{P}[Z_\infty|\mathcal{F}_t].$$

This is a nonnegative, càdlàg supermartingale tending $\mathbb{P}$-a.s. to zero when t goes to infinity. On the other hand, $\mathcal{Q}$-almost everywhere:

$$\lim_{t\to\infty} \frac{\xi_t}{X_t} = \lim_{t\to\infty} \frac{\widetilde{Z}_t}{X_t} = z_\infty - z_\infty = 0.$$

Here, the first equality is due to the fact that $(\mathbb{P}[Z_\infty|\mathcal{F}_t]/X_t)_{t\geq 0}$ tends to zero $\mathcal{Q}$-almost everywhere, by the remark made above on the case where $(Z_t)_{t\geq 0}$ is uniformly integrable. The uniqueness of the decomposition is very easy to check: since $M_t(z'_\infty)$ and ξ'_t tend $\mathbb{P}$-almost surely to zero when $t \to \infty$, Z_t tends $\mathbb{P}$-almost surely to Z'_∞ and then $Z'_\infty = Z_\infty$. Similarly, since ξ'_t/X_t and any càdlàg version of $\mathbb{P}[Z'_\infty|\mathcal{F}_t]/X_t$ tend to zero, $\mathcal{Q}$-almost everywhere, Z_t/X_t tends to z'_∞, which is $\mathcal{Q}$-almost everywhere equal to z_∞. One now deduces that for all $t \geq 0$, $\xi'_t = \xi_t$, $\mathbb{P}$-almost surely, and since ξ and ξ' are càdlàg, they are indistinguishable, which proves the uniqueness of the decomposition. □

As in [11], we can deduce, from Proposition 4.1, the following characterization of the martingales of the form $(M_t(F))_{t\geq 0}$:

Corollary 4.1. *Let us suppose that the assumptions of Theorem 1.1 are satisfied, and that $A_\infty = \infty$, $\mathbb{P}$-almost surely. Then, a càdlàg, nonnegative $\mathbb{P}$-martingale $(Z_t)_{t\geq 0}$ has the form $(M_t(F))_{t\geq 0}$ for a nonnegative, $\mathcal{Q}$-integrable functional F, if and only if:*

$$\mathbb{P}[Z_0] = \mathcal{Q}\left(\lim_{t\to\infty} \frac{Z_t}{X_t}\right). \tag{4.2}$$

Note that, by Proposition 4.1, the limit above necessarily exists $\mathcal{Q}$-almost everywhere.

Proof. By Proposition 4.1, one can write the decomposition:

$$Z_t = M_t(z_\infty) + \mathbb{P}[Z_\infty|\mathcal{F}_t] + \xi_t.$$

Note that in this situation, $(\xi_t)_{t\geq 0}$ is a nonnegative martingale. One has:

$$\mathbb{P}[Z_0] = \mathbb{P}[M_0(z_\infty)] + \mathbb{P}[\,\mathbb{P}[Z_\infty|\mathcal{F}_0]\,] + \mathbb{P}[\xi_0] = \mathcal{Q}[z_\infty] + \mathbb{P}[Z_\infty] + \mathbb{P}[\xi_0].$$

Now, the equation (4.2) is satisfied iff

$$\mathbb{P}[Z_0] = \mathcal{Q}[z_\infty].$$

If this condition holds, one has

$$\mathbb{P}[Z_\infty] = \mathbb{P}[\xi_0] = 0,$$

and then, for all $t \geq 0$,

$$\mathbb{P}[Z_\infty|\mathcal{F}_t] + \xi_t = 0$$

almost surely. Hence, the martingale $(Z_t)_{t\geq 0}$ is a càdlàg modification of $(M_t(z_\infty))_{t\geq 0}$. Since $(M_t(z_\infty))_{t\geq 0}$ is only defined up to càdlàg modification, one can suppose that $(Z_t)_{t\geq 0}$ coincides with $(M_t(z_\infty))_{t\geq 0}$. On the other hand, if $(Z_t)_{t\geq 0}$ has the form $(M_t(F))_{t\geq 0}$, by uniqueness of the decomposition given in Proposition 4.1, $F = z_\infty$, $\mathcal{Q}$-almost everywhere, which implies that $\mathbb{P}[Z_0] = \mathcal{Q}[z_\infty]$, and then (4.2) is satisfied. □

Remark. Let us suppose that, in Proposition 4.1, $(Z_t)_{t\geq 0}$ is a nonnegative martingale. Since the space satisfies the property (NP), there exists a unique finite measure $\mathbb{Q}_Z$ on $(\Omega, \mathcal{F})$, such that for all $t \geq 0$, its restriction to $\mathcal{F}_t$ has density Z_t with respect to $\mathbb{P}$. If one writes the decomposition

$$Z_t = M_t(z_\infty) + \mathbb{P}[Z_\infty|\mathcal{F}_t] + \xi_t,$$

one deduces:

$$\mathbb{Q}_Z = z_\infty\,.\mathcal{Q} + Z_\infty\,.\mathbb{P} + \mathbb{Q}_\xi,$$

where the restriction of $\mathbb{Q}_\xi$ to $\mathcal{F}_t$ has density ξ_t with respect to $\mathbb{P}$. By Radon-Nykodym theorem, one has a decomposition:

$$\mathbb{Q}_\xi = \xi'.\mathbb{P} + \mathbb{Q}'_\xi,$$

where $\mathbb{Q}'_\xi$ is singular with respect to $\mathbb{P}$. Now, if for $t \geq 0$, ξ'_t is the density, with respect to $\mathbb{P}$, of the restriction of $\xi'.\mathbb{P}$ to $\mathcal{F}_t$, then for all $t \geq 0$, $\xi'_t \leq \xi_t$, $\mathbb{P}$-almost surely. Moreover, if $(\xi'_t)_{t\geq 0}$ is supposed to be càdlàg, then almost surely, $\xi'_t \leq \xi_t$ for all $t \geq 0$. By taking the $\mathbb{P}$-almost sure limit for t going to infinity, one deduces that $\xi' = 0$, $\mathbb{P}$-almost surely, therefore, $\mathbb{Q}_\xi = \mathbb{Q}'_\xi$ is singular with respect to $\mathbb{P}$. One can also decompose $\mathbb{Q}_\xi$ as:

$$\mathbb{Q}_\xi = \xi''.\mathcal{Q} + \mathbb{Q}''_\xi,$$

where $\mathbb{Q}''_\xi$ is singular with respect to $\mathcal{Q}$. Now, for all $t \geq 0$, one has, $\mathbb{P}$-almost surely, and then $\mathcal{Q}$-almost everywhere, $M_t(\xi'') \leq \xi_t$. Since $(M_t(\xi''))_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ are right-continuous, one deduces that $\mathcal{Q}$-almost everywhere, $M_t(\xi'') \leq \xi_t$ for all $t \geq 0$. Since $\mathcal{Q}$-almost everywhere, $M_t(\xi'')/X_t$ tends to ξ'' when t goes to infinity, and ξ_t/X_t tends to zero, one has $\xi'' = 0$, $\mathcal{Q}$-almost everywhere and $\mathbb{Q}_\xi = \mathbb{Q}''_\xi$ is singular with respect to t. Hence, we have obtained a decomposition of $\mathbb{Q}_Z$ into three parts:

- A part which is absolutely continuous with respect to $\mathbb{P}$.
- A part which is absolutely continuous with respect to $\mathcal{Q}$.
- A part which is singular with respect to $\mathbb{P}$ and $\mathcal{Q}$.

This decomposition is unique, as a consequence of uniqueness of Radon-Nykodym decomposition (recall that $\mathbb{P}$ and $\mathcal{Q}$ are mutually singular, since $A_\infty = \infty$, $\mathbb{P}$-almost surely, and $A_\infty < \infty$, $\mathcal{Q}$-almost everywhere). This uniqueness can be compared with the uniqueness of the decomposition of the martingale $(Z_t)_{t\geq 0}$ given in Proposition 4.1.

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