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A Reducibility Result for a Class of Linear Wave Equations on $\mathbb{T}^d$

Riccardo Montalto

University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland

Correspondence to be sent to: e-mail: riccardo.montalto@math.uzh.ch

We prove a *reducibility* result for a class of quasi-periodically forced linear wave equations on the d-dimensional torus $\mathbb{T}^d$ of the form

$$\partial_{tt} v - \Delta v + \varepsilon \mathcal{P}(\omega t)[v] = 0,$$

where the perturbation $\mathcal{P}(\omega t)$ is a second order operator of the form $\mathcal{P}(\omega t) = -a(\omega t)\Delta - \mathcal{R}(\omega t)$, the frequency $\omega \in \mathbb{R}^\nu$ is in some Borel set of large Lebesgue measure, the function $a : \mathbb{T}^\nu \to \mathbb{R}$ (independent of the space variable) is sufficiently smooth and $\mathcal{R}(\omega t)$ is a time-dependent finite rank operator. This is the first reducibility result for linear wave equations with unbounded perturbations on the higher dimensional torus $\mathbb{T}^d$. As a corollary, we get that the linearized Kirchhoff equation at a smooth and sufficiently small quasi-periodic function is *reducible*.

1 Introduction and Main Result

We consider a linear quasi-periodically forced wave equation of the form

$$\partial_{tt} v - \Delta v + \varepsilon \mathcal{P}(\omega t)[v] = 0, \quad x \in \mathbb{T}^d \tag{1.1}$$

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where $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, $\varepsilon > 0$ is a small parameter, $\omega \in \Omega \subseteq \mathbb{R}^\nu$, with Ω a closed bounded domain and the operator $\mathcal{P}(\omega t)$ is given by

$$\mathcal{P}(\varphi)[v] := -a(\varphi)\Delta v - \mathcal{R}(\varphi)[v], \quad \varphi \in \mathbb{T}^\nu, \quad v \in L^2_0(\mathbb{T}^d, \mathbb{R}) \tag{1.2}$$

with $\mathcal{R}(\varphi)$ being an operator of the form

$$\mathcal{R}(\varphi)[v] := \sum_{k=1}^{N} b_k(\varphi, x) \int_{\mathbb{T}^d} c_k(\varphi, y) v(y)\, \mathrm{d}y + c_k(\varphi, x) \int_{\mathbb{T}^d} b_k(\varphi, y) v(y)\, \mathrm{d}y, \qquad \varphi \in \mathbb{T}^\nu,$$
$$v \in L^2_0(\mathbb{T}^d, \mathbb{R}). \tag{1.3}$$

Here $\nu, d \geq 1$ are integer numbers, $L^2_0(\mathbb{T}^d, \mathbb{R})$ denotes the space of the real-valued L^2 functions with zero average and the functions $a : \mathbb{T}^\nu \to \mathbb{R}, b_k, c_k : \mathbb{T}^\nu \times \mathbb{T}^d \to \mathbb{R}$, $k = 1, \ldots, N$ are assumed to be sufficiently smooth, namely $a \in \mathcal{C}^q(\mathbb{T}^\nu, \mathbb{R})$, $b_k, c_k \in \mathcal{C}^q(\mathbb{T}^\nu \times \mathbb{T}^d, \mathbb{R})$ for some $q > 0$ large enough. The operator $\mathcal{R}(\varphi)$ is symmetric with respect to the real L^2-inner product. Our aim is to prove a reducibility result for the equation (1.1) for ε small enough and for ω in a suitable Borel set of parameters $\Omega_\varepsilon \subset \Omega$ with asymptotically full Lebesgue measure. The partial differential equation (PDE) (1.1) may be written as the first order system

$$\begin{cases} \partial_t v = \psi \\ \partial_t \psi = (1 + \varepsilon a(\omega t))\, \Delta v + \varepsilon \mathcal{R}(\omega t)[v] \end{cases} \tag{1.4}$$

which is a real Hamiltonian system of the form

$$\begin{cases} \partial_t v = \nabla_\psi H(\omega t, v, \psi) \\ \partial_t \psi = -\nabla_v H(\omega t, v, \psi) \end{cases} \tag{1.5}$$

whose φ-dependent Hamiltonian is given by

$$H(\varphi, v, \psi) := \frac{1}{2} \int_{\mathbb{T}^d} \left(\psi^2 + (1 + \varepsilon a(\varphi))|\nabla v|^2\right) \mathrm{d}x - \varepsilon \frac{1}{2} \int_{\mathbb{T}^d} \mathcal{R}(\varphi)[v]\, v\, \mathrm{d}x. \tag{1.6}$$

In (1.5), $\nabla_\psi H$ and $\nabla_v H$ denote the L^2-gradients of the Hamiltonian H with respect to the variables v and ψ. We assume that the functions $b_k(\varphi, x), c_k(\varphi, x)$, $k = 1, \ldots, N$ have zero average with respect to $x \in \mathbb{T}^d$, namely

$$\int_{\mathbb{T}^d} b_k(\varphi, x)\, \mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} c_k(\varphi, x)\, \mathrm{d}x = 0 \quad \forall \varphi \in \mathbb{T}^\nu, \quad k = 1, \ldots, N. \tag{1.7}$$

In order to precisely state the main result of this article, let us introduce some more notations. For any $s \geq 0$, we define the Sobolev spaces $H^s(\mathbb{T}^d) = H^s(\mathbb{T}^d, \mathbb{C})$, $H^s(\mathbb{T}^d, \mathbb{R})$, respectively, of complex- and real-valued functions

$$H^s(\mathbb{T}^d) := \left\{ u(x) = \sum_{j \in \mathbb{Z}^d} u_j \mathrm{e}^{\mathrm{i} jx} : \|u\|^2_{H^s_x} := \sum_{j \in \mathbb{Z}^d} \langle j \rangle^{2s} |u_j|^2 < +\infty \right\},$$

$$H^s(\mathbb{T}^d, \mathbb{R}) := \left\{ u \in H^s(\mathbb{T}^d) : u = \overline{u} \right\} \tag{1.8}$$

where

$$\langle j \rangle := \max\{1, |j|\}, \quad |j| := \sqrt{j_1^2 + \ldots + j_d^2}, \quad \forall j = (j_1, \ldots, j_d) \in \mathbb{Z}^d.$$

Moreover, we define

$$H^s_0(\mathbb{T}^d) := \left\{ u \in H^s(\mathbb{T}^d) : \int_{\mathbb{T}^d} u(x)\, \mathrm{d}x = 0 \right\}, \quad H^s_0(\mathbb{T}^d, \mathbb{R}) := \left\{ u \in H^s(\mathbb{T}^d, \mathbb{R}) : \int_{\mathbb{T}^d} u(x)\, \mathrm{d}x = 0 \right\} \tag{1.9}$$

and introduce the real subspace $\mathbf{H}^s_0(\mathbb{T}^d)$ of $H^s_0(\mathbb{T}^d) \times H^s_0(\mathbb{T}^d)$

$$\mathbf{H}^s_0(\mathbb{T}^d) := \left\{ \mathbf{u} := (u, \overline{u}) : u \in H^s_0(\mathbb{T}^d) \right\}, \quad \text{equipped with the norm} \quad \|\mathbf{u}\|_{\mathbf{H}^s_x} := \|u\|_{H^s_x}.$$

Given a linear operator $\mathcal{R} : L^2_0(\mathbb{T}^d) \to L^2_0(\mathbb{T}^d)$ (where $L^2_0(\mathbb{T}^d) := H^0_0(\mathbb{T}^d)$), we define its Fourier coefficients with respect to the exponential basis $\{\mathrm{e}^{\mathrm{i} j \cdot x} : j \in \mathbb{Z}^d \setminus \{0\}\}$ of $L^2_0(\mathbb{T}^d)$ as

$$\mathcal{R}_j^{j'} := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathcal{R}[\mathrm{e}^{\mathrm{i} j' \cdot x}] \mathrm{e}^{-\mathrm{i} j \cdot x}\, \mathrm{d}x, \qquad \forall j, j' \in \mathbb{Z}^d \setminus \{0\}. \tag{1.10}$$

We introduce the linear operator $\overline{\mathcal{R}}$, defined by $\overline{\mathcal{R}}[u] = \overline{\mathcal{R}[\overline{u}]}$, for any $u \in L^2_0(\mathbb{T}^d)$.
We say that the operator $\mathcal{R}$ is block diagonal if $\mathcal{R}_j^{j'} = 0$ for any $j, j' \in \mathbb{Z}^d \setminus \{0\}$ with $|j| \neq |j'|$.
Because of the hypothesis (1.7), the Hamiltonian vector field

$$\mathcal{L}(\varphi) := \begin{pmatrix} 0 & 1 \\ \Delta - \varepsilon \mathcal{P}(\varphi) & 0 \end{pmatrix} \overset{(1.2)}{=} \begin{pmatrix} 0 & 1 \\ (1 + \varepsilon a(\varphi))\Delta + \varepsilon \mathcal{R}(\varphi) & 0 \end{pmatrix}, \qquad \varphi \in \mathbb{T}^\nu \tag{1.11}$$

leaves the space of functions with zero average invariant. More precisely for any $0 \leq s \leq q$

$$\mathcal{L}(\varphi) : H^{s+2}_0(\mathbb{T}^d, \mathbb{R}) \times H^{s+1}_0(\mathbb{T}^d, \mathbb{R}) \to H^{s+1}_0(\mathbb{T}^d, \mathbb{R}) \times H^s_0(\mathbb{T}^d, \mathbb{R}), \quad \forall \varphi \in \mathbb{T}^\nu$$

and therefore we can choose $H^1_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})$ as phase space for the Hamiltonian H defined in (1.6). Now we are ready to state the main result of the present paper.

Theorem 1.1. Let ν, d be integer numbers greater or equal than 1. There exists a strictly positive integer $q_0 = q_0(\nu, d) > 1/2$ such that for any $q \geq q_0$ there exists $\varepsilon_q = \varepsilon(q, \nu, d) > 0$ and $\mathfrak{S}_q := \mathfrak{S}(q, \nu, d)$, with $1/2 < \mathfrak{S}_q < q$ such that if $a \in \mathcal{C}^q(\mathbb{T}^\nu, \mathbb{R}), b_k, c_k \in \mathcal{C}^q(\mathbb{T}^\nu \times \mathbb{T}^d, \mathbb{R})$, with b_k, c_k satisfying the hypothesis (1.7) for any $k = 1, \ldots, N$, then for any $\varepsilon \in (0, \varepsilon_q)$ there exists a Borel set $\Omega_\varepsilon \subset \Omega$ of asymptotically full Lebesgue measure, that is

$$|\Omega_\varepsilon| \to |\Omega| \qquad \text{as} \qquad \varepsilon \to 0, \tag{1.12}$$

such that the following holds: for all $\omega \in \Omega_\varepsilon$ and $\varphi \in \mathbb{T}^\nu$, there exists a bounded linear invertible operator $\mathcal{W}_\infty(\varphi) = \mathcal{W}_\infty(\varphi; \omega)$ such that for any $\frac{1}{2} \leq s \leq \mathfrak{S}_q$

$$\mathcal{W}_\infty(\varphi) : \mathbf{H}^s_0(\mathbb{T}^d) \to H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})$$

satisfying the following property: $(v(t, \cdot), \psi(t, \cdot))$ is a solution of (1.4) in $H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})$ if and only if

$$\mathbf{u}(t, \cdot) = (u(t, \cdot), \overline{u}(t, \cdot)) = \mathcal{W}_\infty(\omega t)^{-1}[(v(t, \cdot), \psi(t, \cdot))]$$

is a solution in $\mathbf{H}^s_0(\mathbb{T}^d)$ of the PDE with constant coefficients

$$\partial_t \mathbf{u} = \mathcal{D}_\infty \mathbf{u}, \qquad \mathcal{D}_\infty := \mathrm{i} \begin{pmatrix} -\mathcal{D}_\infty^{(1)} & 0 \\ 0 & \overline{\mathcal{D}_\infty^{(1)}} \end{pmatrix}$$

where for any $s \geq 1$, $\mathcal{D}_\infty^{(1)} : H_0^s(\mathbb{T}^d) \to H_0^{s-1}(\mathbb{T}^d)$ is a linear, time-independent, L^2-self-adjoint, block-diagonal operator. □

The following corollary holds:

Corollary 1.1. For any $\omega \in \Omega_\varepsilon$ and any initial data $(v^{(0)}, \psi^{(0)}) \in H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})$ with $1/2 \leq s \leq \mathfrak{S}_q$, the solution $t \in \mathbb{R} \mapsto (v(t, \cdot), \psi(t, \cdot)) \in H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})$ of the Cauchy problem

$$\begin{cases} \partial_t v = \psi \\ \partial_t \psi = (1 + \varepsilon a(\omega t))\, \Delta v + \varepsilon \mathcal{R}(\omega t)[v] \\ v(0, \cdot) = v^{(0)} \\ \psi(0, \cdot) = \psi^{(0)} \end{cases} \tag{1.13}$$

is stable, namely

$$\sup_{t\in\mathbb{R}}\left(\|v(t,\cdot)\|_{H_x^{s+\frac{1}{2}}}+\|\psi(t,\cdot)\|_{H_x^{s-\frac{1}{2}}}\right)\leq C_q\left(\|v^{(0)}\|_{H_x^{s+\frac{1}{2}}}+\|\psi^{(0)}\|_{H_x^{s-\frac{1}{2}}}\right)$$

for some constant $C_q = C(q, \nu, d) > 0$. □

Remark 1.1. The constants ε_q, $\mathfrak{S}_q$ in Theorem 1.1 and the constant C_q in Corollary 1.1 depend also on the $\|\cdot\|_q$ Sobolev norms of the functions a, b_k, c_k, $k = 1, \ldots, N$ appearing in the definition of the perturbation $\mathcal{P}$ given in (1.2) and (1.3). □

Theorem 1.1 implies a reducibility result for the linearized Kirchhoff equation at a small and sufficiently smooth quasi-periodic function $\varepsilon v_0(\omega t, x)$. The Kirchhoff equation

$$K(v) := \partial_{tt}v - \left(1+\int_{\mathbb{T}^d}|\nabla v|^2\,\mathrm{d}x\right)\Delta v = 0 \tag{1.14}$$

describes nonlinear vibrations of a d-dimensional body (in particular, a string for $d = 1$ and a membrane for $d = 2$). The Cauchy problem for the Kirchhoff equation has been extensively studied, starting from the pioneering paper of Bernstein [11]. Both local and global existence results have been established for initial data in Sobolev and analytic class, see [1, 2, 24, 25, 40, 42, 45] and the recent survey [43]. The existence of periodic solutions for the Kirchhoff equation has been proved by Baldi [3]. This result is proved via Nash–Moser method and thanks to the special structure of the nonlinearity (it is diagonal in space), the linearized operator at any approximate solution can be inverted by Neumann series. This approach does not imply the linear stability of the solutions, since only *the first order Melnikov conditions* are required along the proof. In one space-dimension ($d = 1$), the existence of quasi-periodic solutions and the reducibility of the linearized equation have been established in [44]. In dimension greater or equal than 2, there are no results concerning the existence of quasi-periodic solutions. It is well known that a good strategy for proving the existence and the linear stability of quasi-periodic solutions is to prove the reducibility of the linearized equations at small quasi-periodic approximate solutions obtained along a suitable iterative scheme. Hence our result (Theorem 1.2 below) could be used to prove the existence of quasi-periodic solutions for the nonlinear Kirchhoff equation.

Linearizing the operator K in (1.14) at a quasi-periodic function $\varepsilon v_0(\omega t, x)$ and writing the *linearized equation* $K'(\varepsilon v_0)[v] = 0$ as a first order system, one gets a system of differential equations of the form (1.4) where

$$a(\varphi) = \int_{\mathbb{T}^d} |\nabla v_0(\varphi, x)|^2 \, dx, \quad \mathcal{R}(\varphi)[v] = -2\Delta v_0(\varphi, x) \int_{\mathbb{T}^d} \Delta v_0(\varphi, y) v(y) \, dy, \quad \varphi \in \mathbb{T}^\nu, \quad v \in L^2_0(\mathbb{T}^d, \mathbb{R}).$$

The operator $\mathcal{R}(\varphi)$ defined above has the same form as the one defined in (1.3), by taking $N = 1$, $b_1 = -\Delta v_0$, $c_1 = \Delta v_0$. We point out that Δv_0 has zero average in $x \in \mathbb{T}^d$, hence the hypothesis (1.7) is satisfied. An immediate consequence of Theorem 1.1 and Corollary 1.1 is then the following

Theorem 1.2. Let $q_0, q, \varepsilon_q, \mathfrak{S}_q$ as in Theorem 1.1 and $v_0 \in \mathcal{C}^{q+2}(\mathbb{T}^\nu \times \mathbb{T}^d, \mathbb{R})$. Then the conclusions of Theorem 1.1 and Corollary 1.1 hold for the linearized Kirchhoff equation $K'(\varepsilon v_0)[v] = 0$ at the quasi-periodic function $\varepsilon v_0(\omega t, x)$. □

Now we outline some related works concerning the reducibility of quasi-periodically forced linear partial differential equations. Let us consider a linear differential equation of the form

$$\partial_t u = \mathcal{D} u + \varepsilon \mathcal{P}(\omega t) u, \tag{1.15}$$

where $\mathcal{D}$ is a diagonal operator with discrete spectrum and $\mathcal{P}(\omega t)$ is a linear quasi-periodically forced vector field with nonconstant coefficients. We say that such an equation is *reducible* if there exists a quasi-periodically forced change of variable $u = \Phi(\omega t)[v]$ such that in the new coordinate v, the equation (1.15) is reduced to constant coefficients. Typically, it is necessary to assume that ε (size of the perturbation) is small enough and that the frequency ω, together with the eigenvalues of the operator $\mathcal{D}$, satisfy the so-called *second order Melnikov* non-resonance conditions. These non-resonance conditions involve the differences of the eigenvalues of the operator $\mathcal{D}$. We point out that the reducibility of linear equations is the main ingredient for proving the existence of quasi-periodic solutions (KAM tori) for nonlinear PDEs. Indeed the first reducibility results for linear PDEs have been obtained as a corollary of Kolmogorov-Arnold-Moser (KAM) theorems. We mention the pioneering articles of Kuksin [37], and Wayne [47] concerning the existence of invariant tori for Schrödinger and wave equations in one space dimension with Dirichlet boundary conditions and with bounded perturbations. The first KAM results for PDEs with unbounded perturbations have been obtained by Kuksin [38], Kappeler and Pöschel [36] for analytic perturbations of the KdV equation. Here the unperturbed vector field is ∂_{xxx} and the perturbation contains one space derivative ∂_x. Concerning unbounded perturbations of the quantum Harmonic oscillator on the real line, the first result is due to Bambusi and Graffi [10]. In all these aforementioned results, the perturbation contains derivatives of order $\delta < n-1$, where n is the order of the highest derivative appearing in the linear constant coefficients term. In the case of critical unbounded perturbations, that is $\delta = n-1$, we mention [41, 48] concerning the derivative Non linear Schrödinger equation (NLS) with Dirichlet boundary conditions, in which the authors generalized appropriately the so-called *Kuksin Lemma*, developed in [38]. We also mention the KAM results for the derivative Klein–Gordon equation [12, 13] in which the generalization of the Kuksin Lemma developed in [41, 48] does not apply because of the weaker dispersion relation.

It is well known that the ideas used to deal with the case $\delta \le n-1$ do not apply in the quasi-linear and fully nonlinear case, that is $\delta = n$. The first KAM results in this case have been obtained in [5–7, 32] for quasi-linear perturbations of the Airy, KdV and m-KdV equations, in [30, 31] for quasi-linear Hamiltonian and reversible NLS equations, in [44] for the Kirchhoff equation and in [18, 19] for the water waves equations. The key idea in these series of articles is to split the reduction to constant coefficients of the linearized equation into two parts: the first part is to reduce the equation to another one which is constant coefficients plus a bounded remainder and this is inspired by the breakthrough result of Iooss *et al.* [35]. In a second step, one applies a convergent KAM reducibility scheme which reduces quadratically the size of the perturbation and completes the diagonalization of the equation. This method has been extended also by Bambusi in [8, 9] to deal with unbounded quasi-periodic perturbations of the Schrödinger operator on the real line.

Another difficulty for the reduction procedures and the KAM schemes concerns the multiplicity of the eigenvalues of the unperturbed part of the equation. The first result in this direction is due to Chierchia and You [23] in which the authors prove a KAM result for analytic bounded perturbations of nonlinear wave equations with periodic boundary conditions (double eigenvalues). We mention also the more recent articles [17, 30, 44] concerning Schrödinger and Kirchhoff equations with periodic boundary conditions.

There are very few results for PDEs in higher space dimension since the second order Melnikov non-resonance conditions are violated, typically due to the high multiplicity of the eigenvalues. The first KAM and reducibility results in higher space dimension have been obtained by Eliasson and Kuksin [28, 29] for the linear Schrödinger equation on $\mathbb{T}^d$ with a multiplicative analytic potential and for the nonlinear Schrödinger equation with a convolution potential. The second order Melnikov non-resonance conditions are verified *blockwise*, by introducing the notion of Töplitz-Lipschitz Hamiltonians.

A KAM result for the completely resonant Nonlinear Schrödinger equation on $\mathbb{T}^d$ has been proved by Procesi and Procesi [46], by using Quasi-Töplitz Hamiltonians. We also mention the KAM theorem for the beam equation obtained by Eliasson *et al.* [26]. Recently, Grebert and Paturel [33] proved a reducibility result for the quantum harmonic oscillator on $\mathbb{R}^d$ with an analytic multiplicative potential and in [34] they proved a KAM result for the nonlinear Klein Gordon equation on the d-dimensional sphere. In [14–16], the authors proved the existence of quasi-periodic solutions for Nonlinear wave and Schrödinger equations on $\mathbb{T}^d$ and on Lie groups, by using the *multiscale method*, introduced by Bourgain [20–22] in the analytic setup. This approach does not imply the linear stability of the quasi-periodic solutions since it requires to impose only the *first order Melnikov conditions*.

The reducibility for the quasi-periodically forced Klein–Gordon equation with a small multiplicative potential $\partial_{tt} u - \Delta u + mu + \varepsilon V(\omega t, x) u = 0$ on $\mathbb{T}^d$ is still open. Eliasson *et al.* [27] proved that this equation is *almost reducible* in the sense that it can be reduced to constant coefficients up to a *small remainder*. The aim of the present article is to provide a class of linear wave equations with unbounded perturbations on $\mathbb{T}^d$ which are reducible. We point out that the main difference between Schrödinger and wave (Klein–Gordon) equations is the following: for the Schrödinger equation, the eigenvalues of the linear part of the equation grow like $\sim |j|^2, j \in \mathbb{Z}^d$, whereas the wave equation, written as a first order system in complex coordinates, has eigenvalues growing as $\sim |j|, j \in \mathbb{Z}^d$. It turns out that the second order Melnikov non-resonance conditions

$$|\omega \cdot \ell + \mu_j - \mu_{j'}| \geq \frac{\gamma}{\langle \ell \rangle^{\tau}}, \quad \forall (\ell, j, j') \in \mathbb{Z}^{\nu} \times \mathbb{Z}^d \times \mathbb{Z}^d, \quad (\ell, |j|, |j'|) \neq (0, |j|, |j|) \tag{1.16}$$

in the case of the wave (Klein–Gordon) equation, that is, $\mu_j \sim |j|, j \in \mathbb{Z}^d$ are violated.

In the following, we shall explain the main ideas of the proof of Theorem 1.1. The proof consists in reducing the quasi-periodically forced linear vector field $\mathcal{L}(\omega t)$ defined in (1.11) to a time-independent block-diagonal operator. This reduction procedure is split into two parts.

Regularization of the vector field $\mathcal{L}(\omega t)$. Our first goal is to conjugate the vector field $\mathcal{L}(\omega t)$ to another one which is diagonal up to a sufficiently regularizing perturbation. This is achieved by using a change of variables induced by a reparameterization of time (so that the highest order term has constant coefficients) and time dependent Fourier multipliers (introduced in Section 2.4), see Section 3. We point out that this procedure involve only a reduction in time, since our unbounded perturbation $\mathcal{P}(\omega t)$ is assumed to be diagonal in space up to the finite rank operator $\mathcal{R}(\omega t)$, which is already regularizing, see (1.2) and (1.3).

KAM reducibility scheme. After the preliminary reduction of the order of derivatives, we deal with a time dependent vector field which is a small and regularizing perturbation of a diagonal time-independent vector field. We then perform a KAM reducibility scheme, see Theorem 4.1. The key feature of the scheme is that since the perturbation is regularizing, along the KAM iteration, we can impose non-resonance conditions with a *loss of derivatives* in space, namely

$$|\omega\cdot\ell+\mu_j-\mu_{j'}|\geq\frac{\gamma}{|j|^{\mathtt{d}}|j'|^{\mathtt{d}}\langle\ell\rangle^{\tau}},\quad\forall(\ell,j,j')\in\mathbb{Z}^{\nu}\times(\mathbb{Z}^d\setminus\{0\})\times(\mathbb{Z}^d\setminus\{0\}),\quad(\ell,|j|,|j'|)\neq(0,|j|,|j|)\tag{1.17}$$

for some constant exponents $\mathtt{d}$ and τ large enough and $\gamma\in(0,1)$. Neverthless, all the canonical transformations defined along the iteration will be bounded linear operators (on Sobolev spaces), since the regularizing property of the remainder balances the *loss of space derivatives* in the Melnikov conditions (1.17). This strategy has been used also in [4], to prove a KAM result for gravity water waves in finite depth without capillarity and we implement it within this context.
The conditions (1.17) are much weaker that the ones given in (1.16) and we are able to prove that they are fulfilled for a *large set* of parameters ω. We use the block-decay norm $|\cdot|_s$ (see (2.76)) to estimate the size of the remainders along the iteration. This is convenient since the class of operators having finite block-decay norm is closed under composition (Lemma 2.7), solution of the homological equation (Lemma 4.1) and projections (Lemma 2.9). This norm is well adapted to finite rank operators of the form (1.3) and it gives a strong decay of the blocks arising in the spectral decomposition with respect to the eigenspaces of the operator $\sqrt{-\Delta}$, see Sections 2.2, 2.3.

The article is organized as follows. In Section 2, we introduce some notations and abstract technical tools needed along the proof of Theorem 1.1. The proof of the Theorem is developed in Sections 3–5. In Section 3, we perform the regularization procedure for the linear Hamiltonian vector field $\mathcal{L}$ and we conjugate it to the vector field $\mathcal{L}_4$, defined in (3.70). In Section 4, we prove the block-diagonal reducibility of the vector field $\mathcal{L}_4$, showing that it is conjugated to the block diagonal operator $\mathcal{D}_\infty$ defined in (4.83). In Section 5, we provide the measure estimate of the set of *good parameters* $\Omega_\infty^{2\gamma}$ defined in (4.77). Finally, in Section 6, we conclude the proof of Theorem 1.1 and we prove the Corollary 1.1.

2 Function Spaces, Linear Operators, and Norms

For a function $u \in L^2_0(\mathbb{T}^d) \equiv L^2_0(\mathbb{T}^d, \mathbb{C})$ we consider its Fourier series

$$u(x) = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} u_j e^{\mathrm{i} j \cdot x}, \qquad u_j := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) e^{-\mathrm{i} j \cdot x}\, dx, \quad \forall j \in \mathbb{Z}^d \setminus \{0\}. \tag{2.1}$$

We denote by $\sigma_0(\sqrt{-\Delta})$ the spectrum of the operator $\sqrt{-\Delta}$ restricted to the zero-average functions, that is

$$\sigma_0(\sqrt{-\Delta}) := \left\{ |j| = \sqrt{j_1^2 + \ldots + j_d^2} : j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \setminus \{0\} \right\} \tag{2.2}$$

and for any eigenvalue $\alpha \in \sigma_0(\sqrt{-\Delta})$, we denote by $\mathbb{E}_\alpha$ the corresponding eigenspace, that is

$$\mathbb{E}_\alpha := \mathrm{span}\{ e^{\mathrm{i} j \cdot x} : j \in \mathbb{Z}^d, \quad |j| = \alpha \}. \tag{2.3}$$

Then, any function $u \in L^2_0(\mathbb{T}^d)$ can be written as

$$u(x) = \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \mathrm{u}_\alpha(x), \quad \mathrm{u}_\alpha(x) = \sum_{|j| = \alpha} u_j e^{\mathrm{i} j \cdot x} \in \mathbb{E}_\alpha \tag{2.4}$$

and if $u \in H^s_0(\mathbb{T}^d)$ for some $s \geq 0$, one has

$$\|u\|^2_{H^s_x} = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} |j|^{2s} |u_j|^2 = \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \alpha^{2s} \sum_{|j| = \alpha} |u_j|^2 = \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \alpha^{2s} \|\mathrm{u}_\alpha\|^2_{L^2_x}. \tag{2.5}$$

We also deal with functions $u \in L^2_0(\mathbb{T}^\nu \times \mathbb{T}^d) = L^2(\mathbb{T}^\nu, L^2_0(\mathbb{T}^d))$ which can be regarded as φ-dependent family of functions $u(\varphi, \cdot) \in L^2_0(\mathbb{T}^d)$ that we expand in Fourier series as

$$u(\varphi, x) = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} u_j(\varphi) e^{\mathrm{i} j \cdot x} = \sum_{\substack{\ell \in \mathbb{Z}^\nu \\ j \in \mathbb{Z}^d \setminus \{0\}}} \widehat{u}_j(\ell) e^{\mathrm{i}(\ell \cdot \varphi + j \cdot x)}, \tag{2.6}$$

where

$$u_j(\varphi) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(\varphi, x) e^{-\mathrm{i} j \cdot x}\, dx, \quad \widehat{u}_j(\ell) := \frac{1}{(2\pi)^{\nu + d}} \int_{\mathbb{T}^{\nu + d}} u(\varphi, x) e^{-\mathrm{i}(\ell \cdot \varphi + j \cdot x)}\, d\varphi\, dx.$$

According to (2.4), we can write

$$u(\varphi, x) = \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \mathrm{u}_\alpha(\varphi, x) = \sum_{\substack{\ell \in \mathbb{Z}^\nu \\ \alpha \in \sigma_0(\sqrt{-\Delta})}} \widehat{\mathrm{u}}_\alpha(\ell) e^{\mathrm{i} \ell \cdot \varphi} \tag{2.7}$$

where

$$\mathrm{u}_\alpha(\varphi,x) := \sum_{|j|=\alpha} u_j(\varphi)\mathrm{e}^{\mathrm{i}j\cdot x}, \quad \widehat{\mathrm{u}}_\alpha(\ell) \equiv \widehat{\mathrm{u}}_\alpha(\ell,x) := \frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu} \mathrm{u}_\alpha(\varphi,x)\mathrm{e}^{-\mathrm{i}\ell\cdot\varphi}\,\mathrm{d}\varphi = \sum_{|j|=\alpha}\widehat{u}_j(\ell)\mathrm{e}^{\mathrm{i}j\cdot x}. \tag{2.8}$$

We define for any $s \geq 0$ the Sobolev spaces $H_0^s(\mathbb{T}^{\nu+d}) = H_0^s(\mathbb{T}^{\nu+d},\mathbb{C})$ as

$$H_0^s(\mathbb{T}^{\nu+d}) := \left\{ u \in L_0^2(\mathbb{T}^\nu \times \mathbb{T}^d) : \|u\|_s^2 := \sum_{\substack{\ell\in\mathbb{Z}^\nu \\ j\in\mathbb{Z}^d\setminus\{0\}}} \langle \ell, j\rangle^{2s} |\widehat{u}_j(\ell)|^2 < +\infty \right\}, \tag{2.9}$$

where $\langle \ell, j\rangle := \max\{1, |\ell|, |j|\}$, and for any $\ell = (\ell_1,\ldots,\ell_\nu) \in \mathbb{Z}^\nu$, $|\ell| := \sqrt{\ell_1^2+\ldots+\ell_\nu^2}$. One has

$$\|u\|_s^2 = \sum_{\substack{\ell\in\mathbb{Z}^\nu \\ j\in\mathbb{Z}^d\setminus\{0\}}} \langle \ell, j\rangle^{2s}|\widehat{u}_j(\ell)|^2 = \sum_{\substack{\ell\in\mathbb{Z}^\nu \\ \alpha\in\sigma_0(\sqrt{-\Delta})}} \langle \ell,\alpha\rangle^{2s} \sum_{|j|=\alpha} |\widehat{u}_j(\ell)|^2 = \sum_{\substack{\ell\in\mathbb{Z}^\nu \\ \alpha\in\sigma_0(\sqrt{-\Delta})}} \langle \ell,\alpha\rangle^{2s} \|\widehat{\mathrm{u}}_\alpha(\ell)\|_{L^2}^2 \tag{2.10}$$

where $\langle \ell,\alpha\rangle := \max\{1,|\ell|,\alpha\}$, for any $\ell \in \mathbb{Z}^\nu, \alpha \in \sigma_0(\sqrt{-\Delta})$.
In a similar way, we define the spaces of real valued functions $L_0^2(\mathbb{T}^d,\mathbb{R})$, $L_0^2(\mathbb{T}^{\nu+d},\mathbb{R})$, $H_0^s(\mathbb{T}^d,\mathbb{R})$, $H_0^s(\mathbb{T}^{\nu+d},\mathbb{R})$ and we also deal with Sobolev functions x-independent, belonging to the Sobolev space $H^s(\mathbb{T}^\nu)$ (or $H^s(\mathbb{T}^\nu,\mathbb{R})$). For $u \in H^s(\mathbb{T}^\nu)$ we denote by $\|u\|_s$ its Sobolev norm, given by

$$\|u\|_s := \sum_{\ell\in\mathbb{Z}^\nu} \langle \ell\rangle^{2s}|\widehat{u}(\ell)|^2, \qquad \widehat{u}(\ell) := \frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu} u(\varphi)\mathrm{e}^{-\mathrm{i}\ell\cdot\varphi}\,\mathrm{d}\varphi.$$

Given a Banach space $(E, \|\cdot\|_E)$, we denote by $L^\infty(\mathbb{T}^\nu, E)$ the space of the essentially bounded functions $\mathbb{T}^\nu \to E$ equipped with the norm

$$\|u\|_{L^\infty(\mathbb{T}^\nu,E)} := \mathrm{esssup}_{\varphi\in\mathbb{T}^\nu}\|u(\varphi)\|_E.$$

For any $p \in \mathbb{N}$ we denote by $W^{p,\infty}(\mathbb{T}^\nu, E)$ the space of the p-times weakly differentiable functions $\mathbb{T}^\nu \to E$ equipped with the norm

$$\|u\|_{W^{p,\infty}(\mathbb{T}^\nu,E)} := \max_{|a|\leq p}\|\partial_\varphi^a u\|_{L^\infty(\mathbb{T}^\nu,E)}.$$

In the above formula, for any multi-index $a = (a_1,\ldots,a_\nu) \in \mathbb{N}^\nu$, we use the notations $|a| := a_1+\ldots+a_\nu$ and $\partial_\varphi^a = \partial_{\varphi_1}^{a_1}\ldots\partial_{\varphi_\nu}^{a_\nu}$. We also denote by $\mathcal{C}^0(\mathbb{T}^\nu, E)$ the space of continuous functions $\mathbb{T}^\nu \to E$ equipped with the norm

$$\|u\|_{\mathcal{C}^0(\mathbb{T}^\nu,E)} := \sup_{\varphi\in\mathbb{T}^\nu}\|u(\varphi)\|_E$$

and we denote by $\mathcal{C}^p(\mathbb{T}^\nu, E)$ the space of the p-times differentiable functions with continuous derivatives equipped with the norm

$$\|u\|_{\mathcal{C}^p(\mathbb{T}^\nu, E)} := \max_{|a|\leq p} \|\partial_\varphi^a u\|_{\mathcal{C}^0(\mathbb{T}^\nu, E)}.$$

We recall the standard property

$$W^{p+1,\infty}(\mathbb{T}^\nu, E) \subset \mathcal{C}^p(\mathbb{T}^\nu, E). \tag{2.11}$$

For a function $f : \Omega_o \to E$, $\omega \mapsto f(\omega)$, where $(E, \|\cdot\|_E)$ is a Banach space and Ω_o is a subset of $\mathbb{R}^\nu$, we define the sup-norm and the lipschitz semi-norm as

$$\|f\|^{\sup}_{E,\Omega_o} := \sup_{\omega\in\Omega_o} \|f(\omega)\|_E, \quad \|f\|^{\rm lip}_{E,\Omega_o} := \sup_{\substack{\omega_1,\omega_2\in\Omega_o \\ \omega_1\neq\omega_2}} \frac{\|f(\omega_1) - f(\omega_2)\|_E}{|\omega_1 - \omega_2|} \tag{2.12}$$

and, for $\gamma > 0$, we define the weighted Lipschitz-norm

$$\|f\|^{{\rm Lip}(\gamma)}_{E,\Omega_o} := \|f\|^{\sup}_{E,\Omega_o} + \gamma \|f\|^{\rm lip}_{E,\Omega_o}. \tag{2.13}$$

To shorten the above notations we simply omit to write Ω_o, namely $\|f\|^{\sup}_E = \|f\|^{\sup}_{E,\Omega_o}$, $\|f\|^{\rm lip}_E = \|f\|^{\rm lip}_{E,\Omega_o}$, $\|f\|^{{\rm Lip}(\gamma)}_E = \|f\|^{{\rm Lip}(\gamma)}_{E,\Omega_o}$. If $f : \Omega_o \to \mathbb{C}$, we simply denote $\|f\|^{{\rm Lip}(\gamma)}_{\mathbb{C}}$ by $|f|^{{\rm Lip}(\gamma)}$ and if $E = H^s(\mathbb{T}^{\nu+d})$ we simply denote $\|f\|^{{\rm Lip}(\gamma)}_{H^s} := \|f\|^{{\rm Lip}(\gamma)}_s$. Given two Banach spaces E, F, we denote by $\mathcal{B}(E, F)$ the space of the bounded linear operators $E \to F$. If $E = F$, we simply write $\mathcal{B}(E)$.

Notation. From now on we fix

$$s_0 := \left[\frac{\nu + d}{2}\right] + 1 \tag{2.14}$$

where for any real number $x \in \mathbb{R}$, we denote by $[x]$ its integer part. We write

$$a \lesssim_s b \quad \Longleftrightarrow \quad a \leq C(s) b$$

for some constant $C(s)$ depending on the data of the problem, namely the Sobolev norms $\|a\|_s$, $\|b_k\|_s$, $\|c_k\|_s$ of the functions a, b_k, c_k appearing in (1.2), the number ν of frequencies, the dimension d of the space variable x, the diophantine exponent $\tau > 0$ in the non-resonance conditions, which will be required along the proof. For $s = s_0$ we only write $a \lesssim b$. Also the small constants δ in the sequel depend on the data of the problem.

We recall the classical estimates for the operator $(\omega \cdot \partial_\varphi)^{-1}$ defined as

$$(\omega \cdot \partial_\varphi)^{-1}[1] = 0, \quad (\omega \cdot \partial_\varphi)^{-1}[\mathrm{e}^{\mathrm{i}\ell\cdot\varphi}] = \frac{1}{\mathrm{i}(\omega\cdot\ell)}\mathrm{e}^{\mathrm{i}\ell\cdot\varphi}, \qquad \forall \ell \neq 0, \tag{2.15}$$

for $\omega \in DC(\gamma, \tau)$, where for $\gamma, \tau > 0$,

$$DC(\gamma,\tau) := \left\{ \omega \in \Omega : |\omega \cdot \ell| \geq \frac{\gamma}{|\ell|^\tau}, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\} \right\}. \tag{2.16}$$

If $h(\cdot;\omega) \in H^{s+2\tau+1}(\mathbb{T}^\nu)$, with $\omega \in DC(\gamma,\tau)$, we have

$$\|(\omega\cdot\partial_\varphi)^{-1}h\|_s \leq \gamma^{-1}\|h\|_{s+\tau}, \qquad \|(\omega\cdot\partial_\varphi)^{-1}h\|_s^{\mathrm{Lip}(\gamma)} \leq \gamma^{-1}\|h\|_{s+2\tau+1}^{\mathrm{Lip}(\gamma)}. \tag{2.17}$$

We also recall some classical Lemmas on the composition operators and on the interpolation. Since the variables (φ, x) have the same role, we present it for a generic Sobolev space $H^s(\mathbb{T}^n)$. For any $s \geq 0$ integer, for any domain $A \subseteq \mathbb{R}^n$ we denote by $\mathcal{C}^s(A)$ the space of the s-times continuously differentiable functions equipped by the usual $\|\cdot\|_{\mathcal{C}^s}$ norm.

Lemma 2.1. **(Interpolation)** Let $u, v \in H^s(\mathbb{T}^n)$ with $s \geq s_n$, $s_n := [n/2]+1$. Then, there exists an increasing function $s \mapsto C(s)$ such that

$$\|uv\|_s \leq C(s)\|u\|_s\|v\|_{s_n} + C(s_n)\|u\|_{s_n}\|v\|_s.$$

If $u(\cdot;\omega), v(\cdot;\omega)$, $\omega \in \Omega_o \subseteq \mathbb{R}^\nu$ are ω-dependent families of functions in $H^s(\mathbb{T}^n)$, with $s \geq s_n$ then the same estimate holds replacing $\|\cdot\|_s$ by $\|\cdot\|_s^{\mathrm{Lip}(\gamma)}$. □

Iterating the above inequality one gets that, for some constant $K(s)$, for any $n \geq 0$,

$$\|u^k\|_s \leq K(s)^k \|u\|_{s_0}^{k-1}\|u\|_s \tag{2.18}$$

and if $u(\cdot;\omega) \in H^s$, $s \geq s_n$ is a family of Sobolev functions, the same inequality holds replacing $\|\cdot\|_s$ by $\|\cdot\|_s^{\mathrm{Lip}(\gamma)}$.
We consider the composition operator

$$u(y) \mapsto \mathtt{f}(u)(y) := f(y, u(y)).$$

The following lemma is a classical result due to Moser.

Lemma 2.2 (Composition operator). Let $f \in \mathcal{C}^{s+1}(\mathbb{T}^n \times \mathbb{R}, \mathbb{R})$, with $s \geq s_n := [n/2] + 1$. If $u \in H^s(\mathbb{T}^n)$, with $\|u\|_{s_n} \leq 1$, then $\|\mathtt{f}(u)\|_s \leq C(s, \|f\|_{\mathcal{C}^s})(1 + \|u\|_s)$. If $u(\cdot, \omega) \in H^s(\mathbb{T}^n)$, $\omega \in \Omega_o \subseteq \mathbb{R}^\nu$ is a family of Sobolev functions satisfying $\|u\|_{s_n}^{\mathrm{Lip}(\gamma)} \leq 1$, then, $\|\mathtt{f}(u)\|_s^{\mathrm{Lip}(\gamma)} \leq C(s, \|f\|_{\mathcal{C}^{s+1}})(1 + \|u\|_s^{\mathrm{Lip}(\gamma)})$. □

Now we state the tame properties of the composition operator $u(y) \mapsto u(y+p(y))$ induced by a diffeomorphism of the torus $\mathbb{T}^n$. The Lemma below, can be proved as Lemma 2.21 in [18].

Lemma 2.3 (Change of variables). Let $p := p(\cdot; \omega) : \mathbb{R}^n \to \mathbb{R}^n$, $\omega \in \Omega_o \subset \mathbb{R}^\nu$ be a family of 2π-periodic functions satisfying

$$\|p\|_{\mathcal{C}^{s_n+1}} \leq 1/2, \quad \|p\|_{s_n}^{\mathrm{Lip}(\gamma)} \leq 1 \tag{2.19}$$

where $s_n := [n/2] + 1$. Let $g(y) := y + p(y)$, $y \in \mathbb{T}^n$. Then the composition operator

$$A : u(y) \mapsto (u \circ g)(y) = u(y + p(y))$$

satisfies for all $s \geq s_n$, the tame estimates

$$\|Au\|_{s_n} \lesssim_{s_n} \|u\|_{s_n}, \qquad \|Au\|_s \leq C(s)\|u\|_s + C(s_n)\|p\|_s\|u\|_{s_n+1}. \tag{2.20}$$

Moreover, for any family of Sobolev functions $u(\cdot; \omega)$

$$\|Au\|_{s_n}^{\mathrm{Lip}(\gamma)} \lesssim_{s_n} \|u\|_{s_n+1}^{\mathrm{Lip}(\gamma)}, \tag{2.21}$$

$$\|Au\|_s^{\mathrm{Lip}(\gamma)} \lesssim_s \|u\|_{s+1}^{\mathrm{Lip}(\gamma)} + \|p\|_s^{\mathrm{Lip}(\gamma)}\|u\|_{s_n+2}^{\mathrm{Lip}(\gamma)}, \quad \forall s > s_n. \tag{2.22}$$

The map g is invertible with inverse $g^{-1}(z) = z + q(z)$ and there exists a constant $\delta := \delta(s_n) \in (0, 1)$ such that, if $\|p\|_{2s_n+2}^{\mathrm{Lip}(\gamma)} \leq \delta$, then

$$\|q\|_s \lesssim_s \|p\|_s, \qquad \|q\|_s^{\mathrm{Lip}(\gamma)} \lesssim_s \|p\|_{s+1}^{\mathrm{Lip}(\gamma)}. \tag{2.23}$$

Furthermore, the composition operator $A^{-1}u(z) := u(z + q(z))$ satisfies the estimate

$$\|A^{-1}u\|_s \lesssim_s \|u\|_s + \|p\|_s\|u\|_{s_n+1}, \quad \forall s \geq s_n \tag{2.24}$$

and for any family of Sobolev functions $u(\cdot; \omega)$

$$\|A^{-1}u\|_s^{\mathrm{Lip}(\gamma)} \lesssim_s \|u\|_{s+1}^{\mathrm{Lip}(\gamma)} + \|p\|_{s+1}^{\mathrm{Lip}(\gamma)}\|u\|_{s_n+2}^{\mathrm{Lip}(\gamma)}, \quad \forall s \geq s_n. \tag{2.25}$$

□

2.1 Linear operators

Let $\mathcal{R} \in \mathcal{B}(L^2_0(\mathbb{T}^d))$. The action of this operator on a function $u \in L^2_0(\mathbb{T}^d)$ is given by

$$\mathcal{R}[u] = \sum_{j,j' \in \mathbb{Z}^d \setminus \{0\}} \mathcal{R}_j^{j'} u_{j'} e^{\mathrm{i} j \cdot x} \tag{2.26}$$

where the Fourier coefficients $\mathcal{R}_j^{j'}$ of $\mathcal{R}$ are defined in (1.10). We shall identify the operator $\mathcal{R}$ with the infinite-dimensional matrix of its Fourier coefficients

$$\left(\mathcal{R}_j^{j'}\right)_{j,j' \in \mathbb{Z}^d \setminus \{0\}}. \tag{2.27}$$

We define the conjugated operator $\overline{\mathcal{R}}$ by

$$\overline{\mathcal{R}} u := \overline{\mathcal{R}\bar{u}}. \tag{2.28}$$

One gets easily that the operator $\overline{\mathcal{R}}$ has the matrix representation

$$\left(\overline{\mathcal{R}_{-j}^{-j'}}\right)_{j,j' \in \mathbb{Z}^d \setminus \{0\}}. \tag{2.29}$$

An operator $\mathcal{R}$ is said to be real if it maps real-valued functions on real valued functions and it is easy to see that $\mathcal{R}$ is real if and only if $\mathcal{R} = \overline{\mathcal{R}}$.
We define also the transpose operator $\mathcal{R}^T$ by the relation

$$\langle \mathcal{R}[u], v \rangle_{L^2_x} = \langle u, \mathcal{R}^T[v] \rangle_{L^2_x}, \qquad \forall u, v \in L^2_0(\mathbb{T}^d), \quad \forall \varphi \in \mathbb{T}^\nu \tag{2.30}$$

where

$$\langle u, v \rangle_{L^2_x} := \int_{\mathbb{T}^d} u(x) v(x), \mathrm{d}x, \qquad \forall u, v \in L^2_0(\mathbb{T}^d). \tag{2.31}$$

The operator $\mathcal{R}^T$ has the matrix representation

$$(\mathcal{R}^T)_j^{j'} = \mathcal{R}_{-j'}^{-j}, \quad \forall j, j' \in \mathbb{Z}^d. \tag{2.32}$$

An operator $\mathcal{R}$ is said to be symmetric in $\mathcal{R} = \mathcal{R}^T$.
We define also the adjoint operator $\mathcal{R}^*$ as

$$(\mathcal{R}[u], v)_{L^2_x} = (u, \mathcal{R}^*[v])_{L^2_x}, \quad \forall u, v \in L^2_0(\mathbb{T}^d), \tag{2.33}$$

where $(\cdot,\,\cdot)_{L^2_x}$ is the scalar product on $L^2_0(\mathbb{T}^d)$, namely

$$(u,\,v)_{L^2_x} := \langle u,\,\overline{v}\rangle_{L^2_x} = \int_{\mathbb{T}^d} u(x)\overline{v(x)}\,\mathrm{d}x, \quad \forall u, v \in L^2_0(\mathbb{T}^d). \tag{2.34}$$

An operator $\mathcal{R}$ is said to be self-adjoint if $\mathcal{R} = \mathcal{R}^*$. It is easy to see that $\mathcal{R}^* = \overline{\mathcal{R}}^T$ and its matrix representation is given by

$$(\mathcal{R}^*)_j^{j'} = \overline{\mathcal{R}_{j'}^j}, \quad \forall j, j' \in \mathbb{Z}^d \setminus \{0\}.$$

We also define the *commutator* between two linear operators $\mathcal{R}, \mathcal{T} \in \mathcal{B}(L^2_0(\mathbb{T}^d))$ by $[\mathcal{R}, \mathcal{T}] := \mathcal{R}\mathcal{T} - \mathcal{T}\mathcal{R}$.

In the following we also deal with real operators $G \in \mathcal{B}\big(L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})\big)$, of the form

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.35}$$

where $A, B, C, D \in \mathcal{B}(L^2_0(\mathbb{T}^d, \mathbb{R}))$. By (2.30), the transpose operator G^T with respect to the bilinear form

$$\langle (v_1, \psi_1),\, (v_2, \psi_2)\rangle_{L^2_x} := \langle v_1, v_2\rangle_{L^2_x} + \langle \psi_1,\, \psi_2\rangle_{L^2_x}, \tag{2.36}$$

$\forall (u_1, \psi_1), (u_2, \psi_2) \in L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})$, is given by

$$G^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}. \tag{2.37}$$

Then it is easy to verify that G is symmetric, that is $G = G^T$ if and only if $A = A^T$, $B = C^T$, $D = D^T$. It is also convenient to regard the real operator G in the complex variables

$$(v, \psi) = \mathcal{C}[(u, \overline{u})], \quad (u, \overline{u}) = \mathcal{C}^{-1}[(v, \psi)] \tag{2.38}$$

where

$$\mathcal{C} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \frac{1}{\mathrm{i}} & -\frac{1}{\mathrm{i}} \end{pmatrix} \qquad \mathcal{C}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}. \tag{2.39}$$

The operators $\mathcal{C}, \mathcal{C}^{-1}$ satisfies

$$\mathcal{C} : \mathbf{L}^2_0(\mathbb{T}^d) \to L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R}), \quad \mathcal{C}^{-1} : L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R}) \to \mathbf{L}^2_0(\mathbb{T}^d)$$

where $\mathbf{L}^2_0(\mathbb{T}^d)$ is the real subspace of $L^2_0(\mathbb{T}^d) \times L^2_0(\mathbb{T}^d)$ defined by

$$\mathbf{L}^2_0(\mathbb{T}^d) := \left\{(u, \overline{u}) : u \in L^2_0(\mathbb{T}^d)\right\}. \tag{2.40}$$

If $G \in \mathcal{B}\left(L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})\right)$ is a real operator of the form (2.35), one has that the conjugated operator

$$\mathcal{R} := \mathcal{C}^{-1} G \mathcal{C} : \mathbf{L}^2_0(\mathbb{T}^d) \to \mathbf{L}^2_0(\mathbb{T}^d)$$

has the form

$$\mathcal{R} = \begin{pmatrix} R_1 & R_2 \\ \overline{R}_2 & \overline{R}_1 \end{pmatrix}, \quad R_1 := \frac{A + D - \mathrm{i}(B - C)}{2}, \quad R_2 := \frac{A - D + \mathrm{i}(B + C)}{2}. \tag{2.41}$$

For the sequel, we also introduce for any $s \geq 0$, the real subspace of $H^s_0(\mathbb{T}^d) \times H^s_0(\mathbb{T}^d)$

$$\mathbf{H}^s_0(\mathbb{T}^d) := \left(H^s_0(\mathbb{T}^d) \times H^s_0(\mathbb{T}^d)\right) \cap \mathbf{L}^2_0(\mathbb{T}^d) \tag{2.42}$$

and we set

$$\|\mathbf{u}\|_{\mathbf{H}^s_x} := \|u\|_{H^s_x}, \qquad \forall \mathbf{u} = (u, \overline{u}) \in \mathbf{H}^s_0(\mathbb{T}^d). \tag{2.43}$$

It is straightforward to verify that for any $s \geq 0$

$$\mathcal{C} : \mathbf{H}^s_0(\mathbb{T}^d) \to H^s_0(\mathbb{T}^d, \mathbb{R}) \times H^s_0(\mathbb{T}^d, \mathbb{R}), \quad \mathcal{C}^{-1} : H^s_0(\mathbb{T}^d, \mathbb{R}) \times H^s_0(\mathbb{T}^d, \mathbb{R}) \to \mathbf{H}^s_0(\mathbb{T}^d). \tag{2.44}$$

2.2 Block representation of linear operators

We may regard an operator $\mathcal{R} : L^2_0(\mathbb{T}^d) \to L^2_0(\mathbb{T}^d)$ as a block matrix

$$\left([\mathcal{R}]^\beta_\alpha\right)_{\alpha, \beta \in \sigma_0(\sqrt{-\Delta})} \tag{2.45}$$

where for all $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$ (recall (2.2)), the block-matrix $[\mathcal{R}]^\beta_\alpha$ is defined by

$$[\mathcal{R}]^\beta_\alpha := \left(\mathcal{R}^{j'}_j\right)_{|j| = \alpha, \, |j'| = \beta}. \tag{2.46}$$

The operator $[\mathcal{R}]^\beta_\alpha$ is a linear operator from $\mathbb{E}_\beta$ onto $\mathbb{E}_\alpha$ where for all $\alpha \in \sigma_0(\sqrt{-\Delta})$, the finite dimensional space $\mathbb{E}_\alpha$ is defined in (2.3). We identify the space $\mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ of the linear operators from $\mathbb{E}_\beta$ onto $\mathbb{E}_\alpha$ with the space of the matrices of their Fourier coefficients, namely

$$\mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \simeq \left\{ M = \left(M_j^{j'} \right)_{\substack{j, j' \in \mathbb{Z}^d \setminus \{0\} \\ |j| = \alpha,\, |j'| = \beta}} \right\}. \tag{2.47}$$

Indeed if $M \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$, its action is given by

$$Mu(x) = \sum_{\substack{|j| = \alpha \\ |j'| = \beta}} M_j^{j'} u_{j'} \mathrm{e}^{\mathrm{i} j \cdot x}, \quad \forall u \in \mathbb{E}_\beta, \quad u(x) = \sum_{|j'| = \beta} u_{j'} \mathrm{e}^{\mathrm{i} j' \cdot x}. \tag{2.48}$$

If $\beta = \alpha$, we use the notation $\mathcal{B}(\mathbb{E}_\alpha) = \mathcal{B}(\mathbb{E}_\alpha, \mathbb{E}_\alpha)$ and we denote by $\mathbb{I}_\alpha$ the identity operator on the space $\mathbb{E}_\alpha$, namely

$$\mathbb{I}_\alpha : \mathbb{E}_\alpha \to \mathbb{E}_\alpha, \qquad u \mapsto u. \tag{2.49}$$

According to (2.4), (2.45), and (2.48), we may write the action of an operator $\mathcal{R}$ on a function $u(x)$ as

$$\mathcal{R}u = \sum_{\alpha, \beta \in \sigma_0(\sqrt{-\Delta})} [\mathcal{R}]_\alpha^\beta [\mathrm{u}_\beta]. \tag{2.50}$$

If $[\mathcal{R}]_\alpha^\beta = 0$, for any $\alpha \neq \beta$, we say that $\mathcal{R}$ is *block-diagonal* and we use the notation

$$\mathcal{R} = \mathrm{diag}_{\alpha \in \sigma_0(\sqrt{-\Delta})} [\mathcal{R}]_\alpha^\alpha. \tag{2.51}$$

The action of a block-diagonal operator $\mathcal{R}$ on a function $u \in L_0^2(\mathbb{T}^d)$ is given by

$$\mathcal{R}u = \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} [\mathcal{R}]_\alpha^\alpha [\mathrm{u}_\alpha]. \tag{2.52}$$

Let $M \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$. The transpose operator $M^T \in \mathcal{B}(\mathbb{E}_\alpha, \mathbb{E}_\beta)$ has the matrix representation

$$(M^T)_j^{j'} := M_{-j'}^{-j}, \quad |j| = \beta, \quad |j'| = \alpha. \tag{2.53}$$

The conjugate operator $\overline{M} \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ is given by

$$(\overline{M})_j^{j'} := \overline{M_{-j}^{-j'}}, \quad |j| = \alpha, \quad |j'| = \beta \tag{2.54}$$

and the adjoint operator $M^* \in \mathcal{B}(\mathbb{E}_\alpha, \mathbb{E}_\beta)$ by

$$M^* := \overline{M}^T. \tag{2.55}$$

Let $\alpha, \beta, \lambda \in \sigma_0(\sqrt{-\Delta})$. Given $A \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$, $B \in \mathcal{B}(\mathbb{E}_\lambda, \mathbb{E}_\beta)$, the operator $AB \in \mathcal{B}(\mathbb{E}_\lambda, \mathbb{E}_\alpha)$ has the matrix representation

$$(AB)_j^{j'} := \sum_{|k|=\beta} A_j^k B_k^{j'}, \quad \forall |j| = \alpha, \quad |j'| = \lambda. \tag{2.56}$$

Given an operator $A \in \mathcal{B}(\mathbb{E}_\alpha)$, we define its trace as

$$\mathrm{Tr}(A) := \sum_{|j|=\alpha} A_j^j. \tag{2.57}$$

It is easy to check that if $A, B \in \mathcal{B}(\mathbb{E}_\alpha)$, then

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA). \tag{2.58}$$

For all $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$, the space $\mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ defined in (2.47), is a Hilbert space equipped by the inner product given for any $X, Y \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ by

$$\langle X, Y \rangle := \mathrm{Tr}(XY^*). \tag{2.59}$$

This scalar product induces the *Hilbert-Schmidt* norm

$$\|X\|_{HS} := \sqrt{\mathrm{Tr}(XX^*)} = \left(\sum_{\substack{|j|=\alpha \\ |j'|=\beta}} |X_j^{j'}|^2 \right)^{\frac{1}{2}}. \tag{2.60}$$

For any operator $X \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$, we define also the operator norm as

$$\|X\|_{\mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)} := \sup \left\{ \|Xu\|_{L^2} : u \in \mathbb{E}_\beta, \quad \|u\|_{L^2} \leq 1 \right\}. \tag{2.61}$$

First we recall some preliminary properties of these norms.

Lemma 2.4. (i) Let $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$, $M \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ and $u \in \mathbb{E}_\beta$. Then $\|Mu\|_{L^2} \leq \|M\|_{HS} \|u\|_{L^2}$, implying that $\|M\|_{\mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)} \leq \|M\|_{HS}$.
(ii) Let $\alpha, \beta, \lambda \in \sigma_0(\sqrt{-\Delta})$, $M \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$, $X \in \mathcal{B}(\mathbb{E}_\lambda, \mathbb{E}_\beta)$. Then $\|MX\|_{HS} \leq \|M\|_{HS} \|X\|_{HS}$. □

Proof. The proof is a straightforward application of the Cauchy-Schwartz inequality. ■

Given a linear operator $L : \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \to \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$, we denote by $\|L\|_{\mathrm{Op}(\alpha,\beta)}$ its operator norm, when the space $\mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ is equipped with the Hilbert-Schmidt norm (2.60), namely

$$\|L\|_{\mathrm{Op}(\alpha,\beta)} := \sup\Big\{\|L(M)\|_{HS} : M \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha), \quad \|M\|_{HS} \le 1\Big\}. \tag{2.62}$$

We denote by $\mathbb{I}_{\alpha,\beta}$ the identity operator on $\mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$, namely

$$\mathbb{I}_{\alpha,\beta} : \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \to \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha), \qquad X \mapsto X. \tag{2.63}$$

For any operator $A \in \mathcal{B}(\mathbb{E}_\alpha)$ we denote by $M_L(A) : \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \to \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ the linear operator defined for any $X \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ as

$$M_L(A)X := AX. \tag{2.64}$$

Similarly, given an operator $B \in \mathcal{B}(\mathbb{E}_\beta)$, we denote by $M_R(B) : \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \to \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ the linear operator defined for any $X \in \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ as

$$M_R(B)X := XB. \tag{2.65}$$

By Lemma 2.4-(ii), we have

$$\|M_L(A)\|_{\mathrm{Op}(\alpha,\beta)} \le \|A\|_{HS}, \quad \|M_R(B)\|_{\mathrm{Op}(\alpha,\beta)} \le \|B\|_{HS}. \tag{2.66}$$

For any $\alpha \in \sigma_0(\sqrt{-\Delta})$, we denote by $\mathcal{S}(\mathbb{E}_\alpha)$, the set of the self-adjoint operators form $\mathbb{E}_\alpha$ onto itself, namely

$$\mathcal{S}(\mathbb{E}_\alpha) := \{A \in \mathcal{B}(\mathbb{E}_\alpha) : A = A^*\} \tag{2.67}$$

and given $A \in \mathcal{B}(\mathbb{E}_\alpha)$ denote by $\mathrm{spec}(A)$ the spectrum of A. The next Lemma follows by elementary arguments of linear algebra and hence its proof is omitted.

Lemma 2.5. Let $A \in \mathcal{S}(\mathbb{E}_\alpha)$, $B \in \mathcal{S}(\mathbb{E}_\beta)$, then the following holds:
(i) The operators $M_L(A)$, $M_R(B)$ defined in (2.64) and (2.65) are self-adjoint operators with respect to the scalar product defined in (2.59).
(ii) The spectrum of the operator $M_L(A) \pm M_R(B)$ satisfies

$$\mathrm{spec}\,(M_L(A) \pm M_R(B)) = \{\lambda \pm \mu : \lambda \in \mathrm{spec}(A), \quad \mu \in \mathrm{spec}(B)\}. \qquad \square$$

We also deal with smooth φ-dependent families of linear operators

$$\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(L^2_0(\mathbb{T}^d)), \quad \varphi \mapsto \mathcal{R}(\varphi). \tag{2.68}$$

According to (2.27), for any $\varphi \in \mathbb{T}^\nu$, the operator $\mathcal{R}(\varphi)$ has the matrix representation $(\mathcal{R}_j^{j'}(\varphi))_{j,j'\in\mathbb{Z}^d\setminus\{0\}}$. We can write the Fourier expansions

$$\mathcal{R}(\varphi) = \sum_{\ell\in\mathbb{Z}^\nu} \widehat{\mathcal{R}}(\ell)e^{\mathrm{i}\ell\cdot\varphi}, \quad \mathcal{R}_j^{j'}(\varphi) = \sum_{\ell\in\mathbb{Z}^\nu} \widehat{\mathcal{R}}_j^{j'}(\ell)\mathrm{e}^{\mathrm{i}\ell\cdot\varphi}, \quad \forall \ell \in \mathbb{Z}^\nu, \quad \forall j, j' \in \mathbb{Z}^d \setminus \{0\}$$

where

$$\widehat{\mathcal{R}}(\ell) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} \mathcal{R}(\varphi)\mathrm{e}^{-\mathrm{i}\ell\cdot\varphi}\, \mathrm{d}\varphi \in \mathcal{B}(L^2_0(\mathbb{T}^d)), \quad \forall \ell \in \mathbb{Z}^\nu, \tag{2.69}$$

$$\widehat{\mathcal{R}}_j^{j'}(\ell) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} \mathcal{R}_j^{j'}(\varphi)\mathrm{e}^{-\mathrm{i}\ell\cdot\varphi}\, \mathrm{d}\varphi, \qquad \forall \ell \in \mathbb{Z}^\nu, \quad \forall j, j' \in \mathbb{Z}^d \setminus \{0\}. \tag{2.70}$$

For any $\ell \in \mathbb{Z}^\nu$, the operator $\widehat{\mathcal{R}}(\ell) \in \mathcal{B}(L^2_0(\mathbb{T}^d))$ has the matrix representation

$$\widehat{\mathcal{R}}(\ell) = \left(\widehat{\mathcal{R}}_j^{j'}(\ell)\right)_{j,j'\in\mathbb{Z}^d\setminus\{0\}}. \tag{2.71}$$

Furthermore, by (2.45), for any $\varphi \in \mathbb{T}^\nu$, the operator $\mathcal{R}(\varphi)$ has the block representation $([\mathcal{R}(\varphi)]_\alpha^\beta)_{\alpha,\beta\in\sigma_0(\sqrt{-\Delta})}$ and for any $\ell \in \mathbb{Z}^\nu$, $\widehat{\mathcal{R}}(\ell)$ has the block representation $([\widehat{\mathcal{R}}(\ell)]_\alpha^\beta)_{\alpha,\beta\in\sigma_0(\sqrt{-\Delta})}$. For any $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$, we have the Fourier expansion $[\mathcal{R}(\varphi)]_\alpha^\beta = \sum_{\ell\in\mathbb{Z}^\nu}[\widehat{\mathcal{R}}(\ell)]_\alpha^\beta \mathrm{e}^{\mathrm{i}\ell\cdot\varphi}$ with

$$[\widehat{\mathcal{R}}(\ell)]_\alpha^\beta := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} [\mathcal{R}(\varphi)]_\alpha^\beta \mathrm{e}^{-\mathrm{i}\ell\cdot\varphi}\, \mathrm{d}\varphi = \left(\widehat{\mathcal{R}}_j^{j'}(\ell)\right)_{|j|=\alpha,\, |j'|=\beta} \qquad \forall \ell \in \mathbb{Z}^\nu, \tag{2.72}$$

recall (2.70).

Let $\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(L^2_0(\mathbb{T}^d))$ be differentiable and let $\omega \in \mathbb{R}^\nu$. For any $\varphi \in \mathbb{T}^\nu$, the operator $\omega\cdot\partial_\varphi\mathcal{R}(\varphi)$ is represented by the matrix $(\omega\cdot\partial_\varphi\mathcal{R}_j^{j'}(\varphi))_{j,j'\in\mathbb{Z}^d\setminus\{0\}}$ and its block representation is given by $(\omega\cdot\partial_\varphi[\mathcal{R}(\varphi)]_\alpha^\beta)_{\alpha,\beta\in\sigma_0(\sqrt{-\Delta})}$. We also note that for any $\ell \in \mathbb{Z}^\nu$, the operator $\widehat{\omega\cdot\partial_\varphi\mathcal{R}}(\ell)$ admits the block representation $(\mathrm{i}\omega\cdot\ell[\widehat{\mathcal{R}}(\ell)]_\alpha^\beta)_{\alpha,\beta\in\sigma_0(\sqrt{-\Delta})}$.

Given $\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(L^2_0(\mathbb{T}^d))$, recalling the notation (2.51), we define the block-diagonal operator $\mathcal{R}_{diag}$ as

$$\mathcal{R}_{diag} := \mathrm{diag}_{\alpha\in\sigma_0(\sqrt{-\Delta})}[\widehat{\mathcal{R}}(0)]_\alpha^\alpha \tag{2.73}$$

and for any $N \in \mathbb{N}$, we define the smoothing operator $\Pi_N\mathcal{R}$ by

$$[\widehat{\Pi_N\mathcal{R}}(\ell)]_\alpha^\beta := \begin{cases} [\widehat{\mathcal{R}}(\ell)]_\alpha^\beta & \text{if} \quad \max\{|\ell|, \alpha, \beta\} \leq N \\ 0 & \text{otherwise.} \end{cases} \tag{2.74}$$

It is straightforward to verify that

$$(\Pi_N \mathcal{R})_{diag} = \Pi_N \mathcal{R}_{diag}. \tag{2.75}$$

2.3 Block-decay norm for linear operators

Given a smooth φ-dependent family $\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(L^2_0(\mathbb{T}^d))$, $\varphi \mapsto \mathcal{R}(\varphi)$ as in (2.68), we define the *block-decay* norm

$$|\mathcal{R}|_s := \sup_{\alpha,\beta \in \sigma_0(\sqrt{-\Delta})} \Big(\sum_{\ell \in \mathbb{Z}^\nu} \langle \ell, \alpha, \beta \rangle^{2s} \| [\widehat{\mathcal{R}}(\ell)]_\alpha^\beta \|_{HS}^2 \Big)^{1/2}, \quad \langle \ell, \alpha, \beta \rangle := \max\{1, |\ell|, \alpha, \beta\}. \tag{2.76}$$

For families of operators of the form $\mathcal{R}(\omega) : \varphi \mapsto \mathcal{R}(\varphi; \omega)$, $\omega \in \Omega_o \subset \mathbb{R}^\nu$, we define the norm

$$|\mathcal{R}|_s^{\mathrm{Lip}(\gamma)} := |\mathcal{R}|_s^{\sup} + \gamma |\mathcal{R}|_s^{\mathrm{lip}}, \tag{2.77}$$
$$|\mathcal{R}|_s^{\sup} := \sup_{\omega \in \Omega_0} |\mathcal{R}(\omega)|_s, \quad |\mathcal{R}|_s^{\mathrm{lip}} := \sup_{\substack{\omega_1, \omega_2 \in \Omega_o \\ \omega_1 \neq \omega_2}} \frac{|\mathcal{R}(\omega_1) - \mathcal{R}(\omega_2)|_s}{|\omega_1 - \omega_2|}.$$

Moreover, if $\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(\mathbf{L}^2_0(\mathbb{T}^d))$, that is, $\mathcal{R}$ has the form

$$\mathcal{R}(\varphi) = \begin{pmatrix} \mathcal{R}_1(\varphi) & \mathcal{R}_2(\varphi) \\ \overline{\mathcal{R}_2(\varphi)} & \overline{\mathcal{R}_1(\varphi)} \end{pmatrix}, \tag{2.78}$$

we define

$$|\mathcal{R}|_s := |\mathcal{R}_1|_s + |\mathcal{R}_2|_s, \qquad |\mathcal{R}|_s^{\mathrm{Lip}(\gamma)} := |\mathcal{R}_1|_s^{\mathrm{Lip}(\gamma)} + |\mathcal{R}_2|_s^{\mathrm{Lip}(\gamma)}. \tag{2.79}$$

In the following, we state some properties of this norm. We prove such properties for families of operators $\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(L^2_0(\mathbb{T}^d))$. If $\mathcal{R}$ is an operator of the form (2.78) then the same statements hold with the obvious modifications.

Lemma 2.6.

(i) The norm $|\cdot|_s$ is increasing, namely $|\mathcal{R}|_s \leq |\mathcal{R}|_{s'}$, for $s \leq s'$.

(ii) The operator $\mathcal{R}_{diag}$ defined by (2.73), satisfies $|\mathcal{R}_{diag}|_s \leq |\mathcal{R}|_s$, implying that $\|[\mathcal{R}]_\alpha^\alpha\|_{HS} \leq \alpha^{-s} |\mathcal{R}|_s$ for any $\alpha \in \sigma_0(\sqrt{-\Delta})$

(iii) Items (i), (ii) hold, replacing $|\cdot|_s$ by $|\cdot|_s^{\mathrm{Lip}(\gamma)}$. □

Proof. The proof is elementary. It follows directly by the definitions (2.76) and (2.77), hence we omit it. ■

Lemma 2.7. Let $\mathcal{R}$, $\mathcal{T}$ be operators of the form (2.78). Then for any $s \geq s_0$ (recall (2.14))

$$|\mathcal{R}\mathcal{B}|_s \lesssim_s |\mathcal{R}|_s |\mathcal{B}|_{2s_0} + |\mathcal{R}|_{2s_0} |\mathcal{B}|_s .$$

If $\mathcal{R} = \mathcal{R}(\omega)$, $\mathcal{T} = \mathcal{T}(\omega)$ are Lipschitz with respect to the parameter $\omega \in \Omega_o \subseteq \Omega$, then the same estimate holds replacing $|\cdot|_s$ by $|\cdot|_s^{\mathrm{Lip}(\gamma)}$. □

Proof. According to the notations (2.45) and (2.46), for any $\varphi \in \mathbb{T}^\nu$, the operator $\mathcal{R}(\varphi)\mathcal{B}(\varphi)$ has the block representation

$$\mathcal{R}(\varphi)\mathcal{T}(\varphi) = \big([\mathcal{R}(\varphi)\mathcal{T}(\varphi)]_\alpha^\beta\big)_{\substack{\alpha,\beta \in \sigma_0(\sqrt{-\Delta}) \\ \ell \in \mathbb{Z}^\nu}}, \quad [\mathcal{R}(\varphi)\mathcal{T}(\varphi)]_\alpha^\beta = \sum_{\alpha_1 \in \sigma_0(\sqrt{-\Delta})} [\mathcal{R}(\varphi)]_\alpha^{\alpha_1} [\mathcal{T}(\varphi)]_{\alpha_1}^\beta$$

and for all $\ell \in \mathbb{Z}^\nu$

$$[\widehat{\mathcal{R}\mathcal{T}}(\ell)]_\alpha^\beta = \sum_{\alpha_1 \in \sigma_0(\sqrt{-\Delta}), \ell' \in \mathbb{Z}^\nu} [\widehat{\mathcal{R}}(\ell - \ell')]_\alpha^{\alpha_1} [\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta .$$

Then, using Lemma 2.4-(ii), we get that for any $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$

$$\sum_{\ell \in \mathbb{Z}^\nu} \langle \ell, \alpha, \beta \rangle^{2s} \|[\widehat{\mathcal{R}\mathcal{T}}(\ell)]_\alpha^\beta\|_{HS}^2 \leq \sum_{\ell \in \mathbb{Z}^\nu} \Bigg(\sum_{\substack{\ell' \in \mathbb{Z}^\nu \\ \alpha_1 \in \sigma_0(\sqrt{-\Delta})}} \langle \ell, \alpha, \beta \rangle^s \|[\widehat{\mathcal{R}}(\ell - \ell')]_\alpha^{\alpha_1}\|_{HS} \|[\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta\|_{HS} \Bigg)^2 . \quad (2.80)$$

Using that for any $\alpha, \beta, \alpha_1 \in \sigma_0(\sqrt{-\Delta})$, $\ell, \ell' \in \mathbb{Z}^\nu$, $\langle \ell, \alpha, \beta \rangle^s \lesssim_s \langle \ell - \ell', \alpha, \alpha_1 \rangle^s + \langle \ell', \alpha_1, \beta \rangle^s$, we get

$$(2.80) \lesssim_s (I) + (II) \quad (2.81)$$

where

$$(I) := \sum_{\ell \in \mathbb{Z}^\nu} \Bigg(\sum_{\substack{\ell' \in \mathbb{Z}^\nu \\ \alpha_1 \in \sigma_0(\sqrt{-\Delta})}} \langle \ell - \ell', \alpha, \alpha_1 \rangle^s \|[\widehat{\mathcal{R}}(\ell - \ell')]_\alpha^{\alpha_1}\|_{HS} \|[\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta\|_{HS} \Bigg)^2 \quad (2.82)$$

$$(II) := \sum_{\ell \in \mathbb{Z}^\nu} \Bigg(\sum_{\substack{\ell' \in \mathbb{Z}^\nu \\ \alpha_1 \in \sigma_0(\sqrt{-\Delta})}} \langle \ell', \alpha_1, \beta \rangle^s \|[\widehat{\mathcal{R}}(\ell - \ell')]_\alpha^{\alpha_1}\|_{HS} \|[\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta\|_{HS} \Bigg)^2 . \quad (2.83)$$

Using that, by Lemma A.1-(i), $\sum_{\substack{\ell'\in\mathbb{Z}^\nu\\ \alpha_1\in\sigma_0(\sqrt{-\Delta})}}\langle \ell',\alpha_1\rangle^{-2s_0}$, $\sum_{\alpha_1\in\sigma_0(\sqrt{-\Delta})}\alpha_1^{-2s_0} < +\infty$ (recall that $s_0 > (\nu+d)/2$), applying the Cauchy Schwartz inequality, one gets

$$
\begin{aligned}
(I) &\lesssim \sum_{\ell\in\mathbb{Z}^\nu}\sum_{\substack{\ell'\in\mathbb{Z}^\nu\\ \alpha_1\in\sigma_0(\sqrt{-\Delta})}} \langle \ell-\ell',\alpha,\alpha_1\rangle^{2s}\|[\widehat{\mathcal{R}}(\ell-\ell')]_\alpha^{\alpha_1}\|_{HS}^2\langle \ell',\alpha_1\rangle^{2s_0}\|[\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta\|_{HS}^2\\
&\lesssim_s \sum_{\substack{\ell'\in\mathbb{Z}^\nu\\ \alpha_1\in\sigma_0(\sqrt{-\Delta})}} \langle \ell',\alpha_1\rangle^{2s_0}\|[\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta\|_{HS}^2\sum_{\ell\in\mathbb{Z}^\nu}\langle \ell-\ell',\alpha,\alpha_1\rangle^{2s}\|[\widehat{\mathcal{R}}(\ell-\ell')]_\alpha^{\alpha_1}\|_{HS}^2\\
&\lesssim_s \sum_{\substack{\ell'\in\mathbb{Z}^\nu\\ \alpha_1\in\sigma_0(\sqrt{-\Delta})}} \frac{1}{\alpha_1^{2s_0}}\langle \ell',\alpha_1\rangle^{4s_0}\|[\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta\|_{HS}^2\sum_{\ell\in\mathbb{Z}^\nu}\langle \ell-\ell',\alpha,\alpha_1\rangle^{2s}\|[\widehat{\mathcal{R}}(\ell-\ell')]_\alpha^{\alpha_1}\|_{HS}^2\\
&\lesssim_s \sum_{\alpha_1\in\sigma_0(\sqrt{-\Delta})}\alpha_1^{-2s_0}\Big(\sup_{\alpha_1\in\sigma_0(\sqrt{-\Delta})}\sum_{\ell'\in\mathbb{Z}^\nu}\langle \ell',\alpha_1\rangle^{4s_0}\|[\widehat{\mathcal{T}}(\ell')]_{\alpha_1}^\beta\|_{HS}^2\Big)\\
&\quad\times\Big(\sup_{\alpha,\alpha_1\in\sigma_0(\sqrt{-\Delta})}\sum_{k\in\mathbb{Z}^\nu}\langle k,\alpha,\alpha_1\rangle^{2s}\|[\widehat{\mathcal{R}}(k)]_\alpha^{\alpha_1}\|_{HS}^2\Big)\\
&\overset{(2.76)}{\lesssim_s} |\mathcal{B}|_{2s_0}^2|\mathcal{R}|_s^2.
\end{aligned}
\tag{2.84}
$$

Similarly one proves that $(II)\lesssim_s|\mathcal{T}|_s^2|\mathcal{R}|_{2s_0}^2$ and then, recalling (2.80), (2.81) one proves $|\mathcal{R}\mathcal{T}|_s\lesssim_s|\mathcal{T}|_{2s_0}|\mathcal{R}|_s+|\mathcal{T}|_s|\mathcal{R}|_{2s_0}$. The estimate for the norm $|\cdot|_s^{\mathrm{Lip}(\gamma)}$ follows easily by the previous one, by applying the triangular inequality. ■

For all $n\geq 1$, iterating the estimate of Lemma 2.7 we get

$$
|\mathcal{R}^n|_{2s_0}\leq [C(s_0)]^{n-1}|\mathcal{R}|_{2s_0}^n \quad \text{and} \quad |\mathcal{R}^n|_s\leq nC(s)^n|\mathcal{R}|_{2s_0}^{n-1}|\mathcal{R}|_s,\ \forall s\geq 2s_0, \tag{2.85}
$$

and the same bounds also hold for the norm $|\cdot|_s^{\mathrm{Lip}(\gamma)}$ if $\mathcal{R}$ is Lipschitz continuous with respect to the parameter ω.

Lemma 2.8. Let $\Phi=\exp(\Psi)$ with $\Psi:=\Psi(\omega)$, depending in a Lipschitz way on the parameter $\omega\in\Omega_o\subset\mathbb{R}$, such that $|\Psi|_{2s_0}^{\mathrm{Lip}(\gamma)}\leq 1$, $|\Psi|_s^{\mathrm{Lip}(\gamma)}<+\infty$, with $s\geq 2s_0$. Then

$$
|\Phi^{\pm1}-\mathrm{Id}|_s\lesssim_s|\Psi|_s,\quad |\Phi^{\pm1}-\mathrm{Id}|_s^{\mathrm{Lip}(\gamma)}\lesssim_s|\Psi|_s^{\mathrm{Lip}(\gamma)}. \tag{2.86}
$$

□

Proof. The claimed estimates can be proved by using the Taylor expansion of $\Phi^{\pm1}-\mathrm{Id}=\exp(\pm\Psi)-\mathrm{Id}$, using the condition $|\Psi|_{2s_0}^{\mathrm{Lip}(\gamma)}\leq 1$ and by applying the estimates (2.85). ■

Lemma 2.9. The operator $\Pi_N^\perp \mathcal{R} := \mathcal{R} - \Pi_N \mathcal{R}$ (recall (2.74)) satisfies

$$|\Pi_N^\perp \mathcal{R}|_s \leq N^{-\mathtt{b}} |\mathcal{R}|_{s+\mathtt{b}}, \quad |\Pi_N^\perp \mathcal{R}|_s^{\mathrm{Lip}(\gamma)} \leq N^{-\mathtt{b}} |\mathcal{R}|_{s+\mathtt{b}}^{\mathrm{Lip}(\gamma)}, \quad \mathtt{b} \geq 0, \tag{2.87}$$

where in the second inequality $\mathcal{R}$ is Lipschitz with respect to the parameter $\omega \in \Omega_o \subseteq \Omega$. □

Proof. We have that for all $\mathtt{b} \in \mathbb{N}$, $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$

$$\begin{aligned}\sum_{\ell \in \mathbb{Z}^\nu} \langle \ell, \alpha, \beta \rangle^{2s} \| [\widehat{\Pi_N^\perp \mathcal{R}(\ell)}]_\alpha^\beta \|_{HS}^2 &\overset{(2.74)}{=} \sum_{\{\ell : \langle \ell, \alpha, \beta \rangle > N\}} \langle \ell, \alpha, \beta \rangle^{2s} \| [\widehat{\mathcal{R}}(\ell)]_\alpha^\beta \|_{HS}^2 \\ &\leq N^{-2\mathtt{b}} \sum_{\ell \in \mathbb{Z}^\nu} \langle \ell, \alpha, \beta \rangle^{2(s+\mathtt{b})} \| [\widehat{\mathcal{R}}(\ell)]_\alpha^\beta \|_{HS}^2 \overset{(2.76)}{\leq} N^{-2\mathtt{b}} |\mathcal{R}|_{s+\mathtt{b}}^2,\end{aligned}$$

and the lemma follows. ■

Lemma 2.10. Let us define the operator

$$\mathcal{R}(\varphi)[h] := q(\varphi, x) \int_{\mathbb{T}^d} g(\varphi, y) h(y)\, dy, \quad h \in L_0^2(\mathbb{T}^d) \quad q, g \in H_0^s(\mathbb{T}^{\nu+d}), \quad s \geq s_0. \tag{2.88}$$

Then

$$|\mathcal{R}|_s \lesssim_s \|g\|_{s_0} \|q\|_s + \|g\|_{s+s_0} \|q\|_0.$$

Moreover, if the functions g and q are Lipschitz with respect to the parameter $\omega \in \Omega_o \subseteq \Omega$, then the same estimate holds replacing $|\cdot|_s$ by $|\cdot|_s^{\mathrm{Lip}(\gamma)}$ and $\|\cdot\|_s$ by $\|\cdot\|_s^{\mathrm{Lip}(\gamma)}$. □

Proof. A direct calculation shows that for all $\ell \in \mathbb{Z}^\nu$ and for all $j, j' \in \mathbb{Z}^d \setminus \{0\}$

$$\widehat{\mathcal{R}}_j^{j'}(\ell) = \sum_{\ell' \in \mathbb{Z}^\nu} \widehat{q}_j(\ell - \ell') \widehat{g}_{-j'}(\ell').$$

Using definition (2.60), the Cauchy–Schwartz inequality (using that $\sum_{\ell' \in \mathbb{Z}^\nu} \langle \ell' \rangle^{-2s_0} < +\infty$) we get

$$\begin{aligned}\| [\widehat{\mathcal{R}}(\ell)]_\alpha^\beta \|_{HS}^2 &= \sum_{\substack{|j|=\alpha \\ |j'|=\beta}} |\widehat{\mathcal{R}}_j^{j'}(\ell)|^2 \leq \sum_{\substack{|j|=\alpha \\ |j'|=\beta}} \left(\sum_{\ell'} |\widehat{q}_j(\ell - \ell')| |\widehat{g}_{-j'}(\ell')| \right)^2 \\ &\leq \sum_{|j|=\alpha} \sum_{|j'|=\beta} \sum_{\ell'} |\widehat{q}_j(\ell - \ell')|^2 \langle \ell' \rangle^{2s_0} |\widehat{g}_{-j'}(\ell')|^2 \\ &\overset{(2.4),(2.8)}{=} \sum_{\ell'} \| \widehat{\mathfrak{q}}_\alpha(\ell - \ell') \|_{L^2}^2 \langle \ell' \rangle^{2s_0} \| \widehat{\mathfrak{g}}_\beta(\ell') \|_{L^2}^2.\end{aligned} \tag{2.89}$$

Now for all $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$,

$$\sum_{\ell \in \mathbb{Z}^\nu} \langle \ell, \alpha, \beta \rangle^{2s} \| [\widehat{\mathcal{R}}(\ell)]_\alpha^\beta \|_{HS}^2 \overset{(2.89)}{\leq} \sum_{\ell, \ell' \in \mathbb{Z}^\nu} \langle \ell, \alpha, \beta \rangle^{2s} \| \widehat{\mathfrak{q}}_\alpha(\ell - \ell') \|_{L^2}^2 \langle \ell' \rangle^{2s_0} \| \widehat{\mathfrak{g}}_\beta(\ell') \|_{L^2}^2. \tag{2.90}$$

Using that $\langle \ell, \alpha, \beta \rangle^{2s} \lesssim_s \langle \ell - \ell', \alpha \rangle^{2s} + \langle \ell', \beta \rangle^{2s}$ we get

$$\begin{aligned}
(2.90) \lesssim_s & \sum_{\ell} \sum_{\ell'} \langle \ell - \ell', \alpha \rangle^{2s} \| \widehat{\mathfrak{q}}_\alpha(\ell - \ell') \|_{L^2}^2 \langle \ell' \rangle^{2s_0} \| \widehat{\mathfrak{g}}_\beta(\ell') \|_{L^2}^2 \\
& + \sum_{\ell} \sum_{\ell'} \langle \ell', \beta \rangle^{2s} \| \widehat{\mathfrak{q}}_\alpha(\ell - \ell') \|_{L^2}^2 \langle \ell' \rangle^{2s_0} \| \widehat{\mathfrak{g}}_\beta(\ell') \|_{L^2}^2 \\
\lesssim_s & \sum_{\ell'} \langle \ell' \rangle^{2s_0} \| \widehat{\mathfrak{g}}_\beta(\ell') \|_{L^2}^2 \sum_{\ell} \langle \ell - \ell', \alpha \rangle^{2s} \| \widehat{\mathfrak{q}}_\alpha(\ell - \ell') \|_{L^2}^2 \\
& + \sum_{\ell'} \langle \ell', \beta \rangle^{2(s+s_0)} \| \widehat{\mathfrak{g}}_\beta(\ell') \|_{L^2}^2 \sum_{\ell} \| \widehat{\mathfrak{q}}_\alpha(\ell - \ell') \|_{L^2}^2 \\
\overset{(2.10)}{\lesssim_s} & \, \| g \|_{s_0}^2 \| q \|_s^2 + \| g \|_{s+s_0}^2 \| q \|_{L^2}^2
\end{aligned} \tag{2.91}$$

and hence the lemma follows. ■

For a φ-independent linear operator $\mathcal{R} \in \mathcal{B}(L^2_0(\mathbb{T}^d))$ having the block-matrix representation (2.45), the block-decay norm (2.76) becomes

$$|\mathcal{R}|_s = \sup_{\alpha, \beta \in \sigma_0(\sqrt{-\Delta})} \langle \alpha, \beta \rangle^s \| [\mathcal{R}]_\alpha^\beta \|_{HS}, \quad \langle \alpha, \beta \rangle := \max\{\alpha, \beta\}. \tag{2.92}$$

The following Lemma holds:

Lemma 2.11. (i) Let $\mathcal{R} \in \mathcal{B}(L^2_0(\mathbb{T}^d))$ satisfy $|\mathcal{R}|_{s+2s_0} < +\infty$, for $s \geq 0$. Then $\mathcal{R} \in \mathcal{B}(L^2_0(\mathbb{T}^d), H^s_0(\mathbb{T}^d))$ and $\|\mathcal{R}\|_{\mathcal{B}(L^2_0, H^s_0)} \lesssim |\mathcal{R}|_{s+2s_0}$. As a consequence $\mathcal{R} \in \mathcal{B}(H^s_0)$, with $\|\mathcal{R}\|_{\mathcal{B}(H^s_0)} \leq \|\mathcal{R}\|_{\mathcal{B}(L^2_0, H^s_0)} \lesssim |\mathcal{R}|_{s+2s_0}$.
(ii) Let $k \in \mathbb{N}$ and $\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(L^2_0(\mathbb{T}^d))$ with $|\mathcal{R}|_{s+k+2s_0} < +\infty$. Then $\mathcal{R} \in W^{k,\infty}\big(\mathbb{T}^\nu, \mathcal{B}(L^2_0, H^s_0)\big)$ and for any $a \in \mathbb{N}^\nu$, $|a| \leq k$, one has

$$\|\partial_\varphi^a \mathcal{R}\|_{L^\infty(\mathbb{T}^\nu, \mathcal{B}(H^s_0))} \lesssim \sup_{\varphi \in \mathbb{T}^\nu} |\partial_\varphi^a \mathcal{R}(\varphi)|_{s+2s_0} \lesssim |\mathcal{R}|_{s+|a|+2s_0}.$$ □

Proof. Proof of (i). Let $u \in L^2_0(\mathbb{T}^d)$. By (2.52) and (2.5), one has that

$$\|\mathcal{R}[u]\|_{H^s_x}^2 = \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \alpha^{2s} \Big\| \sum_{\beta \in \sigma_0(\sqrt{-\Delta})} [\mathcal{R}]_\alpha^\beta [\mathrm{u}_\beta] \Big\|_{L^2}^2 \lesssim \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \Big(\sum_{\beta \in \sigma_0(\sqrt{-\Delta})} \alpha^s \| [\mathcal{R}]_\alpha^\beta [\mathrm{u}_\beta] \|_{L^2} \Big)^2. \tag{2.93}$$

Using Lemma 2.4-(i) and recalling (2.92), one gets

$$\|\mathcal{R}[u]\|_{H^s_x}^2 \lesssim \sum_{\alpha\in\sigma_0(\sqrt{-\Delta})} \Big(\sum_{\beta\in\sigma_0(\sqrt{-\Delta})} \frac{\alpha^{s+s_0}\beta^{s_0}}{\alpha^{s_0}\beta^{s_0}} \|[\mathcal{R}]_\alpha^\beta\|_{HS} \|\mathrm{u}_\beta\|_{L^2} \Big)^2$$
$$\lesssim \sum_{\alpha\in\sigma_0(\sqrt{-\Delta})} \frac{1}{\alpha^{2s_0}} \Big(\sum_{\beta\in\sigma_0(\sqrt{-\Delta})} \frac{\langle\alpha,\beta\rangle^{s+2s_0}}{\beta^{s_0}} \|[\mathcal{R}]_\alpha^\beta\|_{HS} \|\mathrm{u}_\beta\|_{L^2} \Big)^2$$
$$\lesssim |\mathcal{R}|_{s+2s_0}^2 \sum_{\alpha\in\sigma_0(\sqrt{-\Delta})} \frac{1}{\alpha^{2s_0}} \Big(\sum_{\beta\in\sigma_0(\sqrt{-\Delta})} \frac{1}{\beta^{s_0}} \|\mathrm{u}_\beta\|_{L^2} \Big)^2 . \tag{2.94}$$

By the Cauchy–Schwartz inequality

$$(2.94) \lesssim |\mathcal{R}|_{s+2s_0}^2 \sum_{\alpha,\beta\in\sigma_0(\sqrt{-\Delta})} \frac{1}{\alpha^{2s_0}\beta^{2s_0}} \sum_{\beta\in\sigma_0(\sqrt{-\Delta})} \|\mathrm{u}_\beta\|_{L^2}^2 \overset{(2.5)}{\lesssim} |\mathcal{R}|_{s+2s_0}^2 \|u\|_{L^2} \tag{2.95}$$

by applying Lemma A.1-(i) (note that $2s_0 = 2([(\nu+d)/2]+1) > \nu+d$) and then the claim follows.

Proof of (ii).For any $\alpha,\beta\in\sigma_0(\sqrt{-\Delta})$ and for any multi-index $a\in\mathbb{N}^\nu$, $|a|\le k$ one has that the operator $\partial_\varphi^a\mathcal{R}(\varphi)$ admits the block-matrix representation

$$\partial_\varphi^a\mathcal{R}(\varphi) = \big(\partial_\varphi^a[\mathcal{R}(\varphi)]_\alpha^\beta\big)_{\alpha,\beta\in\sigma_0(\sqrt{-\Delta})} .$$

Expanding in Fourier series $\partial_\varphi^a[\mathcal{R}(\varphi)]_\alpha^\beta$, one has

$$\partial_\varphi^a[\mathcal{R}(\varphi)]_\alpha^\beta = \sum_{\ell\in\mathbb{Z}^\nu} \mathrm{i}^{|a|}\ell^a[\widehat{\mathcal{R}}(\ell)]_\alpha^\beta \mathrm{e}^{\mathrm{i}\ell\cdot\varphi},$$

and by the Cauchy–Schwartz inequality

$$\|\partial_\varphi^a[\mathcal{R}(\varphi)]_\alpha^\beta\|_{HS} \le \sum_{\ell\in\mathbb{Z}^\nu} |\ell|^{|a|} \|[\widehat{\mathcal{R}}(\ell)]_\alpha^\beta\|_{HS} \lesssim \Big(\sum_{\ell\in\mathbb{Z}^\nu} \langle\ell\rangle^{2(|a|+s_0)} \|[\widehat{\mathcal{R}}(\ell)]_\alpha^\beta\|_{HS}^2 \Big)^{\frac{1}{2}} . \tag{2.96}$$

Thus by (2.96), for any $\alpha,\beta\in\sigma_0(\sqrt{-\Delta})$, for any $\varphi\in\mathbb{T}^\nu$, one has

$$\langle\alpha,\beta\rangle^{2s}\|\partial_\varphi^a[\mathcal{R}(\varphi)]_\alpha^\beta\|_{HS}^2 \overset{(2.96)}{\lesssim} \sum_{\ell\in\mathbb{Z}^\nu} \langle\ell,\alpha,\beta\rangle^{2(s+|a|+s_0)} \|[\widehat{\mathcal{R}}(\ell)]_\alpha^\beta\|_{HS}^2 \overset{(2.76)}{\lesssim} |\mathcal{R}|_{s+|a|+s_0}^2$$

and then the lemma follows by recalling (2.92) and by applying item (i). ∎

2.4 A class of φ-dependent Fourier multipliers

For any $m \in \mathbb{R}$, we define the class S^m of Fourier multipliers of order m as

$$S^m := \left\{ r : \sigma_0(\sqrt{-\Delta}) \to \mathbb{C} : \sup_{\alpha \in \sigma_0(\sqrt{-\Delta})} |r(\alpha)| \alpha^{-m} < +\infty \right\}$$

where we recall that the set $\sigma_0(\sqrt{-\Delta})$ is defined in (2.2). To any symbol $r \in S^m$, we associate the linear operator $\mathrm{Op}(r)$ defined by

$$\mathrm{Op}(r)u(x) := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} r(|j|) u_j e^{\mathrm{i} j \cdot x}, \qquad \forall u \in H_0^m(\mathbb{T}^d). \tag{2.97}$$

We denote by OPS^m the class of the operators associated to the symbols in S^m.

In the following we deal with φ-dependent families of Fourier multipliers $r : \mathbb{T}^\nu \times \sigma_0(\sqrt{-\Delta}) \to \mathbb{C}$, $r(\varphi, \cdot) \in S^m$. The action of the operator $\mathrm{Op}(r) = \mathrm{Op}(r(\varphi, |j|))$ on Sobolev functions $u \in H_0^s(\mathbb{T}^{\nu+d})$ is given by

$$\mathrm{Op}(r)u(\varphi, x) := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} r(\varphi, |j|) u_j(\varphi) e^{\mathrm{i} j \cdot x} = \sum_{\substack{\ell, \ell' \in \mathbb{Z}^\nu \\ j \in \mathbb{Z}^d \setminus \{0\}}} \widehat{r}(\ell - \ell', |j|) \widehat{u}_j(\ell') e^{\mathrm{i}(\ell \cdot \varphi + j \cdot x)}. \tag{2.98}$$

Using the representation (2.7), the action of the operator $\mathrm{Op}(r)$ on a function $u(\varphi, x)$ can be written as

$$\mathrm{Op}(r)u(\varphi, x) = \sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} r(\varphi, \alpha) \mathrm{u}_\alpha(\varphi, x) = \sum_{\substack{\ell, \ell' \in \mathbb{Z}^\nu \\ \alpha \in \sigma_0(\sqrt{-\Delta})}} \widehat{r}(\ell - \ell', \alpha) \widehat{\mathrm{u}}_\alpha(\ell', x) e^{\mathrm{i} \ell \cdot \varphi}. \tag{2.99}$$

The following elementary properties hold:

$$\overline{\mathrm{Op}(r)} = \mathrm{Op}(\overline{r}) = \mathrm{Op}(r)^*, \quad \mathrm{Op}(r)^T = \mathrm{Op}(r) \tag{2.100}$$

(recall (2.28), (2.30), and (2.33)). The above properties imply that

$$\mathrm{Op}(r) = \mathrm{Op}(r)^* \qquad \text{if and only if} \qquad r(\varphi, \alpha) = \overline{r(\varphi, \alpha)}, \qquad \forall (\varphi, \alpha) \in \mathbb{T}^\nu \times \sigma_0(\sqrt{-\Delta}). \tag{2.101}$$

Let $\mathcal{R} = \mathrm{Op}(r) \in OPS^m$, $\mathcal{B} = \mathrm{Op}(b) \in OPS^{m'}$. Then the composition operator $\mathcal{R} \circ \mathcal{B}$ is given by

$$\mathcal{R} \circ \mathcal{B} = \mathrm{Op}(r) \circ \mathrm{Op}(b) = \mathrm{Op}(rb) \in OPS^{m+m'}. \tag{2.102}$$

Note that $\mathcal{R} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{R}$.
For an operator $\mathcal{R} = \mathrm{Op}(r) \in OPS^m$, for any $s \geq 0$, $m \in \mathbb{R}$, we define the family of norms

$$|\mathrm{Op}(r)|_{m,s} := \sup_{\alpha \in \sigma_0(\sqrt{-\Delta})} \|r(\cdot, \alpha)\|_s \alpha^{-m} \tag{2.103}$$

and if $r = r(\varphi, \alpha; \omega)$, $\omega \in \Omega_o \subseteq \Omega$ is Lipschitz with respect to the parameter $\omega \in \Omega_o$ then we define

$$|\mathrm{Op}(r)|^{\mathrm{Lip}(\gamma)}_{m,s} = |\mathrm{Op}(r)|^{\sup}_{m,s} + \gamma |\mathrm{Op}(r)|^{\mathrm{lip}}_{m,s} \tag{2.104}$$

where

$$|\mathrm{Op}(r)|^{\sup}_{m,s} := \sup_{\omega \in \Omega_o} |\mathrm{Op}(r)(\omega)|_{m,s}, \quad |\mathrm{Op}(r)|^{\mathrm{lip}}_{m,s} := \sup_{\substack{\omega_1, \omega_2 \in \Omega_o \\ \omega_1 \neq \omega_2}} \frac{|\mathrm{Op}(r)(\omega_1) - \mathrm{Op}(r)(\omega_2)|_{m,s}}{|\omega_1 - \omega_2|}.$$

We also deal with operators

$$\mathcal{R} = \begin{pmatrix} \mathrm{Op}(r_1) & \mathrm{Op}(r_2) \\ \mathrm{Op}(\overline{r}_2) & \mathrm{Op}(\overline{r}_1) \end{pmatrix}, \qquad r_1, r_2 \in S^m. \tag{2.105}$$

With a slight abuse of notations we still denote by OPS^m the class of operators of the form (2.105). For such operators, we define the norms $|\mathcal{R}|_{m,s} := |\mathrm{Op}(r_1)|_{m,s} + |\mathrm{Op}(r_2)|_{m,s}$ and $|\mathcal{R}|^{\mathrm{Lip}(\gamma)}_{m,s} := |\mathrm{Op}(r_1)|^{\mathrm{Lip}(\gamma)}_{m,s} + |\mathrm{Op}(r_2)|^{\mathrm{Lip}(\gamma)}_{m,s}$. In the following, we state some properties of the norm $|\cdot|_{m,s}$. We prove such properties for operators $\mathcal{R}(\varphi) = \mathrm{Op}(r(\varphi, \cdot))$. If $\mathcal{R}$ is an operator of the form (2.105) then the same statements hold with the obvious modifications.
It is immediate to verify that

$$|\cdot|_{m,s} \leq |\cdot|_{m,s'}, \quad \forall s \leq s', \quad \forall m \in \mathbb{R}, \tag{2.106}$$

$$|\cdot|_{m,s} \leq |\cdot|_{m',s}, \quad \forall m \geq m', \quad \forall s \geq 0 \tag{2.107}$$

and the same inequality holds for the corresponding Lipschitz norms.

Lemma 2.12. Let $\mathcal{R} = \mathrm{Op}(r)$ with $|\mathcal{R}|_{0,s} < +\infty$, $s \geq s_0$. Then for any $u \in H^s_0(\mathbb{T}^{\nu+d})$

$$\|\mathcal{R}u\|_s \lesssim_s |\mathcal{R}|_{0,s} \|u\|_{s_0} + |\mathcal{R}|_{0,s_0} \|u\|_s.$$

The same statements hold, replacing $\|\cdot\|_s$ by $\|\cdot\|^{\mathrm{Lip}(\gamma)}_s$ and $|\cdot|_{0,s}$ by $|\cdot|^{\mathrm{Lip}(\gamma)}_{0,s}$. If $\mathcal{R}$ is an operator of the form (2.105), then a similar estimate holds. □

Proof. The claimed estimate follows by the same arguments used to prove Lemma 2.13 in [18], hence the proof is omitted. Actually our case is even simpler since the symbol r does not depend on the variable $x \in \mathbb{T}^d$. ■

Lemma 2.13. Let $\mathcal{R} = \mathrm{Op}(r)$, with $|\mathcal{R}|_{0,s_0+1} < +\infty$. Then $\mathcal{R} \in \mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(H^s_0))$ for any $s \geq 0$ and $\|\mathcal{R}\|_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(H^s_0))} \lesssim |\mathcal{R}|_{0,s_0+1}$. □

Proof. Let $\mathcal{R} = \mathrm{Op}(r) \in OPS^0$. Since $|\mathcal{R}|_{0,s_0+1} < +\infty$, by the definition (2.103), the symbol $r(\cdot, \alpha)$ is in $H^{s_0+1}(\mathbb{T}^\nu)$ for any $\alpha \in \sigma_0(\sqrt{-\Delta})$. Hence, by the Sobolev embedding $r(\cdot, \alpha) \in \mathcal{C}^1(\mathbb{T}^\nu)$ with $\|r(\cdot, \alpha)\|_{\mathcal{C}^1(\mathbb{T}^\nu)} \lesssim \|r(\cdot, \alpha)\|_{s_0+1} \lesssim |\mathcal{R}|_{0,s_0+1}$ for any $\alpha \in \sigma_0(\sqrt{-\Delta})$. Since $\|\mathcal{R}\|_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(H^s_0))} \leq \sup_{\alpha \in \sigma_0(\sqrt{-\Delta})} \|r(\cdot, \alpha)\|_{\mathcal{C}^1(\mathbb{T}^\nu)}$ for any $s \geq 0$, the claimed statement follows. ■

Lemma 2.14. Let $m, m' \in \mathbb{R}$ and $\mathcal{R} \in OPS^m$, $\mathcal{B} \in OPS^{m'}$ be two operators of the form (2.105) with $|\mathcal{R}|_{m,s}$, $|\mathcal{B}|_{m',s} < \infty$, with $s \geq s_0$. Then the operator $\mathcal{R}\mathcal{B} \in OPS^{m+m'}$ has still the form (2.105) and it satisfies the estimate

$$|\mathcal{R}\mathcal{B}|_{m+m',s} \lesssim_s |\mathcal{R}|_{m,s} |\mathcal{B}|_{m',s_0} + |\mathcal{R}|_{m,s_0} |\mathcal{B}|_{m',s}.$$

The same estimate holds replacing the norm $|\cdot|_{m,s}$ by the norm $|\cdot|^{\mathrm{Lip}(\gamma)}_{m,s}$, if $\mathcal{R}$ and $\mathcal{B}$ are Lipschitz with respect to the parameter $\omega \in \Omega_o$. □

Proof. The claimed statement follows by using the property (2.102), the definition (2.103) and the interpolation Lemma 2.1. ■

The above lemma implies that if $\mathcal{R} \in OPS^m$, then $\mathcal{R}^k \in OPS^{km}$ for any $k \geq 1$ and

$$|\mathcal{R}^k|_{km,s_0} \leq C(s_0)^{k-1} |\mathcal{R}|^k_{m,s_0}, \qquad |\mathcal{R}^k|_{km,s} \leq k C(s)^k |\mathcal{R}|^{k-1}_{m,s_0} |\mathcal{R}|_{m,s}, \quad s \geq s_0. \tag{2.108}$$

The same estimate holds replacing $|\cdot|_{m,s}$ by $|\cdot|^{\mathrm{Lip}(\gamma)}_{m,s}$.

Lemma 2.15. Let $\Psi(\varphi) \in OPS^{-m}$, $\varphi \in \mathbb{T}^\nu$, $m \geq 0$, with

$$|\Psi|_{-m,s_0} \leq 1. \tag{2.109}$$

Then the operator $\Phi(\varphi) := \exp(\Psi(\varphi))$ satisfies $\Phi(\varphi) - \mathrm{Id} \in OPS^{-m}$, $\forall \varphi \in \mathbb{T}^\nu$, with

$$|\Phi - \mathrm{Id}|_{-m,s} \lesssim_s |\Psi|_{-m,s}. \tag{2.110}$$

Moreover, the operator

$$\Phi_{\geq 2}(\varphi) := \sum_{k\geq 2} \frac{\Psi(\varphi)^k}{k!} \in OPS^{-2m}, \quad \forall \varphi \in \mathbb{T}^\nu \tag{2.111}$$

and it satisfies the estimate

$$|\Phi_{\geq 2}|_{-2m,s} \lesssim_s |\Psi|_{-m,s} |\Psi|_{-m,s_0}. \tag{2.112}$$

If the operator Ψ depends in a Lipschitz way on the parameter $\omega \in \Omega_o \subseteq \Omega$ and $|\Psi|^{\mathrm{Lip}(\gamma)}_{-m,s_0} \leq 1$, then the estimates (2.110) and (2.112) hold replacing the norm $|\cdot|_{-m,s}$ by the norm $|\cdot|^{\mathrm{Lip}(\gamma)}_{-m,s}$. □

Proof. The Lemma follows by using the Taylor expansion of the operator $\Phi - \mathrm{Id}$, the definition (2.111), the estimate (2.108) and the condition (2.109). ■

In the next lemma we compare the block-decay norm $|\cdot|_s$ defined in (2.76) with the norm $|\cdot|_{m,s}$ defined in (2.103).

Lemma 2.16. Let $s \geq 0$ and $\mathcal{R}(\varphi) \in OPS^{-s-\frac{d-1}{2}}$, $\varphi \in \mathbb{T}^\nu$. Then

$$|\mathcal{R}|_s \lesssim |\mathcal{R}|_{-s-\frac{d-1}{2},s}.$$

The same estimate holds replacing $|\cdot|_s$ by $|\cdot|^{\mathrm{Lip}(\gamma)}_s$ and $|\cdot|_{-s-\frac{d-1}{2},s}$ by $|\cdot|^{\mathrm{Lip}(\gamma)}_{-s-\frac{d-1}{2},s}$ if the operator $\mathcal{R}$ depends in a Lipschitz way on the parameter $\omega \in \Omega_o \subseteq \Omega$. □

Proof. Let $\mathcal{R} = \mathrm{Op}\,(r)$. By the representation (2.99), for any $\varphi \in \mathbb{T}^\nu$, the operator $\mathcal{R}(\varphi)$ is block-diagonal (recall the definition (2.51)) and it has the block representation

$$\mathcal{R}(\varphi) = \mathrm{diag}_{\alpha\in\sigma_0(\sqrt{-\Delta})}[\mathcal{R}(\varphi)]_\alpha^\alpha, \qquad [\mathcal{R}(\varphi)]_\alpha^\alpha = r(\varphi,\alpha)\mathbb{I}_\alpha, \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta})$$

and for any $\ell \in \mathbb{Z}^\nu$

$$[\widehat{\mathcal{R}}(\ell)]_\alpha^\alpha = \widehat{r}(\ell,\alpha)\mathbb{I}_\alpha, \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}), \quad \forall \ell \in \mathbb{Z}^\nu$$

where we recall that $\mathbb{I}_\alpha : \mathbb{E}_\alpha \to \mathbb{E}_\alpha$ is the identity. Hence, using that $\|\mathbb{I}_\alpha\|_{HS} \lesssim \alpha^{\frac{d-1}{2}}$ (see (2.60)), recalling the definition (2.76), one gets

$$\begin{aligned} |\mathcal{R}|_s^2 &= \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} \sum_{\ell\in\mathbb{Z}^\nu} \langle \ell,\alpha\rangle^{2s} \|[\widehat{\mathcal{R}}(\ell)]_\alpha^\alpha\|_{HS}^2 = \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} \sum_{\ell\in\mathbb{Z}^\nu} \langle \ell,\alpha\rangle^{2s} |\widehat{r}(\ell,\alpha)|^2 \|\mathbb{I}_\alpha\|_{HS}^2 \\ &\lesssim \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} \sum_{\ell\in\mathbb{Z}^\nu} \langle \ell,\alpha\rangle^{2s} |\widehat{r}(\ell,\alpha)|^2 \alpha^{d-1} \lesssim \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} \|r(\cdot,\alpha)\|_s^2 \alpha^{2s+d-1} \lesssim |\mathcal{R}|^2_{-s-\frac{d-1}{2},s} \end{aligned} \tag{2.113}$$

which is the claimed estimate. ■

2.5 Hamiltonian formalism

We define the symplectic form $\mathcal{W}$ as

$$\mathcal{W}[z_1, z_2] := \langle z_1, J z_2 \rangle_{L^2_x}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \forall z_1, z_2 \in L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R}). \tag{2.114}$$

Definition 2.1. A φ-dependent linear vector field $X(\varphi) : L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R}) \to L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})$, $\varphi \in \mathbb{T}^\nu$, is Hamiltonian, if $X(\varphi) = JG(\varphi)$, where J is given in (2.114) and the operator $G(\varphi)$ is symmetric for every $\varphi \in \mathbb{T}^\nu$. □

Definition 2.2. A φ-dependent map $\Phi(\varphi) : L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R}) \to L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})$, $\varphi \in \mathbb{T}^\nu$ is symplectic if for any $\varphi \in \mathbb{T}^\nu$, for any $z_1, z_2 \in L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})$,

$$\mathcal{W}[\Phi(\varphi)[z_1],\, \Phi(\varphi)[z_2]] = \mathcal{W}[z_1, z_2],$$

or equivalently $\Phi(\varphi)^T J \Phi(\varphi) = J$ for any $\varphi \in \mathbb{T}^\nu$. □

Assume to have a differentiable map $\varphi \in \mathbb{T}^\nu \mapsto \Phi(\varphi) \in \mathcal{B}\big(L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})\big)$ and let us consider the quasi-periodically forced linear Hamiltonian PDE

$$\partial_t z = X(\omega t) z, \qquad X(\varphi) := JG(\varphi), \quad \varphi \in \mathbb{T}^\nu, \quad z \in L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R}). \tag{2.115}$$

Under the change of coordinates $z = \Phi(\omega t) h$, the above PDE is transformed into the equation

$$\partial_t h = X_+(\omega t) h, \tag{2.116}$$

where $X_+(\omega t)$ is the transformed vector field under the action of the map $\Phi(\omega t)$ (push-forward), namely

$$X_+(\varphi) = \Phi_{\omega *} X(\varphi) := \Phi(\varphi)^{-1} X(\varphi) \Phi(\varphi) - \Phi(\varphi)^{-1} \omega \cdot \partial_\varphi \Phi(\varphi), \quad \forall \varphi \in \mathbb{T}^\nu. \tag{2.117}$$

It turns out that, since $X(\varphi)$ is a Hamiltonian vector field and $\Phi(\varphi)$ is symplectic, the transformed vector field $X_+(\varphi)$ is still Hamiltonian, namely it has the form given in Definition (2.1).

2.5.1 Hamiltonian formalism in complex coordinates

In this section we describe how the Hamiltonian structure described before, reads in the complex coordinates introduced in (2.38) and (2.39). Let $JG(\varphi)$, $\varphi \in \mathbb{T}^\nu$ be a linear Hamiltonian vector field, with $G(\varphi) \in \mathcal{B}\left(L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})\right)$ being a symmetric operator as in (2.35). The conjugated vector field $\mathcal{R}(\varphi) := \mathcal{C}^{-1} JG(\varphi)\mathcal{C} \in \mathcal{B}(\mathbf{L}^2_0(\mathbb{T}^d))$ has the form

$$\mathcal{R}(\varphi) = \mathrm{i}\begin{pmatrix} R_1(\varphi) & R_2(\varphi) \\ -\overline{R_2(\varphi)} & -\overline{R_1(\varphi)} \end{pmatrix}, \tag{2.118}$$

where

$$R_1(\varphi) := -A(\varphi) - D(\varphi) + \mathrm{i}B(\varphi) - \mathrm{i}B(\varphi)^T, \quad R_2(\varphi) := -A(\varphi) + D(\varphi) - \mathrm{i}B(\varphi) - \mathrm{i}B(\varphi)^T \tag{2.119}$$

(recall that the operator $\overline{R}$ is defined in (2.28)). The operators $R_1(\varphi), R_2(\varphi)$ are linear operators acting on complex valued L^2 functions $L^2_0(\mathbb{T}^d)$. Furthermore, since $G(\varphi)$ is symmetric, that is, $A(\varphi) = A(\varphi)^T$, $B(\varphi) = C(\varphi)^T$, $D(\varphi) = D(\varphi)^T$, it turns out that

$$R_1(\varphi) = R_1(\varphi)^*, \qquad R_2(\varphi) = R_2(\varphi)^T, \qquad \forall \varphi \in \mathbb{T}^\nu. \tag{2.120}$$

We refer to an operator $\mathcal{R}$ of the form (2.118), with R_1 and R_2 satisfying (2.120), as a Hamiltonian vector field in complex coordinates. The operator $\mathcal{R}(\varphi)$ in (2.118) satisfies

$$\mathcal{R}(\varphi)[\mathbf{u}] = \mathrm{i}J\nabla_{\mathbf{u}}\mathcal{H}(\varphi, \mathbf{u}), \quad \mathbf{u} := (u, \bar{u}), \quad \nabla_{\mathbf{u}}\mathcal{H} = (\nabla_u\mathcal{H}, \nabla_{\bar{u}}\mathcal{H}), \tag{2.121}$$

where the real Hamiltonian $\mathcal{H}$ has the form

$$\mathcal{H}(\varphi, \mathbf{u}) := \langle \mathcal{G}(\varphi)[\mathbf{u}], \mathbf{u}\rangle, \quad \mathcal{G}(\varphi) := \begin{pmatrix} \overline{R_2(\varphi)} & \overline{R_1(\varphi)} \\ R_1(\varphi) & R_2(\varphi) \end{pmatrix}, \tag{2.122}$$

that is

$$\mathcal{H}(\varphi, u, \bar{u}) = \int_{\mathbb{T}^d} R_1(\varphi)[u]\bar{u}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d} R_2(\varphi)[u], u\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d} \overline{R_2(\varphi)}[\bar{u}]\,\bar{u}\,\mathrm{d}x \tag{2.123}$$

and

$$\nabla_u\mathcal{H} = \frac{1}{\sqrt{2}}\left(\nabla_v\mathcal{H} - \mathrm{i}\nabla_\psi\mathcal{H}\right), \quad \nabla_{\bar{u}}\mathcal{H} = \frac{1}{\sqrt{2}}\left(\nabla_v\mathcal{H} + \mathrm{i}\nabla_\psi\mathcal{H}\right).$$

By (2.120), we deduce that

$$\mathcal{G}(\varphi) = \mathcal{G}(\varphi)^T, \qquad \forall \varphi \in \mathbb{T}^\nu.$$

The symplectic form $\mathcal{W}$ defined in (2.114) reads in the coordinates $\mathbf{u} = (u, \bar{u})$ as.

$$\Gamma[\mathbf{u}_1, \mathbf{u}_2] = \mathrm{i} \int_{\mathbb{T}^d} (u_1 \bar{u}_2 - \bar{u}_1 u_2)\, \mathrm{d}x = \mathrm{i} \langle \mathbf{u}_1, J\mathbf{u}_2 \rangle_{\mathrm{L}^2_x}, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{L}^2_0(\mathbb{T}^d) \tag{2.124}$$

where

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle_{\mathrm{L}^2_x} := \int_{\mathbb{T}^d} u_1 u_2 + \overline{u}_1 \overline{u}_2\, \mathrm{d}x, \qquad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{L}^2_0(\mathbb{T}^d). \tag{2.125}$$

Definition 2.3. A φ-dependent family of linear operators $\Phi(\varphi) : \mathbf{L}^2_0(\mathbb{T}^d) \to \mathbf{L}^2_0(\mathbb{T}^d)$, $\varphi \in \mathbb{T}^\nu$ is symplectic if

$$\Gamma[\Phi(\varphi)[\mathbf{u}_1], \Phi(\varphi)[\mathbf{u}_2]] = \Gamma[\mathbf{u}_1, \mathbf{u}_2], \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{L}^2_0(\mathbb{T}^d), \quad \forall \varphi \in \mathbb{T}^\nu. \qquad \square$$

It is well known that if $\mathcal{R}(\varphi)$ is an operator of the form (2.118), (2.120),namely by (2.121), it is a linear Hamiltonian vector field associated to the real quadratic Hamiltonian $\mathcal{H}$ in (2.123), the operator $\Phi(\varphi) = \exp(\mathcal{R}(\varphi))$ is a symplectic. Assume that the map $\varphi \in \mathbb{T}^\nu \mapsto \Phi(\varphi) \in \mathcal{B}(\mathbf{L}^2_0(\mathbb{T}^d))$ is a differentiable family of maps and let $\varphi \in \mathbb{T}^\nu \mapsto \mathcal{X}(\varphi) \in \mathcal{B}(\mathbf{L}^2_0(\mathbb{T}^d))$ be a differentiable families of Hamiltonian vector fields, that is, $\mathcal{X}(\varphi) = \mathrm{i}J\mathcal{G}(\varphi)$, $\mathcal{G}(\varphi) = \mathcal{G}(\varphi)^T$ for any $\varphi \in \mathbb{T}^\nu$. Arguing as in (2.115) and (2.116), under the transformation $\mathbf{u} = \Phi(\omega t)\mathbf{h}$, the PDE

$$\partial_t \mathbf{u} = \mathcal{X}(\omega t)\mathbf{u}, \qquad \omega \in \mathbb{R}^\nu, \quad t \in \mathbb{R}, \tag{2.126}$$

transforms into the PDE

$$\partial_t \mathbf{h} = \mathcal{X}_+(\omega t)\mathbf{h}, \quad \mathcal{X}_+(\varphi) := \Phi_{\omega *}\mathcal{X}(\varphi) = \Phi(\varphi)^{-1}\mathcal{X}(\varphi)\Phi(\varphi) - \Phi(\varphi)^{-1}\omega \cdot \partial_\varphi \Phi(\varphi), \quad \forall \varphi \in \mathbb{T}^\nu. \tag{2.127}$$

If $\Phi(\varphi)$ is symplectic then the vector field $\mathcal{X}_+(\varphi)$ is Hamiltonian, that is it satisfies (2.118) and (2.120). In the following, we will consider also reparameterizations of time of the form

$$\tau = t + \alpha(\omega t),$$

where $\alpha : \mathbb{T}^\nu \to \mathbb{R}$ is a sufficiently smooth function with $\|\alpha\|_{\mathcal{C}^1}$ small enough. Then the function $t \mapsto t + \alpha(\omega t)$ is invertible and its inverse is given by

$$t = \tau + \widetilde{\alpha}(\omega \tau).$$

by setting $\mathbf{v}(t) := \mathcal{A}(\omega t)\mathbf{u} := \mathbf{u}(t + \alpha(\omega t))$, the PDE (2.126) is transformed into

$$\partial_\tau \mathbf{v} = J\mathcal{G}_+(\omega\tau)\mathbf{v}, \qquad \mathcal{G}_+(\vartheta) := \frac{1}{\rho(\vartheta)}\mathcal{G}(\vartheta + \omega\tilde{\alpha}(\vartheta)), \quad \rho(\vartheta) := 1 + \omega \cdot \partial_\varphi \alpha\,(\vartheta + \omega\tilde{\alpha}(\vartheta)) \tag{2.128}$$

which is still a Hamiltonian equation.

3 Regularization Procedure of the Vector Field $\mathcal{L}(\varphi)$

As described in Section 1, in this section we carry out the first part of the reduction procedure of the vector field $\mathcal{L}(\varphi)$, defined in (1.11), to a block-diagonal operator with constant coefficients. Our purpose is to transform the vector field $\mathcal{L}(\varphi)$ into the vector field $\mathcal{L}_4(\varphi)$ which is a regularizing perturbation of a time-independent diagonal operator, see (3.70). The regularizing perturbation $\mathcal{R}_4$ defined in (3.71) is the sum of a finite rank operator and a φ-dependent Fourier multiplier of order $-M$ where the constant M is fixed in (3.68). In the following subsections, we describe in details all the steps needed to transform the vector field $\mathcal{L}(\varphi)$ into the vector field $\mathcal{L}_4(\varphi)$.

3.1 Symplectic symmetrization of the highest order

We start by symmetrizing the highest order of the vector field

$$\mathcal{L}(\varphi) = \begin{pmatrix} 0 & 1 \\ (1 + \varepsilon a(\varphi))\Delta + \varepsilon\mathcal{R}(\varphi) & 0 \end{pmatrix}, \quad \varphi \in \mathbb{T}^\nu$$

where we recall the definitions given in (1.11) and (1.3). For any $\varphi \in \mathbb{T}^\nu$, let us consider the transformation

$$\mathcal{S}(\varphi) : H^s_0(\mathbb{T}^d, \mathbb{R}) \times H^s_0(\mathbb{T}^d, \mathbb{R}) \to H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}), \quad \begin{pmatrix} u \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \beta(\varphi)|D|^{-\frac{1}{2}} u \\ \frac{1}{\beta(\varphi)}|D|^{\frac{1}{2}}\psi \end{pmatrix} \tag{3.1}$$

where $\beta : \mathbb{T}^\nu \to \mathbb{R}$ is a function close to 1 to be determined and for all $m \in \mathbb{R}$, the operator $|D|^m$ is defined by

$$|D|^m(e^{\mathrm{i}j\cdot x}) = |j|^m e^{\mathrm{i}j\cdot x} \quad \forall j \neq 0. \tag{3.2}$$

For any $\varphi \in \mathbb{T}^\nu$, the inverse of the operator $\mathcal{S}(\varphi)$ is given by

$$\mathcal{S}(\varphi)^{-1} : H^s_0(\mathbb{T}^d, \mathbb{R}) \times H^{s-1}_0(\mathbb{T}^d, \mathbb{R}) \to H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}), \quad \begin{pmatrix} u \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\beta(\varphi)}|D|^{\frac{1}{2}} u \\ \beta(\varphi)|D|^{-\frac{1}{2}}\psi \end{pmatrix}. \tag{3.3}$$

By (2.117), the push-forward of the vector field $\mathcal{L}(\varphi)$ by means of the transformation $\mathcal{S}(\varphi)$ is given by

$$\mathcal{L}_1(\varphi) := \mathcal{S}_{\omega *}\mathcal{L}(\varphi) = \mathcal{S}(\varphi)^{-1}\mathcal{L}(\varphi)\mathcal{S}(\varphi) - \mathcal{S}(\varphi)^{-1}\omega\cdot\partial_\varphi \mathcal{S}(\varphi)$$
$$= \begin{pmatrix} -\beta^{-1}(\varphi)(\omega\cdot\partial_\varphi\beta(\varphi)) & \beta^{-2}(\varphi)|D| \\ (1+\varepsilon a(\varphi))\beta^2(\varphi)|D|^{-1}\Delta + \varepsilon\beta^2(\varphi)|D|^{-\frac{1}{2}}\mathcal{R}(\varphi)|D|^{-\frac{1}{2}} & -\beta(\varphi)(\omega\cdot\partial_\varphi\beta^{-1}(\varphi)) \end{pmatrix} \tag{3.4}$$

and we look for $\beta : \mathbb{T}^\nu \to \mathbb{R}$ such that

$$\beta^{-2}(\varphi) = (1+\varepsilon a(\varphi))\beta^2(\varphi), \tag{3.5}$$

namely we choose

$$\beta(\varphi) := \frac{1}{[1+\varepsilon a(\varphi)]^{\frac{1}{4}}}. \tag{3.6}$$

Since

$$\beta(\varphi)\omega\cdot\partial_\varphi\beta^{-1}(\varphi) = -\frac{\omega\cdot\partial_\varphi\beta(\varphi)}{\beta(\varphi)} \quad \text{and} \quad -\Delta = |D|^2$$

we get that

$$\mathcal{L}_1(\varphi) = \begin{pmatrix} -a_0(\varphi) & a_1(\varphi)|D| \\ -a_1(\varphi)|D| + \varepsilon\mathcal{R}^{(1)}(\varphi) & a_0(\varphi) \end{pmatrix}, \tag{3.7}$$

where

$$a_0(\varphi) := \frac{\omega\cdot\partial_\varphi\beta(\varphi)}{\beta(\varphi)}, \quad a_1(\varphi) := \sqrt{1+\varepsilon a(\varphi)}, \quad \mathcal{R}^{(1)}(\varphi) := \beta^2(\varphi)|D|^{-\frac{1}{2}}\mathcal{R}(\varphi)|D|^{-\frac{1}{2}}. \tag{3.8}$$

Since β is a real-valued function, the operator $\mathcal{S}(\varphi)$ is real for any $\varphi \in \mathbb{T}^\nu$ and a direct verification shows that it is also symplectic. Hence the transformed vector field $\mathcal{L}_1(\varphi)$ is still real and Hamiltonian. By (3.6) and (3.8), the functions β, a_1 and the operator $\mathcal{R}^{(1)}$ do not depend on the parameter $\omega \in \Omega$, whereas the function $a_0(\varphi) = a_0(\varphi;\omega)$ depends on $\omega \in \Omega$.

Now we give some estimates on the coefficients of the vector field $\mathcal{L}_1(\varphi)$.

Lemma 3.1. Let $q > s_0 + 1$. Then there exists $\delta_q \in (0,1)$ small enough such that for any $\varepsilon \in (0,\delta_q)$, for any $s_0 \le s \le q-1$, the following holds: the functions β, a_0, a_1 defined in (3.6), (3.8) satisfy the estimates

$$\|\beta^{\pm 1} - 1\|_s, \|a_1 - 1\|_s, \|a_0\|_s^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon. \tag{3.9}$$

The remainder $\mathcal{R}^{(1)}(\varphi)$ in (3.8) has the form

$$\mathcal{R}^{(1)}(\varphi)[v] = \sum_{k=1}^{N} b_k^{(1)}(\varphi, x) \int_{\mathbb{T}^d} c_k^{(1)}(\varphi, y) v(y)\, dy + c_k^{(1)}(\varphi, x) \int_{\mathbb{T}^d} b_k^{(1)}(\varphi, y) v(y)\, dy, \tag{3.10}$$

$\varphi \in \mathbb{T}^\nu, v \in L^2_0(\mathbb{T}^d, \mathbb{R})$ (then it is symmetric $\mathcal{R}^{(1)}(\varphi) = \mathcal{R}^{(1)}(\varphi)^T$, for all $\varphi \in \mathbb{T}^\nu$) with

$$\|b_k^{(1)}\|_s, \|c_k^{(1)}\|_s \lesssim_q 1, \qquad \forall k = 1, \ldots, N. \tag{3.11}$$

Furthermore, for any $s \geq 1/2$, the maps

$$\varphi \mapsto \mathcal{S}(\varphi), \quad \mathbb{T}^\nu \to \mathcal{B}\left(H^s_0(\mathbb{T}^d, \mathbb{R}) \times H^s_0(\mathbb{T}^d, \mathbb{R}), H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})\right),$$

$$\varphi \mapsto \mathcal{S}(\varphi)^{-1}, \quad \mathbb{T}^\nu \to \mathcal{B}\left(H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}), H^s_0(\mathbb{T}^d, \mathbb{R}) \times H^s_0(\mathbb{T}^d, \mathbb{R})\right)$$

are $\mathcal{C}^1$ maps. □

Proof. The estimates (3.9) follows by the definitions (3.6) and (3.8) and by Lemmata 2.1 and 2.2. Let us prove the estimates (3.11). By (3.8), recalling the definition of $\mathcal{R}(\varphi)$ given in (1.3), using that $|D|^{-\frac{1}{2}}$ is symmetric, one has that the operator $\mathcal{R}^{(1)}(\varphi)$ has the form (3.10) with

$$b_k^{(1)}(\varphi, x) := \beta(\varphi)|D|^{-\frac{1}{2}} b_k(\varphi, x), \quad c_k^{(1)}(\varphi, x) := \beta(\varphi)|D|^{-\frac{1}{2}} c_k(\varphi, x), \quad k = 1, \ldots, N.$$

Then the claimed estimates follow by applying the estimate (3.9) and applying the interpolation Lemma 2.1. A direct verification shows that $\mathcal{R}^{(1)}(\varphi) = \mathcal{R}^{(1)}(\varphi)^T$ for any $\varphi \in \mathbb{T}^\nu$. ■

3.2 Complex variables

Now we write the vector field $\mathcal{L}_1(\varphi)$ defined in (3.7) in the complex coordinates introduced in (2.38) and (2.39). More precisely, we conjugate the vector field $\mathcal{L}_1(\varphi)$ by means of the transformation $\mathcal{C}$ defined in (2.39). Since $\mathcal{C}$ is φ-independent, we get that by (2.117), the push-forward $\mathcal{L}_2(\varphi) := \mathcal{C}_{\omega*}\mathcal{L}_1(\varphi) = \mathcal{C}^{-1}\mathcal{L}_1(\varphi)\mathcal{C}$ is given by

$$\mathcal{L}_2(\varphi) = \begin{pmatrix} -\mathrm{i}a_1(\varphi)|D| + \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\varphi) & -a_0(\varphi) + \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\varphi) \\ -a_0(\varphi) - \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\varphi) & \mathrm{i}a_1(\varphi)|D| - \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\varphi) \end{pmatrix}, \quad \mathcal{R}^{(2)}(\varphi) := \frac{\mathcal{R}^{(1)}(\varphi)}{\sqrt{2}}. \tag{3.12}$$

Since a_1 and a_0 are real valued functions and $\mathcal{R}^{(1)}(\varphi)$ (and then $\mathcal{R}^{(2)}(\varphi)$) is symmetric and real, the operator $\mathcal{L}_2(\varphi)$ is a Hamiltonian vector field in complex coordinates, in the sense of the Definition (2.118). We recall that the transformations $\mathcal{C}, \mathcal{C}^{-1}$ satisfy the property (2.44).

3.3 Quasi-periodic reparameterization of time

The aim of this Section is to reduce to constant coefficients the term $a_1(\varphi)|D|$ in the operator $\mathcal{L}_2(\varphi)$ defined in (3.12). In order to do this, let us consider a function $\alpha : \mathbb{T}^\nu \to \mathbb{R}$ (to be determined) and define a reparameterization of time of the form

$$\mathbb{R} \to \mathbb{R}, \quad t \mapsto t + \alpha(\omega t), \quad \omega \in \Omega. \tag{3.13}$$

It is easy to verify that if $\|\alpha\|_{C^1}$ is small enough, the above function is invertible and its inverse has the form

$$\tau \mapsto \tau + \widetilde{\alpha}(\omega \tau). \tag{3.14}$$

The reparameterization of time (3.13) induces also a diffeomorphism of the torus $\mathbb{T}^\nu$

$$\mathbb{T}^\nu \to \mathbb{T}^\nu, \quad \varphi \mapsto \varphi + \alpha(\varphi) \tag{3.15}$$

whose inverse is given by

$$\mathbb{T}^\nu \mapsto \mathbb{T}^\nu, \quad \vartheta \mapsto \vartheta + \widetilde{\alpha}(\vartheta). \tag{3.16}$$

The corresponding composition operators A, A^{-1} acting on the periodic functions $h : \mathbb{T}^\nu \times \mathbb{T}^d \to \mathbb{C}$ are given by

$$Ah(\varphi, x) := h(\varphi + \omega\alpha(\varphi), x), \quad A^{-1}h(\vartheta, x) := h(\vartheta + \omega\widetilde{\alpha}(\vartheta), x). \tag{3.17}$$

According to (2.128), under the reparameterization of time defined by

$$\mathcal{A}(\omega t)\mathbf{v}(t, x) := \mathbf{v}(t + \alpha(\omega t), x), \quad \mathcal{A}(\omega t)^{-1}\mathbf{v}(\tau, x) := \mathbf{v}(\tau + \widetilde{\alpha}(\omega\tau), x), \tag{3.18}$$

the vector field $\mathcal{L}_2(\varphi)$ transforms into the vector field

$$\mathcal{L}_3(\vartheta) := \frac{1}{\rho(\vartheta)} \mathcal{L}_2(\vartheta + \omega\tilde{\alpha}(\vartheta))$$
$$= \frac{1}{\rho(\vartheta)} \begin{pmatrix} -\mathrm{i}(A^{-1}a_1)(\vartheta)|D| + \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\vartheta + \omega\widetilde{\alpha}(\vartheta)) & -(A^{-1}a_0)(\vartheta) + \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\vartheta + \omega\widetilde{\alpha}(\vartheta)) \\ -(A^{-1}a_0)(\vartheta) - \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\vartheta + \omega\widetilde{\alpha}(\vartheta)) & +\mathrm{i}(A^{-1}a_1)(\vartheta)|D| - \mathrm{i}\varepsilon\mathcal{R}^{(2)}(\vartheta + \omega\widetilde{\alpha}(\vartheta)) \end{pmatrix} \tag{3.19}$$

where

$$\rho(\vartheta) := 1 + \omega \cdot \partial_\varphi \alpha(\vartheta + \omega\widetilde{\alpha}(\vartheta)) = A^{-1}[1 + \omega \cdot \partial_\varphi \alpha](\vartheta). \tag{3.20}$$

We want to choose the function $\alpha(\varphi)$ so that

$$\frac{(A^{-1}a_1)(\vartheta)}{\rho(\vartheta)} = m, \quad \forall \vartheta \in \mathbb{T}^\nu, \tag{3.21}$$

for some constant $m \in \mathbb{R}$ to be determined. The above equation leads to

$$m\left(1 + \omega \cdot \partial_\varphi \alpha(\varphi)\right) = a_1(\varphi) \quad \forall \varphi \in \mathbb{T}^\nu. \tag{3.22}$$

Integrating on $\mathbb{T}^\nu$ we fix the value of m as

$$m := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} a_1(\varphi)\, \mathrm{d}\varphi \tag{3.23}$$

and then, assuming that $\omega \in DC(\gamma, \tau)$, for some $\gamma, \tau > 0$ (see the definition (2.16)), we get

$$\alpha(\varphi) = (\omega \cdot \partial_\varphi)^{-1} \left[\frac{a_1}{m} - 1\right](\varphi) \tag{3.24}$$

where the operator $(\omega \cdot \partial_\varphi)^{-1}$ is defined by (2.15). Since the function a_1 is real valued, then m is real and α is a real-valued function.

By (3.19)–(3.24), the vector field $\mathcal{L}_3(\vartheta)$ has then the form

$$\mathcal{L}_3(\vartheta) := \begin{pmatrix} -\mathrm{i}m|D| + \mathrm{i}\varepsilon\mathcal{R}^{(3)}(\vartheta) & a_2(\vartheta) + \mathrm{i}\varepsilon\mathcal{R}^{(3)}(\vartheta) \\ a_2(\vartheta) - \mathrm{i}\varepsilon\mathcal{R}^{(3)}(\vartheta) & \mathrm{i}m|D| - \mathrm{i}\varepsilon\mathcal{R}^{(3)}(\vartheta) \end{pmatrix}, \tag{3.25}$$

where

$$a_2(\vartheta) := \rho^{-1}(\vartheta)A^{-1}[a_0](\vartheta), \qquad \mathcal{R}^{(3)}(\vartheta) := \rho(\vartheta)^{-1}\mathcal{R}^{(2)}(\vartheta + \omega\widetilde{\alpha}(\vartheta)). \tag{3.26}$$

The operator $\mathcal{L}_3(\vartheta)$ is still a Hamiltonian vector field in complex coordinates, since $\mathcal{L}_2(\vartheta)$ is Hamiltonian and the reparameterization of time $\mathcal{A}$ preserves the Hamiltonian structure (see Section 2.5.1). We point out that by (3.23) and (3.8), the constant m is independent of the parameter $\omega \in \Omega$, whereas by (3.24), (3.17), (3.20), and (3.26), the functions $\alpha, \widetilde{\alpha}, \rho, a_2$ and the operator $\mathcal{R}^{(3)}$ depend in a Lipschitz way with respect to the parameter $\omega \in DC(\gamma, \tau)$.

Lemma 3.2. Let $\tau > 0$, $\gamma \in (0,1)$ and $\omega \in DC(\gamma,\tau)$ (recall (2.16)). Then there exists a constant $\sigma = \sigma(\tau) > 0$ such that if $q > s_0 + \sigma$, there exists $\delta_q \in (0,1)$ such that if $\varepsilon\gamma^{-1} \leq \delta_q$, for all $s_0 \leq s \leq q - \sigma$ the following estimates hold:

$$|m-1|, \|a_2\|_s^{\mathrm{Lip}(\gamma)}, \|\rho^{\pm 1} - 1\|_s^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon, \quad \|\alpha\|_s^{\mathrm{Lip}(\gamma)}, \|\widetilde{\alpha}\|_s^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon\gamma^{-1} \tag{3.27}$$

The symmetric operator $\mathcal{R}^{(3)}(\vartheta)$ defined in (3.26) has the form

$$\mathcal{R}^{(3)}(\vartheta)[u] = \sum_{k=1}^{N} b_k^{(3)}(\vartheta, x) \int_{\mathbb{T}^d} c_k^{(3)}(\vartheta, y) v(y)\,\mathrm{d}y + c_k^{(3)}(\vartheta, x) \int_{\mathbb{T}^d} b_k^{(3)}(\vartheta, y) v(y)\,\mathrm{d}y, \tag{3.28}$$

$\varphi \in \mathbb{T}^\nu$, $v \in L_0^2(\mathbb{T}^d)$, with

$$\|b_k^{(3)}\|_s^{\mathrm{Lip}(\gamma)}, \|c_k^{(3)}\|_s^{\mathrm{Lip}(\gamma)} \lesssim_q 1, \qquad k = 1, \ldots, N. \tag{3.29}$$

□

Proof. The estimates (3.27) follow by (3.23), (3.24), and (3.26) and by the estimates (3.9) by applying Lemmata 2.1–2.3. The formula (3.28) follows by (3.10), (3.12), and (3.26), by defining $b_k^{(3)} := 2^{-\frac{1}{4}}\rho^{-\frac{1}{2}} b_k^{(1)}$, $c_k^{(3)} := 2^{-\frac{1}{4}}\rho^{-\frac{1}{2}} c_k^{(1)}$, $k = 1, \ldots, N$ and the estimates (3.29) follow by (3.11) and (3.27) and Lemmata 2.1 and 2.3. ■

3.4 Symplectic reduction up to order $|D|^{-M}$

Introducing the notation

$$T := \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \quad \mathrm{Id} : L_0^2(\mathbb{T}^d) \to L_0^2(\mathbb{T}^d) \quad \text{is the identity} \tag{3.30}$$

and renaming the variable $\vartheta = \varphi$, we can write the vector field in (3.25) as

$$\mathcal{L}_3(\varphi) = \mathrm{i} m T |D| + A_2(\varphi) + \varepsilon \mathcal{R}_3(\varphi), \tag{3.31}$$

where

$$A_2(\varphi) := \begin{pmatrix} 0 & a_2(\varphi) \\ a_2(\varphi) & 0 \end{pmatrix}, \quad \mathcal{R}_3(\varphi) := \mathrm{i} \begin{pmatrix} \mathcal{R}^{(3)}(\varphi) & \mathcal{R}^{(3)}(\varphi) \\ -\mathcal{R}^{(3)}(\varphi) & -\mathcal{R}^{(3)}(\varphi) \end{pmatrix}, \quad \varphi \in \mathbb{T}^\nu \tag{3.32}$$

and the operator $\mathcal{R}^{(3)}(\varphi)$, defined in (3.26), has the form (3.28). The aim of this section is to conjugate $\mathcal{L}_3(\varphi)$ to the vector field $\mathcal{L}_4(\varphi)$ defined in (3.70) which is the sum of a diagonal operator and a regularizing remainder. Since the operator $\mathcal{R}^{(3)}(\varphi)$ is finite rank operator of the form (3.28), it is already regularizing. Hence in the following two Sections 3.4.1 and 3.4.2, we neglect the operator $\mathcal{R}_3(\varphi)$ in (3.31) and we work with the vector field

$$L_3^{(0)}(\varphi) := \mathrm{i}mT|D| + A_2(\varphi), \quad \varphi \in \mathbb{T}^\nu. \tag{3.33}$$

We compute the complete conjugation of $\mathcal{L}_3$ in Section 3.31.

3.4.1 Block-decoupling up to order $|D|^{-M}$

Given a positive integer M, our goal is to conjugate the operator $L_3^{(0)}$ in (3.33) to the operator $L_3^{(M)}$ in (3.51) whose off-diagonal part Q_M is an operator of order $-M$. This is achieved by applying iteratively M-times a conjugation map which transforms the off-diagonal block operator into a 1-smoother ones. For such a procedure we will use the class of φ-dependent Fourier multipliers introduced in Section 2.4.

We describe the inductive step of such a procedure. We assume that $q > s_0 + \sigma + M$, where the constant $\sigma = \sigma(\tau)$ is given in Lemma 3.2 and $M \in \mathbb{N}$ is the number of the steps of this regularization procedure. In this section we use the following notation: If $n \in \{1, \ldots, M\}$, $s \geq 0$, we write

$$a \lesssim_{n,s} b \quad \Longleftrightarrow \quad a \leq C(n,s)b$$

for some constant $C(n,s) > 0$ (that may depend also on d, τ, ν).

At the nth step, we have a Hamiltonian vector field

$$L_3^{(n)}(\varphi) = \mathrm{i}mT|D| + R_n(\varphi) + Q_n(\varphi), \tag{3.34}$$

where $R_n(\varphi) = R_n(\varphi;\omega)$, $Q_n(\varphi) = Q_n(\varphi;\omega)$, $\omega \in DC(\gamma,\tau)$ are Hamiltonian vector fields of the form

$$R_n := \mathrm{i} \begin{pmatrix} \mathrm{Op}(r_n) & 0 \\ 0 & -\overline{\mathrm{Op}(r_n)} \end{pmatrix}, \qquad Q_n := \mathrm{i} \begin{pmatrix} 0 & \mathrm{Op}(q_n) \\ -\overline{\mathrm{Op}(q_n)} & 0 \end{pmatrix} \tag{3.35}$$

and $r_n(\varphi,\cdot) \in S^{-1}$, $q_n(\varphi,\cdot) \in S^{-n}$. Moreover they satisfy the estimates

$$|R_n|_{-1,s}^{\mathrm{Lip}(\gamma)}, |Q_n|_{-n,s}^{\mathrm{Lip}(\gamma)} \lesssim_{n,q} \varepsilon, \qquad \forall s_0 \leq s \leq q - n - \sigma \tag{3.36}$$

where $\sigma = \sigma(\tau) > 0$ is given in Lemma 3.2. Recall that the definition of the norm $|\cdot|_{m,s}$ is given in (2.103).

Initialization. The Hamiltonian vector field $L_3^{(0)}(\varphi)$ in (3.33) satisfies the assumptions (3.34)–(3.36), with $R_0(\varphi) = 0$ and $Q_0(\varphi) = A_2(\varphi) \in OPS^0$, by Lemma 3.2.

Inductive step. We consider a symplectic transformation of the form

$$\mathcal{V}_n := \exp(\mathrm{i}V_n) \tag{3.37}$$

where the operator V_n has the form

$$V_n := \begin{pmatrix} 0 & \mathrm{Op}(v_n) \\ -\overline{\mathrm{Op}(v_n)} & 0 \end{pmatrix}, \qquad v_n \in S^{-n-1}. \tag{3.38}$$

We write

$$\mathcal{V}_n = \mathrm{Id} + \mathrm{i}V_n + \mathcal{V}_{n,\geq 2}, \qquad \mathcal{V}_{n,\geq 2} := \sum_{k\geq 2} \frac{\mathrm{i}^k}{k!} V_n^k. \tag{3.39}$$

In the above formula, with a slight abuse of notations we denote by $\mathrm{Id} : \mathbf{L}_0^2(\mathbb{T}^d) \to \mathbf{L}_0^2(\mathbb{T}^d)$ the identity on the space $\mathbf{L}_0^2(\mathbb{T}^d)$. By Lemma 2.15, one gets $\mathcal{V}_{n,\geq 2} \in OPS^{-2(n+1)}$. We now compute the push-forward $(\mathcal{V}_n)_{\omega *} L_3^{(n)}(\varphi)$. By (2.127) one has

$$(\mathcal{V}_n)_{\omega *} L_3^{(n)}(\varphi) = \mathcal{V}_n(\varphi)^{-1} \Big(L_3^{(n)}(\varphi)\mathcal{V}_n(\varphi) - \omega \cdot \partial_\varphi \mathcal{V}_n(\varphi) \Big). \tag{3.40}$$

Since $\omega \cdot \partial_\varphi \mathcal{V}_n(\varphi) = \omega \cdot \partial_\varphi (\mathcal{V}_n(\varphi) - \mathrm{Id})$, by Lemmata 2.14 and 2.15, one has

$$-\mathcal{V}_n(\varphi)^{-1}\omega \cdot \partial_\varphi \mathcal{V}_n(\varphi) = -\mathcal{V}_n(\varphi)^{-1}\omega \cdot \partial_\varphi (\mathcal{V}_n(\varphi) - \mathrm{Id}) \in OPS^{-n-1}. \tag{3.41}$$

Moreover

$$\begin{aligned} L_3^{(n)}(\varphi)\mathcal{V}_n(\varphi) &\overset{(3.34)}{=} \mathrm{i}\mathcal{V}_n(\varphi) mT|D| + [\mathrm{i}mT|D|, \mathrm{i}V_n(\varphi)] + Q_n(\varphi) + R_n(\varphi) \\ &\quad + [\mathrm{i}mT|D|, \mathcal{V}_{n,\geq 2}(\varphi)] + (R_n(\varphi) + Q_n(\varphi))(\mathcal{V}_n(\varphi) - \mathrm{Id}). \end{aligned} \tag{3.42}$$

Note that $[\mathrm{i}mT|D|, \mathcal{V}_{n,\geq 2}(\varphi)] \in OPS^{-2n-1} \subset OPS^{-n-1}$, $(R_n(\varphi)+Q_n(\varphi))(\mathcal{V}_n(\varphi)-\mathrm{Id}) \in OPS^{-n-2} \subset OPS^{-n-1}$, therefore the only off-diagonal term of order $-n$ (which we want to eliminate) is given by $[\mathrm{i}mT|D|, \mathrm{i}V_n(\varphi)] + Q_n(\varphi)$. We want to choose $V_n(\varphi)$ so that

$$[\mathrm{i}mT|D|, \mathrm{i}V_n(\varphi)] + Q_n(\varphi) = 0. \tag{3.43}$$

By a direct calculation, one has

$$\begin{aligned}&[\mathrm{i}mT|D|, \mathrm{i}V_n(\varphi)] + Q_n(\varphi)\\ &\quad = \begin{pmatrix} 0 & \mathrm{Op}\,(2m|j|v_n(\varphi, |j|) + \mathrm{i}q_n(\varphi, |j|)) \\ \overline{\mathrm{Op}\,(2m|j|v_n(\varphi, |j|) + \mathrm{i}q_n(\varphi, |j|))} & 0 \end{pmatrix}.\end{aligned} \tag{3.44}$$

Then $[\mathrm{i}mT|D|, \mathrm{i}V_n] + Q_n = 0$ if we choose the symbol v_n so that

$$v_n(\varphi, \alpha) := -\frac{\mathrm{i}q_n(\varphi, \alpha)}{2m\alpha}, \qquad \forall \varphi \in \mathbb{T}^\nu, \qquad \forall \alpha \in \sigma_0(\sqrt{-\Delta}). \tag{3.45}$$

Since $q_n(\varphi, \cdot) \in S^{-n}$, the symbol $v_n(\varphi, \cdot) \in S^{-n-1}$ for any $\varphi \in \mathbb{T}^\nu$.

Lemma 3.3. For any $s_0 \leq s \leq q - n - \sigma$, the operators $V_n(\varphi), \mathcal{V}_n(\varphi) - \mathrm{Id} \in S^{-n-1}$ and $\mathcal{V}_{n,\geq 2}(\varphi) \in OPS^{-2(n+1)}$, see (3.38) and (3.39) (which depend on the parameter $\omega \in DC(\gamma, \tau)$) satisfy the estimates

$$|V_n|^{\mathrm{Lip}(\gamma)}_{-n-1,s}, \ |\mathcal{V}_n^{\pm 1} - \mathrm{Id}|_{-n-1,s}, \ |\mathcal{V}_{n,\geq 2}|_{-2(n+1),s} \lesssim_{n,q} \varepsilon. \tag{3.46}$$

□

Proof. The estimate for the operator V_n follows by the definitions (3.38) and (3.45) and by the estimates (3.27) and (3.36). The estimates for $\mathcal{V}_n(\varphi) - \mathrm{Id}$ and $\mathcal{V}_{n,\geq 2}(\varphi)$ follow by applying Lemma 2.15, using the estimate on $V_n(\varphi)$. ■

By (3.40)–(3.43), one gets

$$L_3^{(n+1)}(\varphi) = \mathrm{i}mT|D| + R_n(\varphi) + P_n(\varphi), \tag{3.47}$$

where

$$P_n := (\mathcal{V}_n^{-1} - \mathrm{Id})R_n + \mathcal{V}_n^{-1}\left([\mathrm{i}mT|D|, \mathcal{V}_{n,\geq 2}] + (R_n + Q_n)(\mathcal{V}_n - \mathrm{Id}) - \omega \cdot \partial_\varphi(\mathcal{V}_n - \mathrm{Id})\right). \tag{3.48}$$

Note that P_n is the only operator which contains off-diagonal terms. In the next lemma, we provide some estimates on the remainder P_n.

Lemma 3.4. For any $s_0 \le s \le q - \sigma - n - 1$, the operator $P_n(\varphi) = P_n(\varphi;\omega) \in OPS^{-n-1}$, $\omega \in DC(\gamma,\tau)$ satisfies the estimates

$$|P_n|_{-n-1,s}^{\mathrm{Lip}(\gamma)} \lesssim_{n,q} \varepsilon. \tag{3.49}$$

□

Proof. The Lemma follows by Lemma 3.3, the estimates (3.36), by applying the property (2.107) and Lemma 2.14 to estimate all the terms in (3.48). ■

By (3.47) and (3.49), the vector field $L_3^{(n+1)}(\varphi)$ has the same form (3.34) and (3.35) with $R_{n+1}(\varphi)$, $Q_{n+1}(\varphi)$ that satisfy the estimates (3.36) at the step $n+1$. Since $L_3^{(n)}$ is a Hamiltonian vector field and $\mathcal{V}_n$ is symplectic, the vector field $L_3^{(n+1)}$ is still Hamiltonian. We can repeat iteratively the procedure of Lemmata 3.3 and 3.4. Applying it M-times, we derive the following proposition.

Proposition 3.1. Let $\gamma \in (0,1)$, $\tau > 0$, $M \in \mathbb{N}$, $q > s_0 + \sigma + M$. Then there exists a constant $\delta_q \in (0,1)$ (possibly smaller than the one appearing in Lemma 3.2) such that for $\varepsilon\gamma^{-1} \le \delta_q$, for any $s_0 \le s \le q - \sigma - M$, for any $\omega \in DC(\gamma,\tau)$, the following holds: the symplectic invertible map $\widetilde{\mathcal{V}}_M(\varphi) := \mathcal{V}_0(\varphi) \circ \ldots \circ \mathcal{V}_{M-1}(\varphi) \in OPS^0$ satisfies the estimate

$$|\widetilde{\mathcal{V}}_M^{\pm 1}|_{0,s}^{\mathrm{Lip}(\gamma)}, \ |\widetilde{\mathcal{V}}_M^T|_{0,s}^{\mathrm{Lip}(\gamma)} \lesssim_{M,q} 1, \tag{3.50}$$

and the push forward $L_3^{(M)}(\varphi) := (\widetilde{\mathcal{V}}_M)_{\omega *} L_3^{(0)}(\varphi)$ of the Hamiltonian vector field $L_3^{(0)}(\varphi)$ in (3.33) is the Hamiltonian vector field

$$L_3^{(M)}(\varphi) = \mathrm{i} mT|D| + R_M(\varphi) + Q_M(\varphi) \tag{3.51}$$

where $R_M(\varphi) = R_M(\varphi;\omega)$, $Q_M(\varphi) = Q_M(\varphi;\omega)$, $\omega \in DC(\gamma,\tau)$ have the form

$$R_M := \mathrm{i}\begin{pmatrix} \mathrm{Op}(r_M) & 0 \\ 0 & -\overline{\mathrm{Op}(r_M)} \end{pmatrix}, \qquad r_M(\varphi,\cdot) \in S^{-1}, \tag{3.52}$$

$$Q_M := \mathrm{i}\begin{pmatrix} 0 & \mathrm{Op}(q_M) \\ -\overline{\mathrm{Op}(q_M)} & 0 \end{pmatrix}, \qquad q_M(\varphi,\cdot) \in S^{-M} \tag{3.53}$$

and satisfy the estimates

$$|R_M|_{-1,s}^{\mathrm{Lip}(\gamma)}, \ |Q_M|_{-M,q}^{\mathrm{Lip}(\gamma)} \lesssim_{M,s} \varepsilon, \quad \forall s_0 \le s \le q - \sigma - M. \tag{3.54}$$

□

Proof. We need only to prove the estimates (3.50). For any $n = 1, \dots, M-1$ one has

$$|\mathcal{V}_n|_{0,s} \le 1 + |\mathcal{V}_n - \mathrm{Id}|_{0,s} \overset{(2.107)}{\le} 1 + |\mathcal{V}_n - \mathrm{Id}|_{-n-1,s} \overset{(3.46)}{\lesssim_{n,s}} 1,$$

for any $s_0 \le s \le q - n - \sigma$. Since $n \le M$, one has that the above estimate holds for any $s_0 \le s \le q - \sigma - M$. Applying Lemma 2.14 and using the above estimate one gets the estimate (3.50) for $\widetilde{\mathcal{V}}_M$. The estimates for $\widetilde{\mathcal{V}}_M^{-1}$ follow by similar arguments and the estimates for $\widetilde{\mathcal{V}}_M^T$ follow since $|\widetilde{\mathcal{V}}_M^T|_{0,s} \le |\widetilde{\mathcal{V}}_M|_{0,s}$ and then the lemma is proved. ■

The operator $L_3^{(M)}(\varphi)$ in (3.51) is a space-diagonal operator up to the smoothing remainder $Q_M(\varphi) \in OPS^{-M}$. The prize which has been paid is that there is a *loss of regularity* of M derivatives with respect to the variable φ. In any case, the number of regularizing steps M will be fixed in (3.68).

3.4.2 Reduction to constant coefficients of the diagonal reminder R_M

Our next aim is to eliminate the φ dependence from the diagonal remainder $R_M(\varphi)$ of the Hamiltonian vector field $L_3^{(M)}(\varphi)$ defined in (3.51). In order to achieve this purpose, we look for a transformation of the form

$$\mathcal{E}(\varphi) := \exp(\mathrm{i}E(\varphi)), \qquad E(\varphi) := \begin{pmatrix} \mathrm{Op}(e(\varphi, |j|)) & 0 \\ 0 & -\overline{\mathrm{Op}(e(\varphi, |j|))} \end{pmatrix}, \qquad e(\varphi, \cdot) \in S^{-1}. \tag{3.55}$$

For any $\varphi \in \mathbb{T}^\nu$,

$$\mathcal{E}(\varphi)^{\pm 1} = \begin{pmatrix} \mathrm{Op}\,(\exp(\pm \mathrm{i}e(\varphi, |j|))) & 0 \\ 0 & \overline{\mathrm{Op}\,(\exp(\pm \mathrm{i}e(\varphi, |j|)))} \end{pmatrix} \tag{3.56}$$

and

$$\mathcal{E}(\varphi)^{-1}\omega \cdot \partial_\varphi \mathcal{E}(\varphi) = \begin{pmatrix} \mathrm{Op}\,\big(\mathrm{i}\omega \cdot \partial_\varphi e(\varphi, \cdot)\big) & 0 \\ 0 & \overline{\mathrm{Op}\,\big(\mathrm{i}\omega \cdot \partial_\varphi e(\varphi, \cdot)\big)} \end{pmatrix}. \tag{3.57}$$

Therefore by (2.117), (3.56), and (3.57) and recalling the properties stated in (2.100), the vector field $L_4^{(M)}(\varphi) := \mathcal{E}_{\omega *} L_3^{(M)}(\varphi)$ is given by

$$L_4^{(M)} := \mathcal{E}^{-1} L_3^{(M)} \mathcal{E} - \mathcal{E}(\varphi)^{-1}\omega \cdot \partial_\varphi \mathcal{E}$$
$$= \begin{pmatrix} \mathrm{Op}\,(\exp(-\mathrm{i}e))\,(\mathrm{i}m|D| + \mathrm{i}\mathrm{Op}(r_M))\,\mathrm{Op}\,(\exp(\mathrm{i}e)) & 0 \\ 0 & \overline{\mathrm{Op}\,(\exp(-\mathrm{i}e))\,(\mathrm{i}m|D| + \mathrm{i}\mathrm{Op}(r_M))\,\mathrm{Op}\,(\exp(\mathrm{i}e))} \end{pmatrix}$$

$$+ \mathcal{E}^{-1} Q_M \mathcal{E} - \begin{pmatrix} \mathrm{Op}\,(\mathrm{i}\omega \cdot \partial_\varphi e) & 0 \\ 0 & \overline{\mathrm{Op}\,(\mathrm{i}\omega \cdot \partial_\varphi e)} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{i}m|D| + \mathrm{Op}\,(\mathrm{i}r_M - \mathrm{i}\omega \cdot \partial_\varphi e) & 0 \\ 0 & -\mathrm{i}m|D| + \overline{\mathrm{Op}\,(\mathrm{i}r_M - \mathrm{i}\omega \cdot \partial_\varphi e)} \end{pmatrix} + \mathcal{E}^{-1} Q_M \mathcal{E}. \tag{3.58}$$

To shorten notations, in the above chain of equalities, we avoided to write the dependence on φ. In order to eliminate the φ-dependence from the symbol $r_M(\varphi, |j|)$, we need to solve the equation

$$-\omega \cdot \partial_\varphi e(\varphi, |j|) + r_M(\varphi, |j|) = c(|j|) \in \mathbb{R}, \qquad \forall j \in \mathbb{Z}^d \setminus \{0\}, \quad \forall \varphi \in \mathbb{T}^\nu$$

or equivalently

$$-\omega \cdot \partial_\varphi e(\varphi, \alpha) + r_M(\varphi, \alpha) = c(\alpha), \qquad \forall (\varphi, \alpha) \in \mathbb{T}^\nu \times \sigma_0(\sqrt{-\Delta}), \qquad c(\alpha) \in \mathbb{R}. \tag{3.59}$$

Integrating with respect to φ the above equation, we determine the value of the constant $c(\alpha)$, namely

$$c(\alpha) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} r_M(\varphi, \alpha)\, \mathrm{d}\varphi, \qquad \forall \alpha \in \sigma_0(\sqrt{-\Delta}) \tag{3.60}$$

and then we choose

$$e(\varphi, \alpha) := (\omega \cdot \partial_\varphi)^{-1} \left(r_M(\varphi, \alpha) - c(\alpha) \right), \qquad \forall (\varphi, \alpha) \in \mathbb{T}^\nu \times \sigma_0(\sqrt{-\Delta}), \tag{3.61}$$

(note that $\omega \in DC(\gamma, \tau)$ and recall the definition (2.15)). By (3.53), (3.56), (3.58), and (3.59), one gets

$$L_4^{(M)}(\varphi) = \mathcal{E}_{\omega *} L_3^{(M)}(\varphi) = \mathrm{i} D_M T + Q_{M,4}(\varphi), \tag{3.62}$$

where the diagonal operator D_M is defined as

$$D_M := m|D| + \mathrm{Op}(c(|j|)) = \mathrm{diag}_{j \in \mathbb{Z}^d \setminus \{0\}} \left(m|j| + c(|j|) \right) \tag{3.63}$$

and

$$Q_{M,4}(\varphi) := \mathcal{E}(\varphi)^{-1} Q_M(\varphi) \mathcal{E}(\varphi) = \mathrm{i} \begin{pmatrix} 0 & \mathrm{Op}(q_{M,4}) \\ -\overline{\mathrm{Op}(q_{M,4})} & 0 \end{pmatrix}, \qquad q_{M,4} := q_M \exp(-2\mathrm{i}e). \tag{3.64}$$

Lemma 3.5. Let $\gamma \in (0,1)$, $\tau > 0$, $M \in \mathbb{N}$, $q > s_0 + \sigma + 2\tau + M + 1$. Then there exists a constant $\delta_q \in (0,1)$ (possibly smaller than the one appearing in Proposition 3.1) such that for $\varepsilon\gamma^{-1} \leq \delta_q$, for any $s_0 \leq s \leq q - M - \sigma - 2\tau - 1$, the following holds: for any $\alpha \in \sigma_0(\sqrt{-\Delta})$, the constant $c(\alpha) = c(\alpha;\omega)$, given in (3.60), is real and defined for all the parameters $\omega \in DC(\gamma,\tau)$. Furthermore it satisfies the Lipschitz estimate

$$\sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} |c(\alpha)|^{\mathrm{Lip}(\gamma)}\alpha \lesssim_{M,q} \varepsilon. \tag{3.65}$$

The symplectic invertible operator $\mathcal{E}(\varphi) = \mathcal{E}(\varphi;\omega) \in OPS^0$, $\omega \in DC(\gamma,\tau)$, defined in (3.55) satisfies the estimates

$$|\mathcal{E}^{\pm 1}|_{0,s}^{\mathrm{Lip}(\gamma)}, \ |\mathcal{E}^T|_{0,s}^{\mathrm{Lip}(\gamma)} \lesssim_{M,q} 1. \tag{3.66}$$

The Hamiltonian vector field $\mathcal{Q}_{M,4}(\varphi) = \mathcal{Q}_{M,4}(\varphi;\omega) \in OPS^{-M}$, $\omega \in DC(\gamma,\tau)$ defined in (3.64) satisfies the estimates

$$|\mathcal{Q}_{M,4}|_{-M,s}^{\mathrm{Lip}(\gamma)} \lesssim_{M,q} \varepsilon. \tag{3.67}$$

□

Proof. Since the remainder R_M in (3.52) is a Hamiltonian vector field, then $\mathrm{Op}(r_M)$ is self-adjoint, hence by (2.101) the symbol $r_M(\varphi,\alpha)$ is real, implying that, by (3.60), $c(\alpha)$ is real for any $\alpha \in \sigma_0(\sqrt{-\Delta})$. The estimate (3.65) follows by (3.60) and (3.54). The estimates (3.66) follow by (3.56), (3.61), (3.54), and (3.65) (using also Lemma 2.2 to estimate $\|\exp(\mathrm{i}e)\|_s$.) The estimate (3.67) follows by Lemma 2.14 and by the estimates (3.54) and (3.66). ■

3.4.3 Conjugation of the operator $\mathcal{L}_3$ in (3.31)

Now we compute the conjugation of the vector field $\mathcal{L}_3 = L_3^{(0)} + \mathcal{R}_3$ in (3.31) (see (3.32) and (3.33)). First, we link the number of regularization steps with the regularity q of the functions $a(\varphi), b_k(\varphi,x), c_k(\varphi,x)$, $k = 1,\ldots,N$ (recall (1.2), (1.3)). We define

$$M = M(q) := [q/2], \qquad \overline{\mu} = \overline{\mu}(\tau,d) := \frac{d-1}{2} + \sigma + 2\tau + 1 \tag{3.68}$$

and we define the map

$$\mathcal{T} := \widetilde{\mathcal{V}}_M \circ \mathcal{E}. \tag{3.69}$$

By (3.51) and (3.62) one gets that

$$\mathcal{L}_4(\varphi) := (\mathcal{T})_{\omega*}\mathcal{L}_3(\varphi) = \mathrm{i}D_M T + \mathcal{R}_4(\varphi) \tag{3.70}$$

where the diagonal operator D_M is defined in (3.63), T is defined in (3.30) and the operator $\mathcal{R}_4$ is defined by

$$\mathcal{R}_4(\varphi) := Q_{M,4}(\varphi) + \varepsilon \mathcal{T}(\varphi)^{-1}\mathcal{R}_3(\varphi)\mathcal{T}(\varphi), \quad \varphi \in \mathbb{T}^\nu. \tag{3.71}$$

Lemma 3.6. Let $\gamma \in (0,1)$, $\tau > 0$, $q > 2(s_0 + \overline{\mu})$, where $\overline{\mu}$ is defined in (3.68). Then there exists $\delta_q \in (0,1)$ (possibly smaller than the one appearing in Lemma 3.5) such that if $\varepsilon\gamma^{-1} \le \delta_q$, for all $s_0 \le s \le [q/2] - \overline{\mu}$, the following holds: the symplectic invertible operator $\mathcal{T}(\varphi) = \mathcal{T}(\varphi;\omega) \in OPS^0$, $\omega \in DC(\gamma,\tau)$ defined in (3.55) satisfies the estimates

$$|\mathcal{T}^{\pm 1}|_{0,s}^{\mathrm{Lip}(\gamma)}, \, |\mathcal{T}^T|_{0,s}^{\mathrm{Lip}(\gamma)} \lesssim_q 1. \tag{3.72}$$

As a consequence one has $\mathcal{T}^{\pm 1} \in \mathcal{C}^1\left(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0(\mathbb{T}^d))\right)$.
The remainder $\mathcal{R}_4(\varphi) = \mathcal{R}_4(\varphi;\omega)$, $\omega \in DC(\gamma,\tau)$ defined in (3.71) satisfies the estimates

$$|\mathcal{R}_4|_s^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon \tag{3.73}$$

where the block-decay norm $|\cdot|_s^{\mathrm{Lip}(\gamma)}$ is defined in (2.76)-(2.79). □

Proof. By the choices of the constants in (3.68), one has that if $s_0 \le s \le [q/2] - \overline{\mu}$, then

$$s + \frac{d-1}{2} \le M \qquad \text{and} \qquad s_0 \le s \le q - M - \sigma - 2\tau - 1.$$

The estimates (3.72) follow by Lemma 2.14 and by the estimates (3.50) and (3.66). The fact that $\mathcal{T}^{\pm 1} \in \mathcal{C}^1\left(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0(\mathbb{T}^d))\right)$ follows by applying Lemma 2.13.
Now we prove the estimate (3.73). We estimate separately the two terms in (3.71).
Estimate of $Q_{M,4}$. By Lemma 2.16 one gets

$$|Q_{M,4}|_s^{\mathrm{Lip}(\gamma)} \lesssim |Q_{M,4}|_{-s-\frac{d-1}{2},s}^{\mathrm{Lip}(\gamma)}$$

hence we can apply the estimate (3.67), obtaining that $|Q_{M,4}|_{-s-\frac{d-1}{2},s}^{\mathrm{Lip}(\gamma)} \lesssim |Q_{M,4}|_{-M,s}^{\mathrm{Lip}(\gamma)} \lesssim_{M,q} \varepsilon \lesssim_q \varepsilon$, since the constant $M = M(q) = [q/2]$.

Estimate of $\mathcal{T}^{-1}\mathcal{R}_3\mathcal{T}$. Recalling the definition of $\mathcal{R}_3$ given in (3.32) and using that the operator $\mathcal{R}^{(3)}$ has the form (3.28), defining

$$B_{1,k} := (\mathrm{i}b_k^{(3)}, -\mathrm{i}b_k^{(3)}), \quad B_{2,k} := (b_k^{(3)}, b_k^{(3)}), \quad C_{1,k} := (\mathrm{i}c_k^{(3)}, -\mathrm{i}c_k^{(3)}), \quad C_{2,k} := (c_k^{(3)}, c_k^{(3)}),$$
$$k = 1, \ldots, N$$

we have that for $\mathbf{u} = (u, \overline{u}) \in \mathbf{L}^2_0(\mathbb{T}^d)$,

$$\mathcal{R}_3[\mathbf{u}] = \sum_{k=1}^{N} B_{1,k} \langle C_{2,k}, \mathbf{u} \rangle_{\mathbf{L}^2_x} + C_{1,k} \langle B_{2,k}, \mathbf{u} \rangle_{\mathbf{L}^2_x}$$

where we recall that the bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{L}^2_x}$ is defined in (2.125). Thus

$$(\mathcal{T}^{-1}\mathcal{R}_3\mathcal{T})[\mathbf{u}] = \sum_{k=1}^{N} \widetilde{B}_{1,k} \langle \widetilde{C}_{2,k}, \mathbf{u} \rangle_{\mathbf{L}^2_x} + \widetilde{C}_{1,k} \langle \widetilde{B}_{2,k}, \mathbf{u} \rangle_{\mathbf{L}^2_x},$$

$$\widetilde{B}_{1,k} := \mathcal{T}^{-1} B_{1,k}, \quad \widetilde{B}_{2,k} := \mathcal{T}^{T} B_{2,k}, \quad \widetilde{C}_{1,k} := \mathcal{T}^{-1} C_{1,k}, \quad \widetilde{C}_{2,k} := \mathcal{T}^{T} C_{2,k}, \quad k = 1, \ldots, N.$$

The operator $\varepsilon\mathcal{T}^{-1}\mathcal{R}_3\mathcal{T}$ satisfies the claimed inequality, by applying the estimates (3.29), (3.72) and Lemmata 2.12, 2.10. ■

4 Block-Diagonal Reducibility

In this section, we carry out the second part of the reduction of $\mathcal{L}(\varphi)$ to a block-diagonal operator with constant coefficients. Our goal is to block-diagonalize the linear Hamiltonian vector field $\mathcal{L}_4(\varphi)$ obtained in (3.70). We are going to perform an iterative Nash–Moser reducibility scheme for the linear Hamiltonian vector field

$$\mathcal{L}_0(\varphi) := \mathcal{L}_4(\varphi) = \mathcal{D}_0 + \mathcal{R}_0(\varphi), \tag{4.1}$$

where

$$\mathcal{D}_0 = \mathrm{i} \begin{pmatrix} -\mathcal{D}_0^{(1)} & 0 \\ 0 & \mathcal{D}_0^{(1)} \end{pmatrix}, \quad \mathcal{D}_0^{(1)} := D_M = \mathrm{diag}_{j \in \mathbb{Z}^d \setminus \{0\}} \left(m|j| + c(|j|) \right) \tag{4.2}$$

(see (3.63)) and $\mathcal{R}_0(\varphi) := \mathcal{R}_4(\varphi)$, $\varphi \in \mathbb{T}^\nu$, is a Hamiltonian vector field of the form

$$\mathcal{R}_0(\varphi) = \mathrm{i} \begin{pmatrix} \mathcal{R}_0^{(1)}(\varphi) & \mathcal{R}_0^{(2)}(\varphi) \\ -\overline{\mathcal{R}}_0^{(2)}(\varphi) & -\overline{\mathcal{R}}_0^{(1)}(\varphi) \end{pmatrix}, \quad \mathcal{R}_0^{(1)}(\varphi) = \mathcal{R}_0^{(1)}(\varphi)^*, \quad \mathcal{R}_0^{(2)}(\varphi) = \mathcal{R}_0^{(2)}(\varphi)^T \tag{4.3}$$

satisfying, by (3.73), the estimate

$$|\mathcal{R}_0|_s^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon, \qquad \forall s_0 \le s \le [q/2] - \overline{\mu} \tag{4.4}$$

where the constant $\overline{\mu}$ is defined in (3.68). According to the block representation (2.45), the operator $\mathcal{D}_0^{(1)}$ can be written as

$$\mathcal{D}_0^{(1)} = \mathrm{diag}_{\alpha \in \sigma_0(\sqrt{-\Delta})} \mu_\alpha^0 \mathbb{I}_\alpha, \qquad \mu_\alpha^0 := m\,\alpha + c(\alpha), \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}) \tag{4.5}$$

where $\mathbb{I}_\alpha : \mathbb{E}_\alpha \to \mathbb{E}_\alpha$ is the identity (recall (2.3), (2.51)) and the real constants m and $c(\alpha)$ satisfy the estimates (3.27) and (3.65). We define

$$N_{-1} := 1, \quad N_k := N_0^{\chi^k} \ \forall k \geq 0, \quad \chi := 3/2 \tag{4.6}$$

(then $N_{k+1} = N_k^\chi$, $\forall k \geq 0$) and for $\tau, \mathtt{d} > 0$, we define the constants

$$\mathfrak{s}_0 := 2s_0, \quad \mathtt{a} := 4\tau + 8\mathtt{d} + 3, \quad \mathtt{b} := \mathtt{a} + 1, \quad S_q := [q/2] - \overline{\mu} - \mathtt{b}, \quad \text{with} \quad q > 2(\mathfrak{s}_0 + \overline{\mu} + \mathtt{b}). \tag{4.7}$$

In order to state the theorem below, we recall the definition of the space $\mathcal{S}(\mathbb{E}_\alpha), \alpha \in \sigma_0(\sqrt{-\Delta})$ given in (2.67), the definition of the norm $\| \cdot \|_{\mathrm{Op}(\alpha,\beta)}, \alpha, \beta \in \sigma_0(\sqrt{-\Delta})$ given in (2.62), the identity $\mathbb{I}_{\alpha,\beta}, \alpha, \beta \in \sigma_0(\sqrt{-\Delta})$ in (2.63), the definition of $M_L(A)$ in (2.64) and the definition of $M_R(B)$ in (2.65).

Theorem 4.1 (KAM reducibility). Let $\gamma \in (0,1)$, $\tau, \mathtt{d} > 0$ and let q satisfy (4.7). There exist, $N_0 = N_0(q, \tau, \mathtt{d}, \nu, d) \in \mathbb{N}$ large enough, $\delta_q = \delta(q, \tau, \mathtt{d}, \nu, d) \in (0,1)$ (possibly smaller than the one appearing in Lemma 3.6) such that, if

$$\varepsilon \gamma^{-1} \leq \delta_q \tag{4.8}$$

then, for all $k \geq 0$:

$(\mathbf{S1})_k$ There exists a Hamiltonian vector field

$$\mathcal{L}_k(\varphi) := \mathcal{D}_k + \mathcal{R}_k(\varphi), \qquad \varphi \in \mathbb{T}^\nu, \tag{4.9}$$

$$\mathcal{D}_k = \mathrm{i} \begin{pmatrix} -\mathcal{D}_k^{(1)} & 0 \\ 0 & \overline{\mathcal{D}_k^{(1)}} \end{pmatrix}, \qquad \mathcal{D}_k^{(1)} := \mathrm{diag}_{\alpha \in \sigma_0(\sqrt{-\Delta})} [\mathcal{D}_k^{(1)}]_\alpha^\alpha, \quad [\mathcal{D}_k^{(1)}]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha),$$
$$\forall \alpha \in \sigma_0(\sqrt{-\Delta}) \tag{4.10}$$

defined for all $\omega \in \Omega_k^\gamma$, where $\Omega_0^\gamma := DC(\gamma, \tau)$ (see (2.16)) and for $k \geq 1$,

$$\begin{aligned} \Omega_k^\gamma := \Big\{ & \omega \in \Omega_{k-1}^\gamma : \|\mathbb{A}_{k-1}^-(\ell, \alpha, \beta)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \leq \frac{\alpha^{\mathtt{d}} \beta^{\mathtt{d}} \langle \ell \rangle^\tau}{\gamma}, \\ & \forall (\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta}), \\ & (\ell, \alpha, \beta) \neq (0, \alpha, \alpha), \quad \langle \ell, \alpha, \beta \rangle \leq N_{k-1} \quad \text{and} \quad \|\mathbb{A}_{k-1}^+(\ell, \alpha, \beta)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \\ & \leq \frac{\langle \ell \rangle^\tau}{\gamma \langle \alpha + \beta \rangle}, \\ & \forall (\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta}), \quad \langle \ell, \alpha, \beta \rangle \leq N_{k-1} \Big\}. \end{aligned} \tag{4.11}$$

The operators $\mathbb{A}^{\pm}_{k-1}(\ell, \alpha, \beta) : \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \to \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ are defined by

$$\mathbb{A}^{-}_{k-1}(\ell, \alpha, \beta) := \omega \cdot \ell \mathbb{I}_{\alpha,\beta} + M_L([\mathcal{D}^{(1)}_{k-1}]^\alpha_\alpha) - M_R([\mathcal{D}^{(1)}_{k-1}]^\beta_\beta), \tag{4.12}$$

$$\mathbb{A}^{+}_{k-1}(\ell, \alpha, \beta) := \omega \cdot \ell \mathbb{I}_{\alpha,\beta} + M_L([\mathcal{D}^{(1)}_{k-1}]^\alpha_\alpha) + M_R(\overline{[\mathcal{D}^{(1)}_{k-1}]^\beta_\beta}). \tag{4.13}$$

For $k \geq 0$, for all $\alpha \in \sigma_0(\sqrt{-\Delta})$, the self-adjoint operator $[\mathcal{D}^{(1)}_k]^\alpha_\alpha \in \mathcal{S}(\mathbb{E}_\alpha)$ satisfies

$$\|[\mathcal{D}^{(1)}_k - \mathcal{D}^{(1)}_0]^\alpha_\alpha\|^{\mathrm{Lip}(\gamma)}_{HS} \lesssim_q \varepsilon \alpha^{-S_q} \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}). \tag{4.14}$$

The remainder $\mathcal{R}_k$ is Hamiltonian and $\forall s \in [\mathfrak{s}_0, S_q]$,

$$|\mathcal{R}_k|^{\mathrm{Lip}(\gamma)}_s \leq |\mathcal{R}_0|^{\mathrm{Lip}(\gamma)}_{s+\mathrm{b}} N^{-\mathrm{a}}_{k-1}, \quad |\mathcal{R}_k|^{\mathrm{Lip}(\gamma)}_{s+\mathrm{b}} \leq |\mathcal{R}_0|^{\mathrm{Lip}(\gamma)}_{s+\mathrm{b}} N_{k-1}. \tag{4.15}$$

Moreover, for $k \geq 1$,

$$\mathcal{L}_k(\varphi) = (\Phi_k)_{\omega *}\mathcal{L}_{k-1}(\varphi), \quad \Phi_{k-1} := \exp(\Psi_{k-1}) \tag{4.16}$$

where the map Ψ_{k-1} is a Hamiltonian vector field and satisfies

$$|\Psi_{k-1}|^{\mathrm{Lip}(\gamma)}_s \leq |\mathcal{R}_0|^{\mathrm{Lip}(\gamma)}_{s+\mathrm{b}} \gamma^{-1} N^{2\tau+4\mathrm{d}+1}_{k-1} N^{-\mathrm{a}}_{k-2}. \tag{4.17}$$

$(\mathbf{S2})_k$ For all $\alpha \in \sigma_0(\sqrt{-\Delta})$, there exists a Lipschitz extension to the set $DC(\gamma, \tau)$, that we denote by $[\widetilde{\mathcal{D}}^{(1)}_k]^\alpha_\alpha(\cdot) : DC(\gamma, \tau) \to \mathcal{S}(\mathbb{E}_\alpha)$ of $[\mathcal{D}^{(1)}_k]^\alpha_\alpha(\cdot) : \Omega^\gamma_k \to \mathcal{S}(\mathbb{E}_\alpha)$ satisfying, for $k \geq 1$,

$$\|[\widetilde{\mathcal{D}}^{(1)}_k]^\alpha_\alpha - [\widetilde{\mathcal{D}}^{(1)}_{k-1}]^\alpha_\alpha\|^{\mathrm{Lip}(\gamma)}_{HS} \lesssim \alpha^{-S_q} |\mathcal{R}_{k-1}|^{\mathrm{Lip}(\gamma)}_{S_q} \lesssim N^{-\mathrm{a}}_{k-2} \alpha^{-S_q} |\mathcal{R}_0|^{\mathrm{Lip}(\gamma)}_{S_q+\mathrm{b}}. \tag{4.18}$$

□

Remark 4.1. The constants $\tau, \mathrm{d} > 0$ in (4.11) will be fixed in the formula (5.1), in Section 5, in order to prove the measure estimate of the set $\Omega^{2\gamma}_\infty$ defined in (4.77) (see Theorem 5.1). □

4.1 Proof of Theorem 4.1

Proof of $(\mathbf{Si})_0$, $i = 1, 2$. Properties (4.9)–(4.15) in $(\mathbf{S1})_0$ hold by (4.1)–(4.4) with $[\mathcal{D}^{(1)}_0]^\alpha_\alpha$ given in (4.5) (for (4.15) recall that $N_{-1} := 1$, see (4.6)). Moreover, since the constants m and $c(\alpha) = c(\alpha; \omega)$ are real, $[\mathcal{D}^{(1)}_0]^\alpha_\alpha$ is self-adjoint, then there is nothing else to verify.

$(\mathbf{S2})_0$ holds, since the constant m is independent of ω and $c(\alpha) = c(\alpha; \omega)$, $\alpha \in \sigma_0(\sqrt{-\Delta})$, is already defined for all $\omega \in DC(\gamma, \tau)$.

4.2 The reducibility step

We now describe the inductive step, showing how to define a symplectic transformation $\Phi_k := \exp(\Psi_k)$ so that the transformed vector field $\mathcal{L}_{k+1}(\varphi) = (\Phi_k)_{\omega *}\mathcal{L}_k(\varphi)$ has the desired properties. To simplify notations, in this section we drop the index k and we write $+$ instead of $k+1$. At each step of the iteration we have a Hamiltonian vector field

$$\mathcal{L}(\varphi) = \mathcal{D} + \mathcal{R}(\varphi), \tag{4.19}$$

where

$$\mathcal{D} := \mathrm{i}\begin{pmatrix} -\mathcal{D}^{(1)} & 0 \\ 0 & \overline{\mathcal{D}}^{(1)} \end{pmatrix}, \qquad \mathcal{D}^{(1)} := \mathrm{diag}_{\alpha\in\sigma_0(\sqrt{-\Delta})}[\mathcal{D}^{(1)}]_\alpha^\alpha, \quad [\mathcal{D}^{(1)}]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha) \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}) \tag{4.20}$$

and $\mathcal{R}(\varphi)$ is a Hamiltonian vector field, namely it has the form

$$\mathcal{R} = \mathrm{i}\begin{pmatrix} \mathcal{R}^{(1)} & \mathcal{R}^{(2)} \\ -\overline{\mathcal{R}}^{(2)} & -\overline{\mathcal{R}}^{(1)} \end{pmatrix}, \qquad \mathcal{R}^{(1)}(\varphi) = \mathcal{R}^{(1)}(\varphi)^*, \qquad \mathcal{R}^{(2)}(\varphi) = \mathcal{R}^{(2)}(\varphi)^T, \quad \forall \varphi \in \mathbb{T}^\nu. \tag{4.21}$$

Let us consider a transformation

$$\Phi(\varphi) := \exp(\Psi(\varphi)), \qquad \Psi(\varphi) := \mathrm{i}\begin{pmatrix} \Psi^{(1)}(\varphi) & \Psi^{(2)}(\varphi) \\ -\overline{\Psi}^{(2)}(\varphi) & -\overline{\Psi}^{(1)}(\varphi) \end{pmatrix}, \quad \varphi \in \mathbb{T}^\nu \tag{4.22}$$

with $\Psi^{(1)}(\varphi) = \Psi^{(1)}(\varphi)^*$, $\Psi^{(2)}(\varphi) = \Psi^{(2)}(\varphi)^T$, for all $\varphi \in \mathbb{T}^\nu$. Writing

$$\Phi = \mathrm{Id} + \Psi + \Psi_{\geq 2}, \quad \Psi_{\geq 2} := \sum_{k\geq 2} \frac{\Psi^k}{k!}. \tag{4.23}$$

By (2.127) we have $\Phi_{\omega *}\mathcal{L}(\varphi) = \Phi(\varphi)^{-1}\big(\mathcal{L}(\varphi)\Phi(\varphi) - \omega\cdot\partial_\varphi\Phi(\varphi)\big)$. By the expansion (4.23), recalling the definition of the projector operator $\Pi_N\mathcal{R}$ given in (2.74), one gets that

$$\begin{aligned} \mathcal{L}(\varphi)\Phi(\varphi) - \omega\cdot\partial_\varphi\Phi(\varphi) = \Phi(\varphi)\mathcal{D} + \big(-\omega\cdot\partial_\varphi\Psi + [\mathcal{D}, \Psi(\varphi)] + \Pi_N\mathcal{R}(\varphi)\big) + \Pi_N^\perp\mathcal{R}(\varphi) \\ - \omega\cdot\partial_\varphi\Psi_{\geq 2}(\varphi) + [\mathcal{D}, \Psi_{\geq 2}(\varphi)] + \mathcal{R}(\varphi)(\Phi(\varphi) - \mathrm{Id}). \end{aligned} \tag{4.24}$$

We want to determine the operator $\Psi(\varphi)$ so that

$$-\omega\cdot\partial_\varphi\Psi(\varphi) + [\mathcal{D}, \Psi(\varphi)] + \Pi_N\mathcal{R}(\varphi) = \Pi_N\mathcal{R}_{diag}, \tag{4.25}$$

where recalling the definitions (2.73) and (2.74)

$$\Pi_N \mathcal{R}_{diag} := \mathrm{i} \begin{pmatrix} \Pi_N \mathcal{R}^{(1)}_{diag} & 0 \\ 0 & -\Pi_N \overline{\mathcal{R}}^{(1)}_{diag} \end{pmatrix}. \tag{4.26}$$

Lemma 4.1 (Homological equation). For all $\omega \in \Omega^{\gamma}_{k+1}$ (see (4.11)), there exists a solution Ψ of the homological equation (4.25), which is Hamiltonian and satisfies

$$|\Psi|_s^{\mathrm{Lip}(\gamma)} \lesssim N^{2\tau+4\mathrm{d}+1} \gamma^{-1} |\mathcal{R}|_s^{\mathrm{Lip}(\gamma)}. \tag{4.27}$$

□

Proof. Recalling (4.21) and (4.22), the equation (4.25) is split in the two equations

$$-\mathrm{i}\omega \cdot \partial_\varphi \Psi^{(1)}(\varphi) + [\mathcal{D}^{(1)}, \Psi^{(1)}(\varphi)] + \mathrm{i}\Pi_N \mathcal{R}^{(1)}(\varphi) = \mathrm{i}\Pi_N \mathcal{R}^{(1)}_{diag}, \tag{4.28}$$

$$-\mathrm{i}\omega \cdot \partial_\varphi \Psi^{(2)}(\varphi) + (\mathcal{D}^{(1)} \Psi^{(2)}(\varphi) + \Psi^{(2)}(\varphi) \overline{\mathcal{D}}^{(1)}) + \mathrm{i}\Pi_N \mathcal{R}^{(2)}(\varphi) = 0. \tag{4.29}$$

Using the decomposition (2.45) and recalling (2.72), the equations (4.28) and (4.29) become for any $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$, $\ell \in \mathbb{Z}^\nu$

$$\omega \cdot \ell [\widehat{\Psi}^{(1)}(\ell)]_\alpha^\beta + [\mathcal{D}^{(1)}]_\alpha^\alpha [\widehat{\Psi}^{(1)}(\ell)]_\alpha^\beta - [\widehat{\Psi}^{(1)}(\ell)]_\alpha^\beta [\mathcal{D}^{(1)}]_\beta^\beta = -\mathrm{i}[\widehat{\Pi_N \mathcal{R}}^{(1)}(\ell)]_\alpha^\beta + \mathrm{i}[\widehat{\Pi_N \mathcal{R}}^{(1)}_{diag}(\ell)]_\alpha^\beta \tag{4.30}$$

$$\omega \cdot \ell [\widehat{\Psi}^{(2)}(\ell)]_\alpha^\beta + [\mathcal{D}^{(1)}]_\alpha^\alpha [\widehat{\Psi}^{(2)}(\ell)]_\alpha^\beta + [\widehat{\Psi}^{(2)}(\ell)]_\alpha^\beta [\overline{\mathcal{D}}^{(1)}]_\beta^\beta = -\mathrm{i}[\widehat{\Pi_N \mathcal{R}}^{(2)}(\ell)]_\alpha^\beta. \tag{4.31}$$

By the definitions (4.12) and (4.13), namely setting

$$\begin{aligned} \mathbb{A}^-(\ell, \alpha, \beta) &:= \omega \cdot \ell \mathbb{I}_{\alpha,\beta} + M_L([\mathcal{D}^{(1)}]_\alpha^\alpha) - M_R([\mathcal{D}^{(1)}]_\beta^\beta), \ \mathbb{A}^+(\ell, \alpha, \beta) \\ &:= \omega \cdot \ell \mathbb{I}_{\alpha,\beta} + M_L([\mathcal{D}^{(1)}]_\alpha^\alpha) + M_R([\overline{\mathcal{D}}^{(1)}]_\beta^\beta) \end{aligned} \tag{4.32}$$

the equations (4.30) and (4.31) can be written in the form

$$\begin{aligned} \mathbb{A}^-(\ell, \alpha, \beta)[\widehat{\Psi}^{(1)}(\ell)]_\alpha^\beta &= -\mathrm{i}[\widehat{\Pi_N \mathcal{R}}^{(1)}(\ell)]_\alpha^\beta + \mathrm{i}[\widehat{\Pi_N \mathcal{R}}^{(1)}_{diag}(\ell)]_\alpha^\beta, \quad \mathbb{A}^+(\ell, \alpha, \beta)[\widehat{\Psi}^{(2)}(\ell)]_\alpha^\beta \\ &= -\mathrm{i}[\widehat{\Pi_N \mathcal{R}}^{(2)}(\ell)]_\alpha^\beta. \end{aligned}$$

Then, since $\omega \in \Omega^{\gamma}_{k+1}$, recalling the Definition (2.74), we can define for any $(\ell, \alpha, \beta) \in \mathbb{Z}^{\nu} \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$

$$[\widehat{\Psi}^{(1)}(\ell)]_{\alpha}^{\beta} := \begin{cases} \mathrm{i}\mathbb{A}^{-}(\ell, \alpha, \beta)^{-1}[\widehat{\mathcal{R}}^{(1)}(\ell)]_{\alpha}^{\beta} & \text{if} \quad (\ell, \alpha, \beta) \neq (0, \alpha, \alpha), \quad \langle \ell, \alpha, \beta \rangle \leq N \\ 0 & \text{otherwise} \end{cases} \tag{4.33}$$

$$[\widehat{\Psi}^{(2)}(\ell)]_{\alpha}^{\beta} := \begin{cases} \mathrm{i}\mathbb{A}^{+}(\ell, \alpha, \beta)^{-1}[\widehat{\mathcal{R}}^{(2)}(\ell)]_{\alpha}^{\beta} & \text{if} \quad \langle \ell, \alpha, \beta \rangle \leq N \\ 0 & \text{otherwise.} \end{cases} \tag{4.34}$$

We have

$$\|\mathbb{A}^{-}(\ell, \alpha, \beta)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \leq \frac{\alpha^{\mathrm{d}}\beta^{\mathrm{d}}\langle \ell \rangle^{\tau}}{\gamma}, \quad \|\mathbb{A}^{+}(\ell, \alpha, \beta)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \leq \frac{\langle \ell \rangle^{\tau}}{\gamma(\alpha+\beta)}$$

and since $[\widehat{\Psi}^{(1)}(\ell)]_{\alpha}^{\beta}$, $[\widehat{\Psi}^{(2)}(\ell)]_{\alpha}^{\beta}$ are nonzero only if $\langle \ell, \alpha, \beta \rangle \leq N$, we get immediately that

$$\|[\widehat{\Psi}^{(1)}(\ell)]_{\alpha}^{\beta}\|_{HS} \leq N^{\tau+2\mathrm{d}}\gamma^{-1}\|[\widehat{\mathcal{R}}^{(1)}(\ell)]_{\alpha}^{\beta}\|_{HS}, \quad \|[\widehat{\Psi}^{(2)}(\ell)]_{\alpha}^{\beta}\|_{HS} \leq N^{\tau}\gamma^{-1}\|[\widehat{\mathcal{R}}^{(2)}(\ell)]_{\alpha}^{\beta}\|_{HS}. \tag{4.35}$$

Hence, recalling the definition (2.76) of the block-decay norm, one gets that

$$|\Psi^{(1)}|_s \lesssim N^{\tau+2\mathrm{d}}\gamma^{-1}|\mathcal{R}^{(1)}|_s, \quad |\Psi^{(2)}|_s \lesssim N^{\tau}\gamma^{-1}|\mathcal{R}^{(2)}|_s. \tag{4.36}$$

Now, let $\omega_1, \omega_2 \in \Omega^{\gamma}_{k+1}$. As a notation for any function $f = f(\omega)$ depending on the parameter ω, we write $\Delta_{\omega} f := f(\omega_1) - f(\omega_2)$. By (4.33), one has

$$\Delta_{\omega}[\widehat{\Psi}^{(1)}(\ell)]_{\alpha}^{\beta} = \mathrm{i}\Delta_{\omega}\mathbb{A}^{-}(\ell, \alpha, \beta)^{-1}[\widehat{\mathcal{R}}^{(1)}(\ell; \omega_1)]_{\alpha}^{\beta} + \mathrm{i}\mathbb{A}^{-}(\ell, \alpha, \beta; \omega_2)^{-1}\Delta_{\omega}[\widehat{\mathcal{R}}^{(1)}(\ell)]_{\alpha}^{\beta}. \tag{4.37}$$

As in (4.35), one gets

$$\|\mathbb{A}^{-}(\ell, \alpha, \beta; \omega_2)^{-1}\Delta_{\omega}[\widehat{\mathcal{R}}^{(1)}(\ell)]_{\alpha}^{\beta}\|_{HS} \lesssim N^{\tau+2\mathrm{d}}\gamma^{-1}\|\Delta_{\omega}[\widehat{\mathcal{R}}^{(1)}(\ell)]_{\alpha}^{\beta}\|_{HS}, \tag{4.38}$$

hence it remains to estimate only the first term in (4.37). We have

$$\Delta_{\omega}\mathbb{A}^{-}(\ell, \alpha, \beta)^{-1} = -\mathbb{A}^{-}(\ell, \alpha, \beta; \omega_1)^{-1}\left(\Delta_{\omega}\mathbb{A}^{-}(\ell, \alpha, \beta)\right)\mathbb{A}^{-}(\ell, \alpha, \beta; \omega_2)^{-1}, \tag{4.39}$$

Therefore

$$\|\Delta_{\omega}\mathbb{A}^{-}(\ell, \alpha, \beta)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \leq \frac{N^{2\tau}\alpha^{2\mathrm{d}}\beta^{2\mathrm{d}}}{\gamma^2}\|\Delta_{\omega}\mathbb{A}^{-}(\ell, \alpha, \beta)\|_{\mathrm{Op}(\alpha,\beta)}. \tag{4.40}$$

Moreover

$$\Delta_\omega \mathbb{A}^-(\ell,\alpha,\beta) = (\omega_1 - \omega_2)\cdot \ell\ \mathbb{I}_{\alpha,\beta} + M_L(\Delta_\omega[\mathcal{D}^{(1)}]_\alpha^\alpha) - M_R(\Delta_\omega[\mathcal{D}^{(1)}]_\beta^\beta) \tag{4.41}$$

and using that, by (4.5) and (4.14)

$$[\mathcal{D}^{(1)}(\omega)]_\alpha^\alpha = \mu_\alpha^0(\omega)\mathbb{I}_\alpha + [\mathcal{D}^{(1)} - \mathcal{D}_0^{(1)}]_\alpha^\alpha, \quad \text{with} \quad \|[\mathcal{D}^{(1)} - \mathcal{D}_0^{(1)}]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon\alpha^{-S_q}, \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}), \tag{4.42}$$

we get

$$M_L(\Delta_\omega[\mathcal{D}^{(1)}]_\alpha^\alpha) - M_R(\Delta_\omega[\mathcal{D}^{(1)}]_\beta^\beta) = \Delta_\omega\left(\mu_\alpha^0 - \mu_\beta^0\right)\mathbb{I}_{\alpha,\beta} + M_L(\Delta_\omega[\mathcal{D}^{(1)} - \mathcal{D}_0^{(1)}]_\alpha^\alpha) - M_R(\Delta_\omega[\mathcal{D}^{(1)} - \mathcal{D}_0^{(1)}]_\beta^\beta).$$

Using that the constant m is independent of ω, that is $\Delta_\omega m = 0$ and by recalling (4.5) and (3.65), one gets

$$\begin{aligned}|\Delta_\omega(\mu_\alpha^0 - \mu_\beta^0)| &\lesssim |\Delta_\omega c(\alpha)| + |\Delta_\omega c(\beta)| \lesssim \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} |c(\alpha)|^{\mathrm{lip}}|\omega_1 - \omega_2| \\ &\lesssim \gamma^{-1} \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} |c(\alpha)|^{\mathrm{Lip}(\gamma)}|\omega_1 - \omega_2| \\ &\lesssim_q \varepsilon\gamma^{-1}|\omega_1 - \omega_2|. \end{aligned} \tag{4.43}$$

By (4.42), (4.43) and using the property (2.66) one gets

$$\begin{aligned}&\| - M_L(\Delta_\omega[\mathcal{D}^{(1)}]_\alpha^\alpha) + M_R(\Delta_\omega[\mathcal{D}^{(1)}]_\beta^\beta)\|_{\mathrm{Op}(\alpha,\beta)} \\ &\lesssim |\Delta_\omega(\mu_0^\alpha - \mu_0^\beta)|\|\mathbb{I}_{\alpha,\beta}\|_{\mathrm{Op}(\alpha,\beta)} \\ &\quad + \|M_R(\Delta_\omega[\mathcal{D}^{(1)} - \mathcal{D}_0^{(1)}]_\beta^\beta) - M_L(\Delta_\omega[\mathcal{D}^{(1)} - \mathcal{D}_0^{(1)}]_\alpha^\alpha)\|_{\mathrm{Op}(\alpha,\beta)} \\ &\lesssim_q \varepsilon\gamma^{-1}|\omega_1 - \omega_2|. \end{aligned} \tag{4.44}$$

Recalling (4.41), we get the estimate

$$\|\Delta_\omega \mathbb{A}^-(\ell,\alpha,\beta)\|_{\mathrm{Op}(\alpha,\beta)} \le \left(C\langle\ell\rangle + C'(q)\varepsilon\gamma^{-1}\right)|\omega_1 - \omega_2|,$$

for some constants $C, C'(q) > 0$, hence, by (4.40), by taking δ_q in (4.8) small enough (so that $C'(q)\varepsilon\gamma^{-1} \le 1$), one gets that for $\langle \ell,\alpha,\beta\rangle \le N$

$$\|\Delta_\omega \mathbb{A}^-(\ell,\alpha,\beta)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \lesssim N^{2\tau+4\mathrm{d}+1}\gamma^{-2}|\omega_1 - \omega_2|.$$

The above estimate implies that

$$\| \left\{ \Delta_\omega \mathbb{A}^-(\ell, \alpha, \beta)^{-1} \right\} [\widehat{\mathcal{R}}^{(1)}(\ell; \omega_1)]_\alpha^\beta \|_{HS} \lesssim N^{2\tau+4d+1} \gamma^{-2} \| [\widehat{\mathcal{R}}^{(1)}(\ell; \omega_1)]_\alpha^\beta \|_{HS} |\omega_1 - \omega_2|. \qquad (4.45)$$

By (4.37), (4.38), and (4.45) we get the estimate

$$\| \Delta_\omega [\widehat{\Psi}^{(1)}(\ell)]_\alpha^\beta \|_{HS} \lesssim N^{\tau+2d} \gamma^{-1} \| \Delta_\omega [\widehat{\mathcal{R}}^{(1)}(\ell)]_\alpha^\beta \|_{HS} + N^{2\tau+4d+1} \gamma^{-2} \| [\widehat{\mathcal{R}}^{(1)}(\ell; \omega_1)]_\alpha^\beta \|_{HS}. \qquad (4.46)$$

Thus (4.36) and (4.46) and the definitions (2.76) and (2.77) imply

$$|\Psi^{(1)}|_s^{\mathrm{Lip}(\gamma)} \lesssim N^{2\tau+4d+1} \gamma^{-1} |\mathcal{R}^{(1)}|_s^{\mathrm{Lip}(\gamma)}.$$

The estimate of $\Psi^{(2)}$ in terms of $\mathcal{R}^{(2)}$ follows by similar arguments and then (4.27) is proved. ■

By (4.24) and (4.25), we get

$$\mathcal{L}_+(\varphi) := \Phi_{\omega *} \mathcal{L}(\varphi) = \mathcal{D}_+ + \mathcal{R}_+(\varphi), \quad \varphi \in \mathbb{T}^\nu, \qquad (4.47)$$

$$\mathcal{D}_+ := \mathcal{D} + \Pi_N \mathcal{R}_{diag}, \quad \mathcal{R}_+ := (\Phi^{-1} - \mathrm{Id}) \Pi_N \mathcal{R}_{diag}$$
$$+ \Phi^{-1} \left(\Pi_N^\perp \mathcal{R} - \omega \cdot \partial_\varphi \Psi_{\geq 2} + [\mathcal{D}, \Psi_{\geq 2}] + \mathcal{R}(\Phi - \mathrm{Id}) \right). \qquad (4.48)$$

Lemma 4.2 (The new block-diagonal part). The new block-diagonal part is given by

$$\mathcal{D}_+ := \mathcal{D} + \Pi_N \mathcal{R}_{diag} = \mathrm{i} \begin{pmatrix} -\mathcal{D}_+^{(1)} & 0 \\ 0 & \overline{\mathcal{D}}_+^{(1)} \end{pmatrix}, \qquad \mathcal{D}_+^{(1)} := \mathcal{D}^{(1)} + \Pi_N \mathcal{R}_{diag}^{(1)} = \mathrm{diag}_{\alpha \in \sigma_0(\sqrt{-\Delta})} [\mathcal{D}_+^{(1)}]_\alpha^\alpha, \qquad (4.49)$$

where

$$[\mathcal{D}_+^{(1)}]_\alpha^\alpha := \begin{cases} [\mathcal{D}]_\alpha^\alpha + [\widehat{\mathcal{R}}^{(1)}(0)]_\alpha^\alpha & \text{if} \quad \alpha \leq N \\ [\mathcal{D}]_\alpha^\alpha & \text{otherwise.} \end{cases} \qquad (4.50)$$

As a consequence

$$\| [\mathcal{D}_+^{(1)}]_\alpha^\alpha - [\mathcal{D}]_\alpha^\alpha \|_{HS}^{\mathrm{Lip}(\gamma)} \lesssim \alpha^{-S_q} |\mathcal{R}|_{S_q}^{\mathrm{Lip}(\gamma)}, \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}). \qquad (4.51)$$

□

Proof. Notice that, since $\mathcal{R}^{(1)}(\varphi)$ is selfadjoint, the operators $[\widehat{\mathcal{R}}^{(1)}(0)]_\alpha^\alpha : \mathbb{E}_\alpha \to \mathbb{E}_\alpha$ are self-adjoint, that is $[\widehat{\mathcal{R}}^{(1)}(0)]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha)$, for any $\alpha \in \sigma_0(\sqrt{-\Delta})$ and using that $[\mathcal{D}^{(1)}]_\alpha^\alpha$ is self-adjoint, we get that $[\mathcal{D}_+^{(1)}]_\alpha^\alpha$ is self-adjoint for all $\alpha \in \sigma_0(\sqrt{-\Delta})$. The formula (4.50) follows by (4.49) and recalling the definitions (2.73), (2.74). The estimate (4.51) follows by

$$\sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} \alpha^{S_q}\|[\mathcal{D}_+^{(1)}]_\alpha^\alpha - [\mathcal{D}]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)} \overset{(4.50)}{\leq} \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} \alpha^{S_q}\|[\widehat{\mathcal{R}}^{(1)}(0)]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)} \overset{Lemma\ 2.6}{\leq} |\mathcal{R}|_{S_q}^{\mathrm{Lip}(\gamma)} \tag{4.52}$$

which implies the estimate (4.51). ■

4.3 The iteration

Let $k \geq 0$ and let us suppose that $(\mathbf{Si})_k$ are true. We prove $(\mathbf{Si})_{k+1}$. To simplify notations, in this proof we write $|\cdot|_s$ for $|\cdot|_s^{\mathrm{Lip}(\gamma)}$.

Proof of $(\mathbf{S1})_{k+1}$. Since the self-adjoint operators $[\mathcal{D}_k^{(1)}]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha)$ are defined on Ω_k^γ, the set Ω_{k+1}^γ is well-defined and by Lemma 4.1, the following estimates hold on Ω_{k+1}^γ

$$|\Psi_k|_s \lesssim_s N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s \overset{(4.15)}{\lesssim_s} N_k^{2\tau+4\mathrm{d}+1}N_{k-1}^{-\mathrm{a}}\gamma^{-1}|\mathcal{R}_0|_{s+\mathrm{b}}, \quad \forall s \in [\mathfrak{s}_0, [q/2]-\overline{\mu}]. \tag{4.53}$$

In particular, by (4.6)–(4.8), taking δ_q small enough,

$$|\Psi_k|_{\mathfrak{s}_0} \leq 1. \tag{4.54}$$

By (4.54), we can apply Lemma 2.8 to the map $\Phi_k^{\pm1} := \exp(\pm\Psi_k)$, obtaining that

$$|\Phi_k^{\pm1} - \mathrm{Id}|_s \lesssim_s |\Psi_k|_s, \quad |\Phi_k^{\pm1} - \mathrm{Id}|_s \lesssim_s |\Psi_k|_s, \quad \forall s \in [\mathfrak{s}_0, [q/2]-\overline{\mu}]. \tag{4.55}$$

By (4.47), we get $\mathcal{L}_{k+1}(\varphi) := (\Phi_k)_{\omega*}\mathcal{L}_k(\varphi) = \mathcal{D}_{k+1} + \mathcal{R}_{k+1}(\varphi)$, where $\mathcal{D}_{k+1} := \mathcal{D}_k + \Pi_{N_k}(\mathcal{R}_k)_{diag}$ and

$$\mathcal{R}_{k+1} := (\Phi_k^{-1} - \mathrm{Id})\Pi_{N_k}(\mathcal{R}_k)_{diag} + \Phi_k^{-1}\left(\Pi_{N_k}^\perp\mathcal{R}_k - \omega\cdot\partial_\varphi\Psi_{k,\geq2} + [\mathcal{D}_k, \Psi_{k,\geq2}] + \mathcal{R}_k(\Phi_k - \mathrm{Id})\right). \tag{4.56}$$

Since $\mathcal{R}_k$ is defined on Ω_k^γ and Ψ_k is defined on Ω_{k+1}^γ, the remainder $\mathcal{R}_{k+1}$ is defined on Ω_{k+1}^γ too. Since the remainder $\mathcal{R}_k$ is Hamiltonian, the map Ψ_k is Hamiltonian, then Φ_k is symplectic and the operator $\mathcal{L}_{k+1}$ is Hamiltonian.

Now let us prove the estimates (4.15) for $\mathcal{R}_{k+1}$. Applying Lemmata 2.6–2.8 and the estimates (4.53)–(4.55), for any $s \in [\mathfrak{s}_0, [q/2]-\overline{\mu}]$, we get

$$|(\Phi_k^{-1} - \mathrm{Id})\Pi_{N_k}(\mathcal{R}_k)_{diag}|_s, \ |\Phi_k^{-1}\mathcal{R}_k(\Phi_k - \mathrm{Id})|_s \lesssim_s N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0} \tag{4.57}$$

and

$$|\Phi_k^{-1}\Pi_{N_k}^\perp\mathcal{R}_k|_s \lesssim_s |\Pi_{N_k}^\perp\mathcal{R}_k|_s + N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0}. \tag{4.58}$$

Then, it remains to estimate the term $\Phi_k^{-1}\big(-\omega\cdot\partial_\varphi\Psi_{k,\geq 2}+[\mathcal{D}_k,\Psi_{k,\geq 2}]\big)$ in (4.56). A direct calculation shows that for all $n\geq 2$

$$
\begin{aligned}
-\omega\cdot\partial_\varphi(\Psi_k^n)+[\mathcal{D}_k,\Psi_k^n] &= \sum_{i+j=n-1}\Psi_k^i(-\omega\cdot\partial_\varphi\Psi_k+[\mathcal{D}_k,\Psi_k])\Psi_k^j \\
&\overset{(4.25)}{=} \sum_{i+j=n-1}\Psi_k^i\left(\Pi_{N_k}(\mathcal{R}_k)_{diag}-\Pi_{N_k}\mathcal{R}_k\right)\Psi_k^j,
\end{aligned}
\tag{4.59}
$$

therefore using (4.53) and (4.54), Lemmata 2.6 and 2.7, and the estimate (2.85) we get that for any $n\geq 2$, for any $s\in[\mathfrak{s}_0,[q/2]-\overline{\mu}]$

$$
\begin{aligned}
\Big|-\omega\cdot\partial_\varphi(\Psi_k^n)+[\mathcal{D}_k,\Psi_k^n]\Big|_s &\leq n^2C(s)^n\left(|\Psi_k|_{\mathfrak{s}_0}^{n-1}|\mathcal{R}_k|_s+|\Psi_k|_{\mathfrak{s}_0}^{n-2}|\Psi_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0}\right) \\
&\overset{(4.53),(4.54)}{\leq} 2n^2C(s)^nN_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0}.
\end{aligned}
\tag{4.60}
$$

The estimate (4.60) implies that

$$
\begin{aligned}
\Big|\omega\cdot\partial_\varphi\Psi_{k,\geq 2}+[\mathcal{D}_k,\Psi_{k,\geq 2}]\Big|_s &\lesssim \sum_{n\geq 2}\frac{1}{n!}\Big|\omega\cdot\partial_\varphi(\Psi_k^n)+[\mathcal{D}_k,\Psi_k^n]\Big|_s \\
&\overset{(4.60)}{\lesssim} N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0}\sum_{n\geq 2}\frac{C(s)^nn^2}{n!} \\
&\lesssim_s N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0}.
\end{aligned}
\tag{4.61}
$$

Using again (4.53)–(4.55) and Lemma 2.7, we get

$$
\Big|\Phi_k^{-1}\left(-\omega\cdot\partial_\varphi\Psi_{k,\geq 2}+[\mathcal{D}_k,\Psi_{k,\geq 2}]\right)\Big|_s\lesssim_s N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0},\quad \forall s\in[\mathfrak{s}_0,[q/2]-\overline{\mu}]. \tag{4.62}
$$

Collecting the estimates (4.57)–(4.62), we obtain

$$
|\mathcal{R}_{k+1}|_s\lesssim_s|\Pi_{N_k}\mathcal{R}_k|_s+N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0},\quad \forall s\in[\mathfrak{s}_0,[q/2]-\overline{\mu}]. \tag{4.63}
$$

Recalling that $S_q=[q/2]-\overline{\mu}-\mathrm{b}$, see (4.7), using the smoothing property (2.87) and by (4.8) and (4.15), one gets for any $s\in[\mathfrak{s}_0,S_q]$

$$
|\mathcal{R}_{k+1}|_s\lesssim_s N_k^{-\mathrm{b}}|\mathcal{R}_k|_{s+\mathrm{b}}+N_k^{2\tau+4\mathrm{d}+1}\gamma^{-1}|\mathcal{R}_k|_s|\mathcal{R}_k|_{\mathfrak{s}_0},\quad |\mathcal{R}_{k+1}|_{s+\mathrm{b}}\lesssim_s|\mathcal{R}_k|_{s+\mathrm{b}}. \tag{4.64}
$$

By the second inequality in (4.64)

$$
|\mathcal{R}_{k+1}|_{s+\mathrm{b}}\leq C(s)|\mathcal{R}_k|_{s+\mathrm{b}}\overset{(4.15)}{\leq}C(s)|\mathcal{R}_0|_{s+\mathrm{b}}N_{k-1}\leq|\mathcal{R}_0|_{s+\mathrm{b}}N_k
$$

provided $N_{k-1}^{\chi-1} \geq C(s)$ for any $k \geq 0$, which is verified by taking $N_0 > 0$ large enough. Therefore, the second inequality in (4.15) for $\mathcal{R}_{k+1}$ has been proved. Let us prove the first inequality in (4.15) at the step $k+1$. We have

$$|\mathcal{R}_{k+1}|_s \overset{(4.15)}{\lesssim_s} N_k^{-\mathtt{b}} N_{k-1} |\mathcal{R}_0|_{s+\mathtt{b}} + N_k^{2\tau+4\mathtt{d}+1} N_{k-1}^{-2\mathtt{a}} \gamma^{-1} |\mathcal{R}_0|_{\mathfrak{s}_0+\mathtt{b}} |\mathcal{R}_0|_{s+\mathtt{b}} \leq |\mathcal{R}_0|_{s+\mathtt{b}} N_k^{-\mathtt{a}}$$

provided

$$N_k^{\mathtt{b}-\mathtt{a}} N_{k-1}^{-1} \geq 2C(s), \quad \gamma^{-1} |\mathcal{R}_0|_{\mathfrak{s}_0+\mathtt{b}} \leq \frac{N_{k-1}^{2\mathtt{a}} N_k^{-\mathtt{a}-2\tau-4\mathtt{d}-1}}{2C(s)}, \quad \forall k \geq 0$$

which are verified by (4.4), (4.6)–(4.8), by taking $N_0 > 0$ large enough and δ_q small enough. The estimate (4.14) for $[\mathcal{D}_{k+1}^{(1)}]_\alpha^\alpha - [\mathcal{D}_0^{(1)}]_\alpha^\alpha$ follows, since

$$\|[\mathcal{D}_{k+1}^{(1)}]_\alpha^\alpha - [\mathcal{D}_0^{(1)}]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)} \leq \sum_{j=0}^{k} \|[\mathcal{D}_{j+1}^{(1)}]_\alpha^\alpha - [\mathcal{D}_j^{(1)}]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)} \overset{(4.51),(4.15)}{\lesssim_q} \alpha^{-S_q} |\mathcal{R}_0|_{S_q+\mathtt{b}}^{\mathrm{Lip}(\gamma)} \sum_{j\geq 0} N_{j-1}^{-\mathtt{a}} \overset{(4.4)}{\lesssim_q} \alpha^{-S_q} \varepsilon.$$

Proof of $(\mathbf{S2})_{k+1}$. We now construct a Lipschitz extension of the function $\omega \in \Omega_{k+1}^\gamma \mapsto [\mathcal{D}_{k+1}^{(1)}(\omega)]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha)$, for any $\alpha \in \sigma_0(\sqrt{-\Delta})$. We apply Lemma M.5 in [39] to functions with values in $\mathcal{S}(\mathbb{E}_\alpha)$. Recall that the space $\mathcal{S}(\mathbb{E}_\alpha)$ is a Hilbert subspace of $\mathcal{B}(\mathbb{E}_\alpha)$ equipped by the scalar product defined in (2.59), thus Lemma M.5 in [39] can be applied, since it holds for functions with values in a Hilbert space. By the inductive hypothesis, there exists a Lipschitz function $[\widetilde{\mathcal{D}}_k^{(1)}]_\alpha^\alpha : DC(\gamma,\tau) \to \mathcal{S}(\mathbb{E}_\alpha)$, satisfying $[\widetilde{\mathcal{D}}_k^{(1)}(\omega)]_\alpha^\alpha = [\mathcal{D}_k^{(1)}(\omega)]_\alpha^\alpha$, for all $\omega \in \Omega_k^\gamma$. For any $\alpha \in \sigma_0(\sqrt{-\Delta})$, let us define $F_{k,\alpha}(\omega) := [\mathcal{D}_{k+1}^{(1)}(\omega)]_\alpha^\alpha - [\mathcal{D}_k^{(1)}(\omega)]_\alpha^\alpha$, $\omega \in \Omega_{k+1}^\gamma$. By the estimate (4.51) one has that

$$\|F_{k,\alpha}\|_{HS}^{\mathrm{Lip}(\gamma)} \leq \alpha^{-S_q} |\mathcal{R}_k|_{S_q}^{\mathrm{Lip}(\gamma)} \overset{(4.15)}{\leq} \alpha^{-S_q} |\mathcal{R}_0|_{S_q+\mathtt{b}} N_{k-1}^{-\mathtt{a}}$$

and then by Lemma M.5 in [39] there exists a Lipschitz extension $\widetilde{F}_{k,\alpha} : DC(\gamma,\tau) \to \mathcal{S}(\mathbb{E}_\alpha)$ still satisfying the above estimate. Then we define

$$[\widetilde{\mathcal{D}}_{k+1}^{(1)}]_\alpha^\alpha := [\widetilde{\mathcal{D}}_k^{(1)}]_\alpha^\alpha + \widetilde{F}_{k,\alpha}, \qquad \forall \alpha \in \sigma_0(\sqrt{-\Delta})$$

and the claimed estimate (4.18) holds at the step $k+1$.

Corollary 4.1 (KAM transformation). Let $q/2 > \mathfrak{s}_0 + \overline{\mu} + \mathtt{b} + 2s_0 + 2$ (recall (3.68) and (4.7)). Then $\forall \omega \in \cap_{k\geq 0} \Omega_k^\gamma$ the sequence

$$\widetilde{\Phi}_k := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_k \tag{4.65}$$

is in $\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))$ for any $0 \leq s \leq S_q - 2 - 2s_0$ (recall the definition of S_q given in (4.7)) and it converges in $\| \cdot \|^{\mathrm{Lip}(\gamma)}_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))}$ to an operator Φ_∞ which satisfies

$$\left\| \Phi_\infty^{\pm 1} - \mathrm{Id} \right\|^{\mathrm{Lip}(\gamma)}_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))} \lesssim_q \varepsilon \gamma^{-1}. \tag{4.66}$$

Moreover, $\Phi_\infty^{\pm 1}$ is symplectic. □

Proof. To simplify notations, we write $| \cdot |_s$ instead of $| \cdot |_s^{\mathrm{Lip}(\gamma)}$. First, note that for any $k \geq 0$

$$\Phi_k = \exp(\Psi_k) = \mathrm{Id} + \mathcal{M}_k, \qquad \mathcal{M}_k := \sum_{j \geq 1} \frac{\Psi_k^j}{j!} \tag{4.67}$$

with

$$|\mathcal{M}_k|_s \overset{(2.86)}{\lesssim_s} |\Psi_k|_s \overset{(4.17)}{\lesssim_s} |\mathcal{R}_0|^{\mathrm{Lip}(\gamma)}_{s+\mathtt{b}} \gamma^{-1} N_k^{2\tau + 4\mathtt{d} + 1} N_{k-1}^{-\mathtt{a}} \overset{(4.4)}{\lesssim_q} \varepsilon \gamma^{-1} N_k^{2\tau + 4\mathtt{d} + 1} N_{k-1}^{-\mathtt{a}}, \quad \forall \mathfrak{s}_0 \leq s \leq S_q. \tag{4.68}$$

Therefore, by applying Lemma 2.11-(*ii*) one gets that for any $0 \leq s \leq S_q - 2 - 2s_0$, $\mathcal{M}_k \in W^{2,\infty}(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))$ with $\|\mathcal{M}_k\|_{W^{2,\infty}(\mathbb{T}^\nu, \mathcal{L}(\mathbf{H}^s_0))} \lesssim_q \varepsilon \gamma^{-1} N_k^{2\tau + 4\mathtt{d} + 1} N_{k-1}^{-\mathtt{a}}$. By the property (2.11), applied with $p = 1$ and $E = \mathcal{B}(\mathbf{H}^s_0)$, one gets that $\mathcal{M}_k \in \mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))$ and

$$\|\mathcal{M}_k\|_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))} \leq \|\mathcal{M}_k\|_{W^{2,\infty}(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))} \lesssim_q \varepsilon \gamma^{-1} N_k^{2\tau + 4\mathtt{d} + 1} N_{k-1}^{-\mathtt{a}}, \quad \forall 0 \leq s \leq S_q - 2 - 2s_0. \tag{4.69}$$

Therefore, one gets that $\Phi_k \in \mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))$ and hence $\widetilde{\Phi}_k \in \mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))$ for any $k \geq 0$, using the algebra property of the space $\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))$. By (4.65)–(4.67), for any $k \geq 0$, one gets

$$\widetilde{\Phi}_{k+1} = \widetilde{\Phi}_k \Phi_{k+1} = \widetilde{\Phi}_k + \widetilde{\Phi}_k \mathcal{M}_{k+1}, \tag{4.70}$$

therefore (4.69) imply that

$$\|\widetilde{\Phi}_{k+1}\|_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))} \leq \|\widetilde{\Phi}_k\|_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))} (1 + \varepsilon_k(q)), \quad \varepsilon_k(q) := C(q) \varepsilon \gamma^{-1} N_{k+1}^{2\tau + 4\mathtt{d} + 1} N_k^{-\mathtt{a}}. \tag{4.71}$$

Iterating the above inequality, one then prove that for any $k \geq 0$

$$\|\widetilde{\Phi}_k\|_{\mathcal{C}^1(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}^s_0))} \leq \prod_{j=0}^{k-1} (1 + \varepsilon_j(q)). \tag{4.72}$$

Using that

$$\ln\left(\prod_{j=0}^{k-1}(1+\varepsilon_j(q))\right)=\sum_{j=0}^{k-1}\ln(1+\varepsilon_j(q))\le\sum_{j\ge 0}\varepsilon_j(q)\overset{(4.71),(4.7),(4.8)}{\le}C_1(q),$$

one gets that

$$\|\widetilde{\Phi}_k\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}\le\exp(C_1(q))=:C_2(q),\quad\forall\nu\ge 0. \tag{4.73}$$

Now we show that $(\widetilde{\Phi}_k)_{k\ge 0}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}$. One has

$$\begin{aligned}\|\widetilde{\Phi}_{k+j}-\widetilde{\Phi}_k\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}&\le\sum_{i=k}^{k+j-1}\|\widetilde{\Phi}_{i+1}-\widetilde{\Phi}_i\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}\overset{(4.70)}{\lesssim}\sum_{i=k}^{k+j-1}\|\widetilde{\Phi}_i\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}\|\mathcal{M}_{i+1}\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}\\&\overset{(4.73),(4.69)}{\lesssim_q}\varepsilon\gamma^{-1}\sum_{i\ge k}N_{i+1}^{2\tau+4\mathrm{d}+1}N_i^{-\mathrm{a}}\lesssim_q\varepsilon\gamma^{-1}N_{k+1}^{2\tau+4\mathrm{d}+1}N_k^{-\mathrm{a}}\to 0\end{aligned} \tag{4.74}$$

by using (4.6) and (4.7). Thus $\widetilde{\Phi}_k$ converges with respect to the norm $\|\cdot\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}$ to an operator Φ_∞ which satisfies the estimate

$$\|\Phi_\infty-\mathrm{Id}\|_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}\lesssim_q\varepsilon\gamma^{-1}.$$

Similarly, one can show that

$$\widetilde{\Phi}_k^{-1}=\Phi_k^{-1}\circ\ldots\circ\Phi_0^{-1}$$

is a Cauchy sequence and since $\widetilde{\Phi}_k^{-1}\widetilde{\Phi}_k=\mathrm{Id}$ for any $k\ge 0$, $\widetilde{\Phi}_k^{-1}$ converges to Φ_∞^{-1} and the estimate (4.66) for Φ_∞^{-1} holds. Since Φ_k is symplectic for any $k\ge 0$, Φ_∞ is a symplectic map too. ■

Let us define for all $\alpha\in\sigma_0(\sqrt{-\Delta})$, for all $\omega\in DC(\gamma,\tau)$, the self-adjoint blocks $[\mathcal{D}_\infty^{(1)}(\omega)]_\alpha^\alpha$ as

$$[\mathcal{D}_\infty^{(1)}(\omega)]_\alpha^\alpha:=\lim_{\nu\to+\infty}[\widetilde{\mathcal{D}}_\nu^{(1)}(\omega)]_\alpha^\alpha. \tag{4.75}$$

It could happen that $\Omega_{k_0}^\gamma=\emptyset$ (see (4.11)) for some k_0. In such a case the iterative process of Theorem 4.1 stops after finitely many steps. However, we can always set $[\widetilde{\mathcal{D}}_k^{(1)}]_\alpha^\alpha:=[\widetilde{\mathcal{D}}_{k_0}^{(1)}]_\alpha^\alpha$, $\forall k\ge k_0$, for all $\alpha\in\sigma_0(\sqrt{-\Delta})$ and the functions $[\mathcal{D}_\infty^{(1)}(\cdot)]_\alpha^\alpha:DC(\gamma,\tau)\to\mathcal{S}(\mathbb{E}_\alpha)$ are always well defined.

Corollary 4.2 (Final blocks). For any $k \geq 0, \alpha \in \sigma_0(\sqrt{-\Delta})$,

$$\|[\mathcal{D}_\infty^{(1)}]_\alpha^\alpha - [\widetilde{\mathcal{D}}_k^{(1)}]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)} \lesssim_q \alpha^{-S_q} \varepsilon N_{k-1}^{-\mathrm{a}}, \quad \|[\mathcal{D}_\infty^{(1)}]_\alpha^\alpha - [\widetilde{\mathcal{D}}_0^{(1)}]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)} \lesssim_q \alpha^{-S_q} \varepsilon. \tag{4.76}$$

□

Proof. The bound (4.76) follows by (4.18), (4.15), and (4.4) by summing the telescoping series. ■

Now we define the set

$$\begin{aligned}\Omega_\infty^{2\gamma} := \Big\{ & \omega \in DC(\gamma, \tau) : \|\mathbb{A}_\infty^-(\ell, \alpha, \beta; \omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \leq \frac{\alpha^{\mathrm{d}} \beta^{\mathrm{d}} \langle \ell \rangle^\tau}{2\gamma}, \\ & \forall (\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta}), \\ & (\ell, \alpha, \beta) \neq (0, \alpha, \alpha), \ \|\mathbb{A}_\infty^+(\ell, \alpha, \beta; \omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \leq \frac{\langle \ell \rangle^\tau}{2\gamma \langle \alpha + \beta \rangle}, \\ & \forall (\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta}) \Big\}\end{aligned} \tag{4.77}$$

where the operators $\mathbb{A}_\infty^\pm(\ell, \alpha, \beta) = \mathbb{A}_\infty^\pm(\ell, \alpha, \beta; \omega) : \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \to \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ are defined for any $\omega \in DC(\gamma, \tau)$, $(\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$ as

$$\mathbb{A}_\infty^-(\ell, \alpha, \beta) := \omega \cdot \ell \mathbb{I}_{\alpha,\beta} + M_L([\mathcal{D}_\infty^{(1)}]_\alpha^\alpha) - M_R([\mathcal{D}_\infty^{(1)}]_\beta^\beta) \tag{4.78}$$

$$\mathbb{A}_\infty^+(\ell, \alpha, \beta) := \omega \cdot \ell \mathbb{I}_{\alpha,\beta} + M_L([\mathcal{D}_\infty^{(1)}]_\alpha^\alpha) + M_R([\overline{\mathcal{D}}_\infty^{(1)}]_\beta^\beta). \tag{4.79}$$

Lemma 4.3. One has

$$\Omega_\infty^{2\gamma} \subset \cap_{k \geq 0} \Omega_k^\gamma. \tag{4.80}$$

□

Proof. It suffices to show that for any $k \geq 0$, $\Omega_\infty^{2\gamma} \subseteq \Omega_k^\gamma$. We argue by induction. For $k = 0$, since $\Omega_0^\gamma = DC(\gamma, \tau)$, it follows from the definition (4.77) that $\Omega_\infty^{2\gamma} \subseteq \Omega_0^\gamma$. Assume that $\Omega_\infty^{2\gamma} \subseteq \Omega_k^\gamma$ for some $k \geq 0$ and let us prove that $\Omega_\infty^{2\gamma} \subseteq \Omega_{k+1}^\gamma$. Let $\omega \in \Omega_\infty^{2\gamma}$. By the inductive hypothesis $\omega \in \Omega_k^\gamma$, hence by Theorem 4.1, the operators $[\mathcal{D}_k^{(1)}(\omega)]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha)$ are well defined for all $\alpha \in \sigma_0(\sqrt{-\Delta})$ and $[\mathcal{D}_k^{(1)}(\omega)]_\alpha^\alpha = [\widetilde{\mathcal{D}}_k^{(1)}(\omega)]_\alpha^\alpha$.
Let $(\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$ with $(\ell, \alpha, \beta) \neq (0, \alpha, \alpha)$, $\langle \ell, \alpha, \beta \rangle \leq N_k$. By the definitions (4.12) and (4.13), also the operators $\mathbb{A}_k^\pm(\ell, \alpha, \beta; \omega)$ are well defined. Since $\omega \in \Omega_\infty^{2\gamma}$, the operator $\mathbb{A}_\infty^-(\ell, \alpha, \beta; \omega)$ is invertible and we may write

$$\begin{aligned}\mathbb{A}_k^-(\ell, \alpha, \beta; \omega) &= \mathbb{A}_\infty^-(\ell, \alpha, \beta; \omega) + \Delta_\infty^-(\ell, \alpha, \beta; \omega) \\ &= \mathbb{A}_\infty^-(\ell, \alpha, \beta; \omega) \big(\mathbb{I}_{\alpha,\beta} + \mathbb{A}_\infty^-(\ell, \alpha, \beta; \omega)^{-1} \Delta_\infty^-(\ell, \alpha, \beta; \omega)\big)\end{aligned}$$

where

$$\Delta_\infty^-(\ell,\alpha,\beta;\omega) := M_L\left([\mathcal{D}_k^{(1)}(\omega)]_\alpha^\alpha - [\mathcal{D}_\infty^{(1)}(\omega)]_\alpha^\alpha\right) - M_R\left([\mathcal{D}_k^{(1)}(\omega)]_\beta^\beta - [\mathcal{D}_\infty^{(1)}(\omega)]_\beta^\beta\right).$$

By the property (2.66) and by the estimate (4.76)

$$\|\Delta_\infty^-(\ell,\alpha,\beta;\omega)\|_{\mathrm{Op}(\alpha,\beta)} \lesssim_q N_{k-1}^{-\mathtt{a}}\varepsilon(\alpha^{-\mathfrak{S}_q}+\beta^{-\mathfrak{S}_q}). \tag{4.81}$$

Since $\langle \ell,\alpha,\beta\rangle \le N_k$, one has

$$\begin{aligned}\|\mathbb{A}_\infty^-(\ell,\alpha,\beta;\omega)^{-1}\Delta_\infty^-(\ell,j,j';\omega)\|_{\mathrm{Op}(\alpha,\beta)} &\lesssim_q \frac{\langle \ell\rangle^\tau \alpha^{\mathtt{d}}\beta^{\mathtt{d}}}{\gamma} N_{k-1}^{-\mathtt{a}}\varepsilon(\alpha^{-\mathfrak{S}_q}+\beta^{-\mathfrak{S}_q})\\ &\lesssim_q N_k^{\tau+2\mathtt{d}}N_{k-1}^{-\mathtt{a}}\varepsilon\gamma^{-1} \overset{(4.7)-(4.8)}{\le} \frac{1}{2}\end{aligned} \tag{4.82}$$

for $N_0 > 0$ in (4.8) large enough and δ_q in (4.8) small enough. Thus the operator $\mathbb{A}_k^-(\ell,\alpha,\beta;\omega)$ is invertible, with inverse given by the Neumann series. Hence

$$\begin{aligned}\|\mathbb{A}_k^-(\ell,\alpha,\beta;\omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} &\le \frac{\|\mathbb{A}_\infty^-(\ell,\alpha,\beta;\omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)}}{1-\|\mathbb{A}_\infty^-(\ell,\alpha,\beta;\omega)^{-1}\Delta_\infty^-(\ell,\alpha,\beta;\omega)\|_{\mathrm{Op}(\alpha,\beta)}}\\ &\overset{(4.82)}{\le} 2\|\mathbb{A}_\infty^-(\ell,\alpha,\beta;\omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \overset{(4.77)}{\le} \frac{\langle \ell\rangle^\tau\alpha^{\mathtt{d}}\beta^{\mathtt{d}}}{\gamma}.\end{aligned}$$

By similar arguments, one can also obtain that $\|\mathbb{A}_k^+(\ell,\alpha,\beta;\omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} \le \frac{\langle\ell\rangle^\tau}{\gamma(\alpha+\beta)}$, for any $(\ell,\alpha,\beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$ with $\langle \ell,\alpha,\beta\rangle \le N_k$, then $\omega \in \Omega_{k+1}^\gamma$ and the proof is concluded. ■

To state the main result of this section, we introduce the operator

$$\mathcal{D}_\infty = \mathcal{D}_\infty(\omega) := \mathrm{i}\begin{pmatrix} -\mathcal{D}_\infty^{(1)}(\omega) & 0\\ 0 & \overline{\mathcal{D}_\infty^{(1)}}(\omega)\end{pmatrix}, \quad \mathcal{D}_\infty^{(1)}(\omega) := \mathrm{diag}_{\alpha\in\sigma_0(\sqrt{-\Delta})}[\mathcal{D}_\infty^{(1)}(\omega)]_\alpha^\alpha, \tag{4.83}$$

for any $\omega \in DC(\gamma,\tau)$, where the self-adjoint operators $[\mathcal{D}_\infty^{(1)}(\omega)]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha)$, $\alpha \in \sigma_0(\sqrt{-\Delta})$, are defined in (4.75). For any $\omega \in DC(\gamma,\tau)$, the vector field $\mathcal{D}_\infty(\omega)$ is a φ-independent block-diagonal bounded linear operator $\mathcal{D}_\infty(\omega) : \mathbf{H}_0^s \to \mathbf{H}_0^{s-1}$, for any $s \ge 1$.

Theorem 4.2. Let $q/2 > \mathfrak{s}_0 + \bar\mu + \mathtt{b} + 2s_0 + 2$. Then there exists a constant $\delta_q = \delta(q,\tau,\mathtt{d},\nu,d) > 0$ (possibly smaller than the one in (4.8)) such that if

$$\varepsilon\gamma^{-1} \le \delta_q, \tag{4.84}$$

on the set $\Omega_\infty^{2\gamma}$, the Hamiltonian vector field $\mathcal{L}_0(\varphi)$ in (4.1) is conjugated to the Hamiltonian vector field $\mathcal{D}_\infty$ by Φ_∞, namely for all $\omega \in \Omega_\infty^{2\gamma}$,

$$\mathcal{D}_\infty(\omega) = (\Phi_\infty)_{\omega *}\mathcal{L}_0(\varphi;\omega). \tag{4.85}$$

□

Proof. Since $\Omega_\infty^{2\gamma} \overset{(4.80)}{\subseteq} \cap_{k\geq 0}\Omega_k^\gamma$, the estimate (4.66) holds on the set $\Omega_\infty^{2\gamma}$, and

$$\|\Phi_\infty^{\pm 1} - \mathrm{Id}\|^{\mathrm{Lip}(\gamma)}_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))} \lesssim_q \varepsilon\gamma^{-1}, \qquad \forall 0 \leq s \leq S_q - 2s_0 - 2.$$

By (4.16) and (4.65), for any $k \geq 1$, we get

$$\mathcal{L}_k(\varphi) = (\widetilde{\Phi}_{k-1})_{\omega *}\mathcal{L}_0 = \widetilde{\Phi}_k(\varphi)^{-1}\left(\mathcal{L}_0(\varphi)\widetilde{\Phi}_k(\varphi) - \omega\cdot\partial_\varphi\widetilde{\Phi}_k(\varphi)\right) = \mathcal{D}_k + \mathcal{R}_k(\varphi), \qquad \widetilde{\Phi}_k = \Phi_0\circ\ldots\circ\Phi_k. \tag{4.86}$$

For all $k \geq 0$, for any $s \in [0, S_q]$

$$|\mathcal{D}_\infty^{(1)} - \mathcal{D}_k^{(1)}|_s^{\mathrm{Lip}(\gamma)} \leq |\mathcal{D}_\infty^{(1)} - \mathcal{D}_k^{(1)}|_{S_q}^{\mathrm{Lip}(\gamma)} = \sup_{\alpha\in\sigma_0(\sqrt{-\Delta})} \alpha^{S_q}\|[\mathcal{D}_k^{(1)}]_\alpha^\alpha - [\mathcal{D}_\infty^{(1)}]_\alpha^\alpha\|_{HS}^{\mathrm{Lip}(\gamma)}$$
$$\overset{(4.76)}{\lesssim_q} \varepsilon N_{k-1}^{-\mathtt{a}} \overset{k\to+\infty}{\to} 0 \quad \text{and} \quad |\mathcal{R}_k|_s^{\mathrm{Lip}(\gamma)} \overset{(4.15),(4.4)}{\lesssim_q} \varepsilon N_{k-1}^{-\mathtt{a}} \overset{k\to+\infty}{\to} 0. \tag{4.87}$$

Hence, $|\mathcal{L}_k - \mathcal{D}_\infty|_s^{\mathrm{Lip}(\gamma)} \overset{k\to+\infty}{\to} 0$ for all $\mathfrak{s}_0 \leq s \leq S_q$. By applying Lemma 2.11 and the property (2.11), $\mathcal{R}_k \in W^{1,\infty}(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}_0^s)) \subseteq \mathcal{C}^0(\mathbb{T}^\nu, \mathcal{B}(\mathbf{H}_0^s))$ for any $0 \leq s \leq S_q - 2s_0 - 1$ with

$$\|\mathcal{R}_k\|_{\mathcal{C}^0(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))} \leq \|\mathcal{R}_k\|_{W^{1,\infty}(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))} \lesssim |\mathcal{R}_k|_{s+2s_0+1} \to 0$$

and

$$\|\mathcal{D}_k - \mathcal{D}_\infty\|_{\mathcal{B}(\mathbf{H}_0^s)} \leq |\mathcal{D}_k - \mathcal{D}_\infty|_{s+2s_0} \to 0.$$

Thus, $\mathcal{L}_k \to \mathcal{D}_\infty$ with respect to the norm $\|\cdot\|_{\mathcal{C}^0(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}$, for any $0 \leq s \leq S_q - 2s_0 - 1$. Since, by Lemma 4.1, $\widetilde{\Phi}_k^{\pm 1} \overset{k\to+\infty}{\to} \Phi_\infty^{\pm 1}$ with respect to the norm $\|\cdot\|^{\mathrm{Lip}(\gamma)}_{\mathcal{C}^1(\mathbb{T}^\nu,\mathcal{B}(\mathbf{H}_0^s))}$, formula (4.85) follows by taking the limit for $k \to +\infty$ in (4.86). ■

5 Measure Estimates

In this section, we estimate the measure of the set $\Omega_\infty^{2\gamma}$ defined in (4.77). We fix the constants τ and $\mathtt{d}$ in (4.77) as

$$\mathtt{d} := 2d, \qquad \tau := \nu + 4d. \tag{5.1}$$

We prove the following theorem:

Theorem 5.1. Under the same assumptions of Theorem 4.2, one has

$$|\Omega \setminus \Omega_\infty^{2\gamma}| = O(\gamma). \qquad \square$$

The rest of this section is devoted to the proof of Theorem 5.1.
By the definition (4.77), one can write that

$$\Omega \setminus \Omega_\infty^{2\gamma} = (\Omega \setminus DC(\gamma,\tau)) \cup \left(DC(\gamma,\tau) \setminus \Omega_\infty^{2\gamma}\right). \tag{5.2}$$

By a standard volume estimate one has

$$|\Omega \setminus DC(\gamma,\tau)| \lesssim \gamma. \tag{5.3}$$

Using again the definition (4.77), we write

$$DC(\gamma,\tau) \setminus \Omega_\infty^{2\gamma} = \bigcup_{\substack{(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta}) \\ (\ell,\alpha,\beta)\neq(0,\alpha,\alpha)}} R(\ell,\alpha,\beta) \bigcup_{(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta})} Q(\ell,\alpha,\beta), \tag{5.4}$$

where for any $(\ell,\alpha,\beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$, $(\ell,\alpha,\beta) \neq (0,\alpha,\alpha)$,

$$\begin{aligned} R(\ell,\alpha,\beta) := \Big\{&\omega \in DC(\gamma,\tau) : \mathbb{A}_\infty^-(\ell,\alpha,\beta;\omega) \text{ is not invertible or it is invertible and} \\ &\|\mathbb{A}_\infty^-(\ell,\alpha,\beta;\omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} > \frac{\alpha^{\mathtt{d}}\beta^{\mathtt{d}}\langle\ell\rangle^\tau}{2\gamma}\Big\} \end{aligned} \tag{5.5}$$

and for any $(\ell,\alpha,\beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$

$$\begin{aligned} Q(\ell,\alpha,\beta) := \Big\{&\omega \in DC(\gamma,\tau) : \mathbb{A}_\infty^+(\ell,\alpha,\beta;\omega) \text{ is not invertible or it is invertible and} \\ &\|\mathbb{A}_\infty^+(\ell,\alpha,\beta;\omega)^{-1}\|_{\mathrm{Op}(\alpha,\beta)} > \frac{\langle\ell\rangle^\tau}{2\gamma(\alpha+\beta)}\Big\}. \end{aligned} \tag{5.6}$$

By (4.5), for any $\alpha \in \sigma_0(\sqrt{-\Delta})$, we can write

$$[\mathcal{D}_\infty^{(1)}]_\alpha^\alpha = \mu_\alpha^0 \mathbb{I}_\alpha + R_{\infty,\alpha}, \qquad R_{\infty,\alpha} := [\mathcal{D}_\infty^{(1)}]_\alpha^\alpha - [\mathcal{D}_0^{(1)}]_\alpha^\alpha \in \mathcal{S}(\mathbb{E}_\alpha)$$

which is self-adjoint and Lipschitz continuous with respect to the parameter $\omega \in DC(\gamma,\tau)$. We set

$$\mathrm{spec}(R_{\infty,\alpha}(\omega)) := \left\{r_k^{(\alpha)}(\omega), k = 1,\ldots,d_\alpha\right\} \quad \text{with} \quad r_1^{(\alpha)}(\omega) \le r_2^{(\alpha)}(\omega) \le \ldots \le r_{n_\alpha}^{(\alpha)}(\omega), \tag{5.7}$$

where n_α is the dimension of the finite dimensional space $\mathbb{E}_\alpha$

$$n_\alpha = \mathrm{card}\left\{j \in \mathbb{Z}^d \setminus \{0\} : |j| = \alpha\right\} \simeq \alpha^{d-1}. \tag{5.8}$$

By the property (A.2 in Appendix), one has that

$$|r_k^{(\alpha)}(\omega)| \le \|R_{\infty,\alpha}\|_{\mathcal{B}(\mathbb{E}_\alpha)} \overset{Lemma\, 2.4-(i)}{\le} \|R_{\infty,\alpha}\|_{HS} \overset{(4.76)}{\lesssim_q} \varepsilon \alpha^{-S_q} \tag{5.9}$$

uniformly for any $\omega \in DC(\gamma, \tau)$.

Furthermore, by Lemma A.2-(i) the functions $\omega \mapsto r_k^{(\alpha)}(\omega)$ are Lipschitz with respect to ω, since

$$\begin{aligned} |r_k^{(\alpha)}(\omega_1) - r_k^{(\alpha)}(\omega_2)| &\le \|R_{\infty,\alpha}(\omega_1) - R_{\infty,\alpha}(\omega_2)\|_{\mathcal{B}(\mathbb{E}_\alpha)} \overset{Lemma\, 2.4-(i)}{\le} \|R_{\infty,\alpha}(\omega_1) - R_{\infty,\alpha}(\omega_2)\|_{HS} \\ &\le \|R_{\infty,\alpha}\|_{HS}^{\mathrm{lip}} |\omega_1 - \omega_2| \overset{(4.76)}{\lesssim_q} \varepsilon \gamma^{-1} \alpha^{-S_q} |\omega_1 - \omega_2|. \end{aligned} \tag{5.10}$$

We also set

$$\mathrm{spec}([\mathcal{D}_\infty^{(1)}(\omega)]_\alpha^\alpha) := \left\{\lambda_k^{(\alpha)}(\omega), k = 1, \ldots, n_\alpha\right\} \quad \text{with} \quad \lambda_1^{(\alpha)}(\omega) \le \lambda_2^{(\alpha)}(\omega) \le \ldots \le \lambda_{n_\alpha}^{(\alpha)}(\omega).$$

By Lemma A.2-(ii), we have that

$$\lambda_k^{(\alpha)}(\omega) = \mu_\alpha^0(\omega) + r_k^{(\alpha)}(\omega) \overset{(4.5)}{=} m\,\alpha + \mathtt{r}_k^{(\alpha)}(\omega), \qquad \mathtt{r}_k^{(\alpha)} := c(\alpha) + r_k^{(\alpha)}, \quad \forall k = 1, \ldots, n_\alpha. \tag{5.11}$$

By the estimates (3.65), (5.9), and (5.10), one gets

$$|\mathtt{r}_k^{(\alpha)}|^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon \alpha^{-1}, \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}), \quad \forall k = 1, \ldots, n_\alpha. \tag{5.12}$$

By the definitions (4.78), (4.79) and by Lemmata 2.5, A.2-(ii), the operators $\mathbb{A}_\infty^\pm(\ell, \alpha, \beta) : \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha) \to \mathcal{B}(\mathbb{E}_\beta, \mathbb{E}_\alpha)$ are self-adjoint with respect to the scalar product (2.59) and the following holds:

for any $(\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$, $(\ell, \alpha, \beta) \neq (0, \alpha, \alpha)$

$$\mathrm{spec}\left(\mathbb{A}_\infty^-(\ell, \alpha, \beta; \omega)\right) = \left\{\omega \cdot \ell + \lambda_k^{(\alpha)}(\omega) - \lambda_j^{(\beta)}(\omega), \quad k = 1, \ldots, n_\alpha, \quad j = 1, \ldots, n_\beta\right\}$$

and for any $(\ell, \alpha, \beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$

$$\mathrm{spec}\left(\mathbb{A}_\infty^+(\ell, \alpha, \beta; \omega)\right) = \left\{\omega \cdot \ell + \lambda_k^{(\alpha)}(\omega) + \lambda_j^{(\beta)}(\omega), \quad k = 1, \ldots, n_\alpha, \quad j = 1, \ldots, n_\beta\right\}.$$

Therefore, recalling the definitions (5.5) and (5.6) and also by applying Lemma A.2-(*iii*), one has

$$R(\ell,\alpha,\beta)\subseteq \widetilde{R}(\ell,\alpha,\beta):=\bigcup_{k=1}^{n_\alpha}\bigcup_{j=1}^{n_\beta}\widetilde{R}_{kj}(\ell,\alpha,\beta),\quad \forall(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta}),$$
$$(\ell,\alpha,\beta)\neq(0,\alpha,\alpha), \tag{5.13}$$

$$Q(\ell,\alpha,\beta)\subseteq \widetilde{Q}(\ell,\alpha,\beta):=\bigcup_{k=1}^{n_\alpha}\bigcup_{j=1}^{n_\beta}\widetilde{Q}_{kj}(\ell,\alpha,\beta),\quad \forall(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta}) \tag{5.14}$$

where for any $(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta})$, $(\ell,\alpha,\beta)\neq(0,\alpha,\alpha)$, $k=1,\dots,n_\alpha$, $j=1,\dots,n_\beta$

$$\widetilde{R}_{kj}(\ell,\alpha,\beta):=\left\{\omega\in DC(\gamma,\tau):|\omega\cdot\ell+\lambda_k^{(\alpha)}(\omega)-\lambda_j^{(\beta)}(\omega)|<\frac{2\gamma}{\langle\ell\rangle^\tau\alpha^{\mathrm{d}}\beta^{\mathrm{d}}}\right\} \tag{5.15}$$

and for any $(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta})$, $k=1,\dots,n_\alpha$, $j=1,\dots,n_\beta$

$$\widetilde{Q}_{kj}(\ell,\alpha,\beta):=\left\{\omega\in DC(\gamma,\tau):|\omega\cdot\ell+\lambda_k^{(\alpha)}(\omega)+\lambda_j^{(\beta)}(\omega)|<\frac{2\gamma(\alpha+\beta)}{\langle\ell\rangle^\tau}\right\}. \tag{5.16}$$

Thus, by (5.4) one has

$$DC(\gamma,\tau)\setminus\Omega_\infty^{2\gamma}\subseteq\bigcup_{\substack{(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta})\\(\ell,\alpha,\beta)\neq(0,\alpha,\alpha)}}\widetilde{R}(\ell,\alpha,\beta)\quad\bigcup_{(\ell,\alpha,\beta)\in\mathbb{Z}^\nu\times\sigma_0(\sqrt{-\Delta})\times\sigma_0(\sqrt{-\Delta})}\widetilde{Q}(\ell,\alpha,\beta). \tag{5.17}$$

Lemma 5.1. (*i*) If $\widetilde{R}(\ell,\alpha,\beta)\neq\emptyset$, then $|\alpha-\beta|\lesssim\langle\ell\rangle$. Moreover, for any $\alpha,\beta\in\sigma_0(\sqrt{-\Delta})$, $\alpha\neq\beta$, then $\widetilde{R}(0,\alpha,\beta)=\emptyset$.
(*ii*) If $\widetilde{Q}(\ell,\alpha,\beta)\neq\emptyset$, then $\alpha,\beta\lesssim\langle\ell\rangle$. Moreover, for any $\alpha,\beta\in\sigma_0(\sqrt{-\Delta})$ then $\widetilde{Q}(0,\alpha,\beta)=\emptyset$. □

Proof. We prove item (*i*). The proof of item (*ii*) is similar. Assume that $\widetilde{R}(\ell,\alpha,\beta)\neq\emptyset$. Then there exist $k\in\{1,\dots,n_\alpha\}$, $j\in\{1,\dots,n_\beta\}$ such that $\widetilde{R}_{kj}(\ell,\alpha,\beta)\neq\emptyset$. For any $\omega\in\widetilde{R}_{kj}(\ell,\alpha,\beta)$, one has

$$|\omega\cdot\ell+\lambda_k^{(\alpha)}(\omega)-\lambda_j^{(\beta)}(\omega)|<\frac{2\gamma}{\langle\ell\rangle^\tau\alpha^{\mathrm{d}}\beta^{\mathrm{d}}}$$

and using (5.11) and the estimates (3.27) and (5.12), for ε small enough, one gets that

$$|\lambda_k^{(\alpha)}-\lambda_j^{(\beta)}|\geq\frac{1}{2}|\alpha-\beta|-C(q)\varepsilon(\alpha^{-1}+\beta^{-1}) \tag{5.18}$$

implying that

$$|\alpha-\beta| \le |\omega||\ell| + \frac{2\gamma}{\langle \ell \rangle^{\tau} \alpha^{\mathtt{d}} \beta^{\mathtt{d}}} + C(q)\varepsilon(\alpha^{-1}+\beta^{-1}) \lesssim \langle \ell \rangle .$$

Now we show that if $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$ with $\alpha \neq \beta$, then $\widetilde{R}_{kj}(0,\alpha,\beta) = \emptyset$ for any $k \in \{1,\ldots,n_\alpha\}, j \in \{1,\ldots,n_\beta\}$. By using (5.18) and Lemma A.1-(ii), for ε small enough one gets

$$|\lambda_k^{(\alpha)} - \lambda_j^{(\beta)}| \ge C_1 \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \overset{\alpha,\beta\ge 1}{\ge} \frac{C_1}{\alpha\beta}, \tag{5.19}$$

for some constant $C_1 > 0$ implying that $\widetilde{R}_{kj}(0,\alpha,\beta) = \emptyset$ by the definition (5.15), since $\mathtt{d} > 1$ and taking $0 < \gamma < C_1$. Item (i) then follows by recalling the definition of $\widetilde{R}(\ell,\alpha,\beta)$ in (5.13). ■

Lemma 5.2. For $\varepsilon\gamma^{-1}$ small enough, the following holds:
(i) For any $(\ell,\alpha,\beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$, $(\ell,\alpha,\beta) \neq (0,\alpha,\alpha)$, if $\widetilde{R}(\ell,\alpha,\beta) \neq \emptyset$ then $|\widetilde{R}(\ell,\alpha,\beta)| \lesssim \gamma \alpha^{d-1-\mathtt{d}} \beta^{d-1-\mathtt{d}} \langle \ell \rangle^{-\tau-1}$.
(ii) For any $(\ell,\alpha,\beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$, if $\widetilde{Q}(\ell,\alpha,\beta) \neq \emptyset$ then $|\widetilde{Q}(\ell,\alpha,\beta)| \lesssim \gamma \alpha^{d-1} \beta^{d-1} \langle \alpha+\beta \rangle \langle \ell \rangle^{-\tau-1}$. □

Proof. Let us prove item (i). The proof of item (ii) can be done by using similar arguments. Let $(\ell,\alpha,\beta) \in \mathbb{Z}^\nu \times \sigma_0(\sqrt{-\Delta}) \times \sigma_0(\sqrt{-\Delta})$ with $(\ell,\alpha,\beta) \neq (0,\alpha,\alpha)$. By (5.13), it is enough to estimate the measure of the set $\widetilde{R}_{kj}(\ell,\alpha,\beta)$ for any $k = 1,\ldots,n_\alpha, j = 1,\ldots,n_\beta$. Since, by Lemma 5.1-$(i)$, $\ell \neq 0$, we can write

$$\omega = \frac{\ell}{|\ell|} s + v, \quad \text{with} \quad v \cdot \ell = 0$$

and we define

$$\phi(s) := |\ell| s + \lambda_k^{(\alpha)}(s) - \lambda_j^{(\beta)}(s), \tag{5.20}$$

$$\lambda_k^{(\alpha)}(s) := \lambda_k^{(\alpha)}\left(\frac{\ell}{|\ell|} s + v\right), \quad \forall \alpha \in \sigma_0(\sqrt{-\Delta}), \quad \forall k = 1,\ldots,n_\alpha$$

and according to (5.11) and (5.12)

$$\lambda_k^{(\alpha)}(s) = m\,\alpha + \mathtt{r}_k^{(\alpha)}(s), \qquad |\mathtt{r}_k^{(\alpha)}|^{\mathrm{Lip}(\gamma)} \lesssim_q \varepsilon\,\alpha^{-1}. \tag{5.21}$$

Using that $|\cdot|^{\rm lip} \le \gamma^{-1}|\cdot|^{{\rm Lip}(\gamma)}$, recalling that m does not depend on ω (see Section 3.3), one gets

$$\begin{aligned} |\phi(s_1)-\phi(s_2)| &\ge \Big(|\ell| - (|\mathtt{r}_j^{(\alpha)}|^{\rm lip} + |\mathtt{r}_k^{(\beta)}|^{\rm lip})\Big)\,|s_1-s_2| \\ &\ge \Big(|\ell| - \gamma^{-1}(|\mathtt{r}_j^{(\alpha)}|^{{\rm Lip}(\gamma)} + |\mathtt{r}_k^{(\beta)}|^{{\rm Lip}(\gamma)})\Big)\,|s_1-s_2| \\ &\overset{(5.21)}{\ge} \big(|\ell| - C(q)\varepsilon\gamma^{-1}\big)\,|s_1-s_2| \ge \frac{|\ell|}{2}|s_1-s_2| \end{aligned} \tag{5.22}$$

for $\varepsilon\gamma^{-1}$ small enough. The above estimate implies that

$$\Big|\Big\{s : \frac{\ell}{|\ell|}s + v \in \widetilde{R}_{kj}(\ell,\alpha,\beta)\Big\}\Big| \lesssim \frac{\gamma}{\alpha^{\rm d}\beta^{\rm d}\langle\ell\rangle^{\tau+1}}$$

and by Fubini Theorem we get $|\widetilde{R}_{kj}(\ell,\alpha,\beta)| \lesssim \frac{\gamma}{\alpha^{\rm d}\beta^{\rm d}\langle\ell\rangle^{\tau+1}}$. Finally recalling (5.8) and (5.13), we get the claimed estimate for the measure of $\widetilde{R}(\ell,\alpha,\beta)$ and the proof is concluded. ■

Proof of Theorem 5.1 concluded. By (5.17), by applying Lemmata 5.1 and 5.2 and recalling the definitions of the constants τ and ${\rm d}$ made in (5.1), one gets the estimate

$$|DC(\gamma,\tau)\setminus\Omega_\infty^{2\gamma}| \lesssim \sum_{\ell\in\mathbb{Z}^\nu, j,j'\in\mathbb{Z}^d} \frac{\gamma}{\langle j\rangle^{{\rm d}+1-d}\langle j'\rangle^{{\rm d}+1-d}\langle\ell\rangle^{\tau+1}} + \sum_{\substack{\ell\in\mathbb{Z}^\nu, j,j'\in\mathbb{Z}^d \\ |j|,|j'|\lesssim\langle\ell\rangle}} \frac{\gamma}{\langle\ell\rangle^{\tau+1-2d}} \lesssim \gamma. \tag{5.23}$$

Hence, the Theorem 5.1 follows by (5.2), (5.3), (5.23).

6 Proof of Theorem 1.1 and Corollary 1.1

In this section, we prove Theorem 1.1 and Corollary 1.1. We define

$$\mathcal{W}_1(\varphi) := \mathcal{S}(\varphi)\circ\mathcal{C}, \qquad \mathcal{W}_2(\varphi) := \mathcal{T}(\varphi)\circ\Phi_\infty(\varphi), \quad \varphi\in\mathbb{T}^\nu \tag{6.1}$$

where the maps $\mathcal{S}, \mathcal{C}, \mathcal{T}$ are defined in (3.1), (2.39), and(3.69) and the map Φ_∞ is given in Corollary 4.1. We define the constants

$$\overline{q} = \overline{q}(\nu,d) := 2(\mathfrak{s}_0 + \overline{\mu} + {\rm b} + 2s_0 + 2)$$

and for any $q > \overline{q}$, we define

$$\mathfrak{S}_q = \mathfrak{S}(q,\nu,d) := S_q - 2 - 2s_0 = [q/2] - \overline{\mu} - {\rm b} - 2s_0 - 2$$

where we recall the definitions (3.68), (4.7), (5.1). By Lemmata 3.1, 3.6, 2.13, and Corollary 4.1, one gets that for $\varepsilon\gamma^{-1} \leq \delta_q$ (for some constant δ_q small enough depending on q, ν, d), for any $\varphi \in \mathbb{T}^\nu$, for any $\omega \in \Omega_\infty^{2\gamma}$ the maps $\mathcal{W}_i(\varphi) = \mathcal{W}_i(\varphi;\omega)$, $i = 1, 2$ are bounded and invertible with

$$\mathcal{W}_1(\varphi) : \mathbf{H}_0^s(\mathbb{T}^d) \to H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}), \ \mathcal{W}_1(\varphi)^{-1} : H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \to \mathbf{H}_0^s(\mathbb{T}^d),$$

for any $1/2 \leq s \leq \mathfrak{S}_q$ and

$$\mathcal{W}_2(\varphi)^{\pm 1} : \mathbf{H}_0^s(\mathbb{T}^d) \to \mathbf{H}_0^s(\mathbb{T}^d), \qquad \forall 0 \leq s \leq \mathfrak{S}_q.$$

Let $1/2 \leq s \leq \mathfrak{S}_q$ and $(v^{(0)}, \psi^{(0)}) \in H^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})$. For any $\omega \in \Omega_\infty^{2\gamma}$, defining $\mathcal{W}_\infty(\varphi) := \mathcal{W}_1(\varphi) \circ \mathcal{A} \circ \mathcal{W}_2(\varphi)$, by the change of variable

$$(v(t,\cdot), \psi(t,\cdot)) = \mathcal{W}_\infty(\omega t)[\mathbf{u}(t,\cdot)], \quad \mathbf{u} = (u, \overline{u}) \tag{6.2}$$

(recall that $\mathcal{A}$ is the reparameterization of time defined in (3.18)), the Cauchy problem

$$\begin{cases} (\partial_t v, \partial_t \psi) = \mathcal{L}(\omega t)[(u, \psi)]. \\ (v(0,\cdot), \psi(0,\cdot)) = (v^{(0)}, \psi^{(0)}) \end{cases} \tag{6.3}$$

is transformed into

$$\begin{cases} \partial_t \mathbf{u} = \mathcal{D}_\infty \mathbf{u} \\ \mathbf{u}(0,\cdot) = \mathbf{u}^{(0)} \end{cases}, \quad \mathbf{u}^{(0)} = (u^{(0)}, \overline{u}^{(0)}) = \mathcal{W}_2(0)^{-1} \circ \mathcal{W}_1(0)^{-1}[(v^{(0)}, \psi^{(0)})] \tag{6.4}$$

where the operator $\mathcal{D}_\infty = \begin{pmatrix} -\mathrm{i}\mathcal{D}_\infty^{(1)} & 0 \\ 0 & \mathrm{i}\overline{\mathcal{D}_\infty^{(1)}} \end{pmatrix}$ is defined in (4.83). Since for any $\alpha \in \sigma_0(\sqrt{-\Delta})$, the block $[\mathcal{D}_\infty^{(1)}]_\alpha^\alpha$ is self-adjoint, one has that the operator $\mathcal{D}_\infty^{(1)}$ is self-adjoint, that is.

$$\mathcal{D}_\infty^{(1)} = \big(\mathcal{D}_\infty^{(1)}\big)^*. \tag{6.5}$$

Then, we consider the Cauchy problem

$$\begin{cases} \partial_t u = -\mathrm{i}\mathcal{D}_\infty^{(1)} u \\ u(0,\cdot) = u^{(0)}. \end{cases} \tag{6.6}$$

We prove that

$$\|u(t,\cdot)\|_{H_x^s} = \|u^{(0)}\|_{H_x^s}, \qquad \forall t \in \mathbb{R}. \tag{6.7}$$

Since $\mathcal{D}_\infty^{(1)}$ is a block-diagonal operator, one can easily verify that the commutator $[|D|^s, \mathcal{D}_\infty^{(1)}] = 0$ and therefore

$$\partial_t \|h(t,\cdot)\|_{H_x^s}^2 = -\big(\mathrm{i}(\mathcal{D}_\infty^{(1)} - \big(\mathcal{D}_\infty^{(1)}\big)^*)|D|^s h, |D|^s h\big)_{L_x^2} \overset{(6.5)}{=} 0$$

which implies (6.7).

Now, by (6.2) one has that for any $1/2 \le s \le \mathfrak{S}_q$

$$\begin{aligned}
&\|(u(t,\cdot), \psi(t,\cdot))\|_{H_x^{s+\frac{1}{2}} \times H_x^{s-\frac{1}{2}}} \\
&\lesssim_q \|\mathcal{A} \circ \mathcal{W}_2(\omega t)[\mathbf{u}(t,\cdot)]\|_{\mathbf{H}_x^s} \overset{(3.18)}{\lesssim_q} \|\mathcal{W}_2(\omega\tau + \omega\alpha(\omega\tau))[\mathbf{u}(\tau + \alpha(\omega\tau),\cdot)]\|_{\mathbf{H}_x^s} \\
&\lesssim_q \|\mathbf{u}(\tau + \alpha(\omega\tau),\cdot)\|_{\mathbf{H}_x^s} \overset{(6.7)}{\lesssim_q} \|\mathbf{u}_0\|_{\mathbf{H}_x^s} \overset{(6.4)}{\lesssim_q} \|(v^{(0)}, \psi^{(0)})\|_{H_x^{s+\frac{1}{2}} \times H_x^{s-\frac{1}{2}}}.
\end{aligned}$$

Set $\gamma = \varepsilon^a$, with $0 < a < 1$ and $\Omega_\varepsilon := \Omega_\infty^{2\gamma}$. Then $\varepsilon\gamma^{-1} = \varepsilon^{1-a}$ and hence the smallness condition $\varepsilon\gamma^{-1} \le \delta_q$ is fulfilled by taking ε small enough. Furthermore, by Theorem 5.1, since $\gamma = \varepsilon^a$, we get that (1.12) holds and therefore Theorem 1.1 and Corollary 1.1 have been proved.

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Appendix

We prove some elementary properties of the set $\sigma_0(\sqrt{-\Delta})$ defined in (2.2).

Lemma A.1. (i) Let $p > d$. Then $\sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \alpha^{-p} < +\infty$. If $p > d + \nu$, $\sum_{\substack{\ell \in \mathbb{Z}^\nu \\ \alpha \in \sigma_0(\sqrt{-\Delta})}} \langle \ell, \alpha \rangle^{-p} < +\infty$.

(ii) Let $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$ with $\alpha \neq \beta$. Then there exists a constant $C > 0$ such that $|\alpha - \beta| \ge C(\alpha^{-1} + \beta^{-1})$. □

Proof of (i). By the definition (2.2) one has that

$$\sum_{\alpha \in \sigma_0(\sqrt{-\Delta})} \alpha^{-p} \le \sum_{j \in \mathbb{Z}^d} \langle j \rangle^{-p}, \qquad \sum_{\substack{\ell \in \mathbb{Z}^\nu \\ \alpha \in \sigma_0(\sqrt{-\Delta})}} \langle \ell, \alpha \rangle^{-p} \le \sum_{\substack{\ell \in \mathbb{Z}^\nu \\ j \in \mathbb{Z}^d}} \langle \ell, j \rangle^{-p}$$

the first series on the right-hand side converges if $p > d$ and the second one for $p > \nu + d$.
Proof of (ii). First, we note that if $x, y \in \mathbb{N}$, $x \neq y$ one has that

$$|\sqrt{x} - \sqrt{y}| \ge \max\left\{\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}}\right\} \ge C\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\right),$$

for some constant $C > 0$. Since by the definition of $\sigma_0(\sqrt{-\Delta})$, if $\alpha, \beta \in \sigma_0(\sqrt{-\Delta})$, $\alpha \neq \beta$, they are square roots of integer numbers, that is $\alpha^2, \beta^2 \in \mathbb{N}$, the claimed inequality follows. ■

Now we recall some well-known facts concerning linear self-adjoint operators on finite dimensional Hilbert spaces. Let $\mathcal{H}$ a finite dimensional Hilbert space of dimension n equipped by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Let us denote by $\mathcal{B}(\mathcal{H})$ the space of the linear operators from $\mathcal{H}$ onto itself, equipped by the operator norm $\| \cdot \|_{\mathcal{B}(\mathcal{H})}$. For any self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$, we order its eigenvalues as

$$\mathrm{spec}(A) := \{\lambda_1(A) \le \lambda_2(A) \le \ldots \le \lambda_n(A)\}. \tag{A.1}$$

We recall the well-known property

$$\|A\|_{\mathcal{B}(\mathcal{H})} = \max_{\lambda \in \mathrm{spec}(A)} |\lambda|. \tag{A.2}$$

Moreover, the following lemma holds

Lemma A.2. Let $\mathcal{H}$ be a Hilbert space of dimension n. Then the following holds:

(i) Let $A_1, A_2 : \mathcal{H} \to \mathcal{H}$ be self-adjoint operators. Then their eigenvalues, ranked as in (A.1), satisfy the Lipschitz property

$$|\lambda_k(A_1) - \lambda_k(A_2)| \le \|A_1 - A_2\|_{\mathcal{B}(\mathcal{H})}, \qquad \forall k = 1, \ldots, n.$$

(ii) Let $A = \eta \mathrm{Id}_{\mathcal{H}} + B$, where $\eta \in \mathbb{R}$, $\mathrm{Id}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is the identity and $B : \mathcal{H} \to \mathcal{H}$ is selfadjoint. Then

$$\lambda_k(A) = \eta + \lambda_k(B), \qquad \forall k = 1, \ldots, n.$$

(iii) Let $A : \mathcal{H} \to \mathcal{H}$ be self-adjoint and assume that $\mathrm{spec}(A) \subset \mathbb{R} \setminus \{0\}$. Then A is invertible and its inverse satisfies

$$\|A^{-1}\|_{\mathcal{B}(\mathcal{H})} = \frac{1}{\min_{k=1,\dots,n} |\lambda_k(A)|}. \qquad \square$$

References

[1] Arosio, A. and S. Panizzi. "On the well-posedness of the Kirchhoff string." *Trans. Amer. Math. Soc.* 348, no. 1 (1996): 305–30.

[2] Arosio, A. and S. Spagnolo. "Global solutions of the Cauchy problem for a nonlinear hyperbolic equation." In *Nonlinear PDEÕs and Their Applications*, edited by H. Brezis and J. L. Lions. Collége de France Seminar, vol. VI, 1Ð26, Research Notes Math. 109. Boston: Pitman, 1984.

[3] Baldi, P. "Periodic solutions of forced Kirchhoff equations." *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* 8 (2009): 117–41.

[4] Baldi, P., M. Berti, E. Haus, and R. Montalto. "Time quasi-periodic gravity water waves in finite depth." (in preparation).

[5] Baldi, P., M. Berti, and R. Montalto. "KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation." *Math. Ann.* 359 (2014): 471–536.

[6] Baldi, P., M. Berti, and R. Montalto. "KAM for autonomous quasi-linear perturbations of mKdV." *Boll. Unione Mat. Ital* 9 (2016): 143–88.

[7] Baldi, P., M. Berti, and R. Montalto. "KAM for autonomous quasi-linear perturbations of KdV." *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33 (2016): 1589–638.

[8] Bambusi, D. "Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, I." *Trans. Amer. Math. Soc.*, doi:10.1090/tran/7135, 2017.

[9] Bambusi, D. "Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, II." *Commun. Math. Phys*, doi:10.1007/s00220-016-2825-2, 2017.

[10] Bambusi, D. and S. Graffi. "Time quasi-periodic unbounded perturbations of Schrödinger operators and KAM methods." *Comm. Math. Phys.* 219 (2001): 465–80.

[11] Bernstein, S. N. "Sur une classe d' équations fonctionelles aux dérivées partielles." *Izv. Ross. Akad. Nauk Ser. Mat.* 4 (1940) 17–26.

[12] Berti, M., L. Biasco P., and M. Procesi. "KAM theory for the Hamiltonian DNLW." *Ann. Sci. Éc. Norm. Supér.* (*4*) 46 (2013): 301–73.

[13] Berti, M., L. Biasco, and M. Procesi. "KAM theory for the reversible derivative wave equation." *Arch. Rational Mech. Anal.* 212 (2014) 905–55.

[14] Berti, M. and P. Bolle. "Sobolev quasi periodic solutions of multidimensional wave equations with a multiplicative potential." *Nonlinearity* 25 (2012): 2579–613.

[15] Berti M. and P. Bolle "Quasi-periodic solutions with Sobolev regularity of NLS on $\mathbb{T}^d$ with a multiplicative potential." *Eur. J. Math.* 15 (2013) 229–86.

[16] Berti M., L. Corsi, and M. Procesi. “An abstract Nash-Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds.” *Comm. Math. Phys*. 334, no. 3 (2015): 1413–54.

[17] Berti, M., T. Kappeler, and R. Montalto. “Large KAM tori for perturbations of the dNLS equation.” Preprint arXiv:1603.09252, 2016.

[18] Berti, M. and R. Montalto. “Quasi-periodic standing wave solutions for gravity-capillary water waves.” To appear on *Mem. Amer. Math. Soc*. Preprint arXiv:1602.02411v1, 2016.

[19] Berti, M. and R. Montalto. “Quasi-periodic water waves.” *J. Fixed Point Theory Appl.* 19, no. 1 (2017): 129–56.

[20] Bourgain, J. “Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE.” *Int. Math. Res. Not.* 1994, no. 11 (1994): 475–97.

[21] Bourgain, J. “Quasi-periodic solutions of Hamiltonian perturbations of 2*D* linear Schrödinger equations.” *Ann. Math*. 148 (1998): 363–439.

[22] Bourgain, J. *Green’s Function Estimates for Lattice Schrödinger Operators and Applications*. Annals of Mathematics Studies, vol. 158. Princeton: Princeton University Press, 2005.

[23] Chierchia, L. and J. You. “KAM tori for 1D nonlinear wave equations with periodic boundary conditions.” *Comm. Math. Phys*. 211 (2000): 497–525.

[24] D’Ancona, P. and S. Spagnolo. “Global solvability for the degenerate Kirchhoff equation with real analytic data.” *Invent. Math*. 108 (1992): 247–62.

[25] Dickey, R. W. “Infinite systems of nonlinear oscillation equations related to the string.” *Proc. Amer. Math. Soc*. 23, no. 3 (1969): 459–68.

[26] Eliasson, L. H., B. Grebert, and S. Kuksin. “KAM for the nonlinear beam equation.” *Geom. Funct. Anal*. 26 (2016): 1588–715.

[27] Eliasson, L. H., B. Grebert, and S. Kuksin. “Almost reducibility for Klein Gordon equations with quasiperiodic in time potentials.” (in preparation).

[28] Eliasson, L. H. and S. Kuksin. “On reducibility of Schrödinger equations with quasiperiodic in time potentials.” *Comm. Math. Phys*. 286 (2009): 125–35.

[29] Eliasson, L. H. and S. Kuksin. “KAM for non-linear Schrödinger equation.” *Ann. Math*. 172 (2010): 371–435.

[30] Feola, R. “KAM for quasi-linear forced Hamiltonian NLS.” Preprint arXiv:1602.01341, 2016.

[31] Feola, R. and M. Procesi. “Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations.” *J. Differential Equations* 259, no. 7 (2015): 3389–447.

[32] Giuliani, F. “Quasi-periodic solutions for quasi-linear generalized KdV equations.” *J. Differential Equations* 262 (2017) 5052–132.

[33] Grebert, B. and E. Paturel. “On reducibility of quantum harmonic oscillator on $\mathbb{R}^d$ with quasiperiodic in time potential.” Preprint arXiv:1603.07455, 2016.

[34] Grebert, B. and E. Paturel. “KAM for the nonlinear Klein Gordon equation on $\mathbb{S}^d$.” *Boll. Unione Mat. Ital.* 9 (2016): 237–88.

[35] Iooss, G. P. I. Plotnikov, and J. F. Toland. “Standing waves on an infinitely deep perfect fluid under gravity.” *Arch. Ration. Mech. Anal.* 177, no. 3 (2005) 367–478.

[36] Kappeler, T. and J. Pöschel. *KAM and KdV*. Berlin: Springer, 2003.

[37] Kuksin, S. “Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum.” *Funktsional. Anal. i Prilozhen*. 21, no. 3 (1987): 22–37, 95.

[38] Kuksin, S. “A KAM theorem for equations of the Korteweg-de Vries type.” *Rev. Math. Phys.* 10, no. 3 (1998) 1–64.

[39] Kuksin, S. and J. Pöschel. “Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation.” *Ann. Math.* (*2*) 143 (1996): 149–79.

[40] Lions, J. L. “On some questions in boundary value problems of mathematical physics.” *Contemporary Developments in Continuum Mechanics and PDEs*, edited by G. M. de la Penha and L. A. Medeiros. Amsterdam: North-Holland, 1978.

[41] Liu, J. and X. Yuan. “A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations.” *Comm. Math. Phys*. 307, no. 3 (2011): 629–73.

[42] Manfrin, R. “On the global solvability of Kirchhoff equation for non-analytic initial data.” *J. Differential Equations* 211 (2005): 38–60.

[43] Matsuyama, T. and M. Ruzhansky. “Global Well-Posedness of the Kirchhoff Equation and Kirchhoff Systems.” *Analytic Methods in Interdisciplinary Applications*, 81–96. Springer Proceedings in Mathematics and Statistics, vol. 116, Cham: Springer, 2015.

[44] Montalto, R. “Quasi-periodic solutions of forced Kirchhoff equation.” *NoDEA Nonlinear Differential Equations Appl.* 24, no. 9 (2017). doi:10.1007/s00030-017-0432-3.

[45] Pokhozhaev, S. I. “On a class of quasilinear hyperbolic equations.” *Mat. Sb.* 96 (1975): 152–66. (English transl.: Mat. USSR Sbornik 25, 145–58, 1975).

[46] Procesi, C. and M. Procesi. “A KAM algorithm for the resonant non-linear Schrödinger equation.” *Adv. Math.* 272 (2015): 399–470.

[47] Wayne, E. “Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory.” *Comm. Math. Phys*. 127 (1990): 479–528.

[48] Zhang, J., M. Gao, and X. Yuan. “KAM tori for reversible partial differential equations.” *Nonlinearity* 24 (2011): 1189–228.