On the growth of Sobolev norms for a class of linear Schrödinger equations on the torus with superlinear dispersion

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Abstract: In this paper we consider time dependent Schrödinger equations on the one-dimensional torus $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ of the form $\partial_t u = \mathrm{i}\mathcal{V}(t)[u]$ where $\mathcal{V}(t)$ is a time dependent, self-adjoint pseudo-differential operator of the form $\mathcal{V}(t) = V(t,x)|D|^M + \mathcal{W}(t)$, $M > 1$, $|D| := \sqrt{-\partial_{xx}}$, V is a smooth function uniformly bounded from below and $\mathcal{W}$ is a time-dependent pseudo-differential operator of order strictly smaller than M. We prove that the solutions of the Schrödinger equation $\partial_t u = \mathrm{i}\mathcal{V}(t)[u]$ grow at most as t^ε, $t \to +\infty$ for any $\varepsilon > 0$. The proof is based on a reduction to constant coefficients up to smoothing remainders of the vector field $\mathrm{i}\mathcal{V}(t)$ which uses Egorov type theorems and pseudo-differential calculus.

Keywords: Growth of Sobolev norms, Linear Schrödinger equations, Pseudo-differential operators.

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Contents

1 Introduction and main result

In this paper we consider linear Schrödinger equations of the form

$$\partial_t u + \mathrm{i}\mathcal{V}(t)[u] = 0\,, \quad x \in \mathbb{T} \tag{1.1}$$

where $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ is the 1-dimensional torus, $\mathcal{V}(t)$ is a L^2 self-adjoint, time dependent, pseudo-differential Schrödinger operator of the form

$$\mathcal{V}(t) := V(t,x)|D|^M + \mathcal{W}(t)\,, \quad |D| := \sqrt{-\partial_{xx}}\,, \quad M > 1\,. \tag{1.2}$$

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We assume that V is a real valued $\mathcal{C}^\infty$ function defined on $\mathbb{R} \times \mathbb{T}$ with all derivatives bounded satisfying $\inf_{(t,x)\in\mathbb{R}\times\mathbb{T}} V(t,x) > 0$ and $\mathcal{W}(t)$ is a time-dependent pseudo differential operator of order strictly smaller than M. Our main goal is to show that given $t_0 \in \mathbb{R}$, $s \geq 0$, $u_0 \in H^s(\mathbb{T})$, the Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{i}\mathcal{V}(t)[u] = 0 \\ u(t_0, x) = u_0(x) \end{cases} \tag{1.3}$$

admits a unique solution $u(t)$ satisfying, for any $\varepsilon > 0$, the bound $\|u(t)\|_{H^s} \leq C(s,\varepsilon)(1 + |t - t_0|)^\varepsilon \|u_0\|_{H^s}$ for some constant $C(s,\varepsilon) > 0$. Here, $H^s(\mathbb{T})$ denotes the standard Sobolev space on the 1-dimensional torus $\mathbb{T}$ equipped with the norm $\|\cdot\|_{H^s}$.

There is a wide literature concerning the problem of estimating the high Sobolev norms of linear partial differential equations. For the Schrödinger operator $\mathcal{V}(t) = -\Delta + V(t,x)$ on the d-dimensional torus $\mathbb{T}^d$, the growth $\sim t^\varepsilon$ of the $\|\cdot\|_{H^s}$ norm of the solutions of $\partial_t u = \mathrm{i}\mathcal{V}(t)[u]$ has been proved by Bourgain in [9] for smooth quasi-periodic in time potentials and in [10] for smooth and bounded time dependent potentials. In the case where the potential V is analytic and quasi-periodic in time, Bourgain [9] proved also that $\|u(t)\|_{H^s}$ grows like a power of $\log(t)$. Moreover, this bound is optimal, in the sense that he constructed an example for which $\|u(t)\|_{H^s}$ is bounded from below by a power of $\log(t)$. The result obtained in [10] has been extended by Delort [11] for Schrödinger operators on Zoll manifolds. Furthermore, the logarithmic growth of $\|u(t)\|_{H^s}$ proved in [9] has been extended by Wang [19] in dimension 1, for any real analytic and bounded potential. The key idea in these series of papers is to use the so-called *spectral gap condition* for the operator $-\Delta$. Such a condition states that the spectrum of $-\Delta$ can be enclosed in disjont clusters $(\sigma_j)_{j\geq 0}$ such that the distance between σ_j and σ_{j+1} tends to $+\infty$ for $j \to +\infty$.

All the aforementioned results deal with the Schrödinger operator with a multiplicative potential. The first result in which the growth of $\|u(t)\|_{H^s}$ is exploited for Schrödinger operators with unbounded perturbations is due to Maspero-Robert [17]. More precisely, they prove the growth $\sim t^\varepsilon$ of $\|u(t)\|_{H^s}$, for Schrödinger equations of the form $\mathrm{i}\partial_t u = L(t)u$ where $L(t) = H + P(t)$, H is a time-independent operator of order $\mu + 1$ satisfying the *spectral gap condition* and $P(t)$ is an operator of order $\nu \leq \mu/(\mu + 1)$ (see Theorem 1.8 in [17]). The purpose of this paper is to provide a generalization of the result obtained in [17], at least for Schrödinger operators on the 1-dimensional torus, when the order of H is the same as the order of $P(t)$. Note that the operator defined in (1.2) can be written in the form $H + P(t)$ where $H = |D|^M$, $P(t) = (V(t,x) - 1)|D|^M + \mathcal{W}(t)$ and the operator $|D|^M$ fullfills the *spectral gap condition* since $M > 1$ (superlinear growth of the eigenvalues). Another generalization of [17] has been obtained independently and at the same time as our paper by Bambusi-Grebert-Maspero-Robert [6] in the case in which the order of $P(t)$ is strictly smaller than the one of H. This result covers also several applications in higher space dimension. We also mention that in the case of quasi-periodic systems $\mathrm{i}\partial_t u = L(\omega t)[u]$, $L(\omega t) = H + \varepsilon P(\omega t)$ it is often possible to prove that $\|u(t)\|_{H^s}$ is uniformly bounded in time for ε small enough and for a *large* set of frequencies ω. The general strategy to deal with these quasi-periodic systems is called *reducibility*. It consists in costructing, for most values of the frequencies ω and for ε small enough, a bounded quasi-periodic change of variable $\Phi(\omega t)$ which transforms the equation $\mathrm{i}\partial_t u = L(\omega t)u$ into a time independent system $\mathrm{i}\partial_t v = \mathcal{D}v$ whose solution preserves the Sobolev norms $\|v(t)\|_{H^s}$. We mention the results of Eliasson-Kuksin [12] which proved the reducibility of the Schrödinger equation on $\mathbb{T}^d$ with a small, quasi-periodic in time analytic potential and Grebert-Paturel [16] which proved the reducibility of the quantum harmonic oscillator on $\mathbb{R}^d$. Concerning KAM-reducibility with unbounded perturbations, we mention Bambusi [3], [4] for the reducibility of the quantum harmonic oscillator with unbounded perturbations (see also [5] in any dimension), [1], [2], [15] for fully non-linear KdV-type equations, [13], [14] for fully-nonlinear Schrödinger equations, [7], [8] for the water waves system and [18] for the Kirchhoff equation. Note that in [1], [2], [15], [7], [8], [18] the reducibility of the linearized equations is obtained as a consequence of the KAM theorems proved for the corresponding nonlinear equations.

We now state in a precise way the main results of this paper. First, we introduce some notations. For any function $u \in L^2(\mathbb{T})$, we introduce its Fourier coefficients

$$\widehat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{T}} u(x) e^{-\mathrm{i}x\xi}\, dx\,, \qquad \forall \xi \in \mathbb{Z}\,. \tag{1.4}$$

For any $s \geq 0$, we introduce the Sobolev space of complex valued functions $H^s \equiv H^s(\mathbb{T})$, as

$$H^s := \Big\{ u \in L^2(\mathbb{T}) : \|u\|_{H^s}^2 := \sum_{\xi \in \mathbb{Z}} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 < +\infty \Big\} , \quad \langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}} . \tag{1.5}$$

Given two Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, we denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators from X to Y equipped with the usual operator norm $\|\cdot\|_{\mathcal{B}(X,Y)}$. If $X = Y$, we simply write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$.
Given a linear operator $\mathcal{R} \in \mathcal{B}(L^2(\mathbb{T}))$, we denote by $\mathcal{R}^*$ the adjoint operator of $\mathcal{R}$ with respect to the standard L^2 inner product

$$\langle u, v \rangle_{L^2} := \int_{\mathbb{T}} u(x) \overline{v(x)} \, dx , \quad \forall u, v \in L^2(\mathbb{T}) . \tag{1.6}$$

We say that the operator $\mathcal{R}$ is self-adjoint if $\mathcal{R} = \mathcal{R}^*$.
Given a Banach space $(X, \|\cdot\|_X)$, for any $k \in \mathbb{N}$, for any $-\infty \leq T_1 < T_2 \leq +\infty$ we consider the space $\mathcal{C}^k([T_1, T_2], X)$ of the k-times continuously differentiable functions with values in X. We denote by $\mathcal{C}_b^k([T_1, T_2], X)$ the space of functions in $\mathcal{C}^k([T_1, T_2], X)$ having bounded derivatives, equipped with the norm

$$\|u\|_{\mathcal{C}_b^k([T_1,T_2],X)} := \max_{j=1,\ldots,k} \sup_{t \in [T_1, T_2]} \|\partial_t^j u(t)\|_X . \tag{1.7}$$

For any domain $\Omega \subset \mathbb{R}^d$, we also denote by $\mathcal{C}_b^\infty(\Omega)$ the space of the $\mathcal{C}^\infty$ functions on Ω with all the derivatives bounded.
Since the equation we deal with is a Hamiltonian PDE, we briefly describe the Hamiltonian formalism. We define the symplectic form $\Omega : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to \mathbb{R}$ by

$$\Omega[u_1, u_2] := \mathrm{i} \int_{\mathbb{T}} (u_1 \bar{u}_2 - \bar{u}_1 u_2) \, dx , \quad \forall u_1, u_2 \in L^2(\mathbb{T}) . \tag{1.8}$$

Given a family of linear operators $\mathcal{R} : \mathbb{R} \to \mathcal{B}(L^2)$ such that $\mathcal{R}(t) = \mathcal{R}(t)^*$ for any $t \in \mathbb{R}$, we define the time-dependent quadratic Hamiltonian associated to $\mathcal{R}$ as

$$\mathcal{H}(t, u) := \langle \mathcal{R}(t)[u] \, , \, u \rangle_{L^2_x} = \int_{\mathbb{T}} \mathcal{R}(t)[u] \, \overline{u} \, dx , \qquad \forall u \in L^2(\mathbb{T}) .$$

The Hamiltonian vector field associated to the Hamiltonian $\mathcal{H}$ is defined by

$$X_{\mathcal{H}}(t, u) := \mathrm{i} \nabla_{\overline{u}} \mathcal{H}(t, u) = \mathrm{i} \mathcal{R}(t) \tag{1.9}$$

where the gradient $\nabla_{\overline{u}}$ stands for

$$\nabla_{\overline{u}} := \frac{1}{\sqrt{2}} (\nabla_v + \mathrm{i} \nabla_\psi) , \quad v = \mathrm{Re}(u) , \quad \psi := \mathrm{Im}(u) .$$

We say that $\Phi : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{T}))$ is symplectic if and only if

$$\Omega \Big[\Phi(t)[u_1], \Phi(t)[u_2] \Big] = \Omega[u_1, u_2] , \qquad \forall u_1, u_2 \in L^2(\mathbb{T}) , \quad \forall t \in \mathbb{R} .$$

We recall the classical thing that if $X_{\mathcal{H}}$ is a Hamiltonian vector field, then $\exp(X_{\mathcal{H}})$ is symplectic.

Let us consider a time dependent vector field $X : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{T}))$ and a differentiable family of invertible maps $\Phi : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{T}))$. Under the change of variables $u = \Phi(t)[v]$, the equation $\partial_t u = X(t)[u]$ transforms into the equation $\partial_t v = X_+(t)[v]$ where the *push-forward* $X_+(t)$ of the vector field $X(t)$ is defined by

$$X_+(t) := \Phi_* X(t) := \Phi(t)^{-1} \Big(X(t) \Phi(t) - \partial_t \Phi(t) \Big) , \quad t \in \mathbb{R} . \tag{1.10}$$

It is well known that if Φ is symplectic and $X(t)$ is a Hamiltonian vector field, then the push-forward $X_+(t) = \Phi_* X(t)$ is still a Hamiltonian vector field.
In the next two definitions, we also define time dependent pseudo differential operators on $\mathbb{T}$.

Definition 1.1 (**The symbol class** S^m)**.** *Let $m \in \mathbb{R}$. We say that a $\mathcal{C}^\infty$ function $a : \mathbb{R} \times \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ belongs to the symbol class S^m if and only if for any $\alpha, \beta, \gamma \in \mathbb{N}$ there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that*

$$|\partial_t^\alpha \partial_x^\beta \partial_\xi^\gamma a(t,x,\xi)| \leq C_{\alpha,\beta,\gamma} \langle \xi \rangle^{m-\gamma}, \quad \forall (t,x,\xi) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}. \tag{1.11}$$

We define the class of smoothing symbols $S^{-\infty} := \cap_{m \in \mathbb{R}} S^m$.

Definition 1.2. **(the class of operators** OPS^m**)** *Let $m \in \mathbb{R}$ and $a \in S^m$. We define the time-dependent linear operator $A(t) = \mathrm{Op}\big(a(t,x,\xi)\big) = a(t,x,D)$ as*

$$A(t)[u](x) := \sum_{\xi \in \mathbb{Z}} a(t,x,\xi) \widehat{u}(\xi) e^{\mathrm{i} x \xi}, \qquad \forall u \in \mathcal{C}^\infty(\mathbb{T}).$$

We say that the operator A is in the class OPS^m.
We define the class of smoothing operators $OPS^{-\infty} := \cap_{m \in \mathbb{R}} OPS^m$.

Now, we are ready to state the main results of this paper. We make the following assumptions.

(H1) The operator $\mathcal{V}(t) = V(t,x)|D|^M + \mathcal{W}(t)$ in (1.2) is L^2 self-adjoint for any $t \in \mathbb{R}$.

(H2) The function $V(t,x)$ in (1.2) is in $\mathcal{C}_b^\infty(\mathbb{R} \times \mathbb{T}, \mathbb{R})$, strictly positive and bounded from below, i.e. $\delta := \inf_{(t,x) \in \mathbb{R} \times \mathbb{T}} V(t,x) > 0$.

(H3) The operator $\mathcal{W}(t)$ is a time-dependent pseudo-differential operator $\mathcal{W}(t) = \mathrm{Op}(w(t,x,\xi))$, with symbol $w \in S^{M-\mathfrak{e}}$ for some $\mathfrak{e} > 0$.

The main result of this paper is the following

Theorem 1.3 (**Growth of Sobolev norms**)**.** *Assume the hypotheses* **(H1)**-**(H3)**. *Let $s > 0$, $u_0 \in H^s(\mathbb{T})$, $t_0 \in \mathbb{R}$. Then there exists a unique global solution $u \in \mathcal{C}^0(\mathbb{R}, H^s(\mathbb{T}))$ of the Cauchy problem*

$$\begin{cases} \partial_t u + \mathrm{i}\mathcal{V}(t)[u] = 0 \\ u(t_0,x) = u_0(x) \end{cases} \tag{1.12}$$

and for any $\varepsilon > 0$ there exists a constant $C(s,\varepsilon) > 0$ such that

$$\|u(t)\|_{H^s} \leq C(s,\varepsilon)(1 + |t - t_0|^\varepsilon)\|u_0\|_{H^s}, \qquad \forall t \in \mathbb{R}. \tag{1.13}$$

This theorem will be proved in Section 5 and it will be deduced by the following

Theorem 1.4 (**Normal-form theorem**)**.** *Assume the hypotheses* **(H1)**-**(H3)**. *For any $K > 0$ there exists a time-dependent symplectic differentiable invertible map $t \mapsto \mathcal{T}_K(t)$ satisfying*

$$\sup_{t \in \mathbb{R}} \|\mathcal{T}_K(t)^{\pm 1}\|_{\mathcal{B}(H^s)} + \sup_{t \in \mathbb{R}} \|\partial_t \mathcal{T}_K(t)^{\pm 1}\|_{\mathcal{B}(H^{s+1}, H^s)} < +\infty, \qquad \forall s \geq 0 \tag{1.14}$$

such that the following holds: the vector field $\mathrm{i}\mathcal{V}(t)$ is transformed, by the map $\mathcal{T}_K$, into the vector field

$$\mathrm{i}\mathcal{V}_K(t) := (\mathcal{T}_K)_*(\mathrm{i}\mathcal{V})(t) = \mathrm{i}\Big(\lambda_K(t,D) + \mathcal{W}_K(t)\Big) \tag{1.15}$$

where $\lambda_K(t,D) := \mathrm{Op}(\lambda_K(t,\xi))$ is a space-diagonal operator with symbol λ_K which satisfies

$$\lambda_K \in S^M, \quad \lambda_K(t,\xi) = \overline{\lambda_K(t,\xi)}, \quad \forall (t,\xi) \in \mathbb{R} \times \mathbb{R} \tag{1.16}$$

and

$$\mathcal{W}_K(t) = \mathrm{Op}\Big(w_K(t,x,\xi)\Big), \quad w_K \in S^{-K} \tag{1.17}$$

is L^2 self-adjoint.

In the remaining part of the section, we shall explain the main ideas needed to prove Theorems 1.3, 1.4. In order to prove Theorem 1.3, we need to estimate the Sobolev norm $\|u(t)\|_{H^s}$, $s>0$, for the solutions $u(t)$ of (1.12). Choosing the integer $K \simeq s$ in Theorem 1.4, we transform the PDE $\partial_t u = \mathrm{i}\mathcal{V}(t)u$ into the PDE $\partial_t v = \mathrm{i}\mathrm{Op}\big(\lambda_s(t,\xi)\big)v + O(|D|^{-s})v$. The Hamiltonian structure guarantees that the symbol $\lambda_s(t,\xi)$ is real. Writing the Duhamel formula for the latter equation, one easily gets that $\|v(t)\|_{H^s} \lesssim_s \|v(t_0)\|_{H^s} + |t-t_0|\|v(t_0)\|_{L^2}$ which implies that the same estimate holds for $u(t)$, i.e. $\|u(t)\|_{H^s} \lesssim_s \|u(t_0)\|_{H^s} + |t-t_0|\|u(t_0)\|_{L^2}$ (in this paper, we use the standard notation $A \lesssim_s B$ if and only if $A \le C(s)B$ for some constant $C(s)>0$). Using that $\|u(t)\|_{L^2} = \|u(t_0)\|_{L^2}$ (since $\mathcal{V}(t)$ is self-adjoint), applying the classical interpolation Theorem 2.8, one obtains the growth $\sim |t-t_0|^\varepsilon$ of the Sobolev norm $\|u(t)\|_{H^s}$ for any $s, \varepsilon > 0$.

The proof of Theorem 1.4 is based on a *normal form* procedure, which transforms the vector field $\mathrm{i}\mathcal{V}(t)$ into another one which is an arbitrarily regularizing perturbation of a *space diagonal* vector field. Such a procedure is developed in Section 3 and it is based on symbolic calculus and Egorov type Theorems (see Theorems 2.14-2.16). We describe below our method in more detail.

1. *Reduction of the highest order.* Our first aim is to transform the vector field $\mathrm{i}\mathcal{V}(t) = \mathrm{i}\big(V(t,x)|D|^M + \mathcal{W}(t)\big)$ into another vector field $\mathrm{i}\mathcal{V}_1(t)$ whose highest order is x-independent, i.e. $\mathcal{V}_1(t) = \lambda(t)|D|^M + \mathcal{W}_1(t)$ with $\mathcal{W}_1 \in OPS^{M-\mathfrak{e}}$. This is done in Section 3.1. In order to achieve this purpose, we transform the vector field $\mathrm{i}\mathcal{V}(t)$ by means of the time 1-flow map of the transport equation

$$\partial_\tau u = b_\alpha(\tau;t,x)\partial_x u + \frac{(\partial_x b_\alpha)}{2}u\,, \quad b_\alpha(\tau;t,x) := -\frac{\alpha(t,x)}{1+\tau\alpha_x(t,x)}\,, \qquad \tau\in[0,1]$$

   where $\alpha(t,x)$ is a function in $\mathcal{C}^\infty_b(\mathbb{R}\times\mathbb{T},\mathbb{R})$ (to be determined) satisfying $\inf_{(t,x)\in\mathbb{R}\times\mathbb{T}}\Big(1+(\partial_x\alpha)(t,x)\Big) > 0$. This condition guarantees that $\mathbb{T}\to\mathbb{T}$, $x\mapsto x+\alpha(t,x)$ is a diffeomorphism of the torus with inverse given by $\mathbb{T}\to\mathbb{T}$, $y\mapsto y+\widetilde{\alpha}(t,y)$ and $\widetilde{\alpha}\in\mathcal{C}^\infty_b(\mathbb{R}\times\mathbb{T},\mathbb{R})$ satisfying $\inf_{(t,y)\in\mathbb{R}\times\mathbb{T}}\Big(1+(\partial_y\widetilde{\alpha})(t,y)\Big)>0$ (see Lemma 2.12). The transformed vector field $\mathrm{i}\mathcal{V}_1(t)$, $\mathcal{V}_1(t) = \mathrm{Op}\Big(v_1(t,x,\xi)\Big)$ is analyzed by using Theorems 2.14, 2.15 and its final expansion is provided in Lemma 3.2. It turns out that the principal part of the operator $\mathcal{V}_1(t)$ is given by

$$\Big[V(t,y)\big(1+\widetilde{\alpha}_y(t,y)\big)^M\Big]_{y=x+\alpha(t,x)}|D|^M\,.$$

   The function $\widetilde{\alpha}$ is choosen in such a way that $V(t,y)\big(1+\widetilde{\alpha}_y(t,y)\big)^M = \lambda(t)$ where $\lambda\in\mathcal{C}^\infty_b(\mathbb{R},\mathbb{R})$ is independent of x (see (3.16)–(3.19)). The hypothesis **(H2)** on $V(t,x)$, i.e. $\inf_{(t,x)\in\mathbb{R}\times\mathbb{T}}V(t,x)>0$ ensures that $\inf_{t\in\mathbb{R}}\lambda(t)>0$ and $\inf_{(t,y)\in\mathbb{R}\times\mathbb{T}}\Big(1+(\partial_y\widetilde{\alpha})(t,y)\Big)>0$ and hence also $\inf_{(t,x)\in\mathbb{R}\times\mathbb{T}}\Big(1+(\partial_x\alpha)(t,x)\Big)>0$, by Lemma 2.12.

2. *Reduction of the lower order terms.* After the first reduction described above, we deal with a vector field $\mathrm{i}\mathcal{V}_1(t)$ where $\mathcal{V}_1(t) = \lambda(t)|D|^M + \mathcal{W}_1(t)$ and $\mathcal{W}_1\in OPS^{M-\bar{\mathfrak{e}}}$ for some constant $\bar{\mathfrak{e}}>0$. The next step is to transform such a vector field into another one of the form $\mathrm{i}\Big(\lambda(t)|D|+\mu_N(t,D)+\mathcal{W}_N(t)\Big)$ where $\mu_N(t,D)$ is a time-dependent Fourier multiplier of order $M-\bar{\mathfrak{e}}$ and $\mathcal{W}_N\in OPS^{M-N\bar{\mathfrak{e}}}$ for any integer $N>0$. This is proved by means of an iterative procedure developed in Section 3.2, see Proposition 3.3. At the n-th step of such a procedure, we deal with a vector field $\mathrm{i}\mathcal{V}_n(t)$, $\mathcal{V}_n(t) = \lambda(t)|D|^M + \mu_n(t,D) + \mathcal{W}_n(t)$, $\mu_n\in S^{M-\bar{\mathfrak{e}}}$, $\mathcal{W}_n\in OPS^{M-n\bar{\mathfrak{e}}}$. We transform such a vector field by means of the time-1 flow map of the PDE
$$\partial_\tau u = \mathrm{i}\mathcal{G}_n(t)[u] \quad \text{where} \quad \mathcal{G}_n(t) = \mathcal{G}_n(t)^*\,, \quad \mathcal{G}_n\in OPS^{1-n\bar{\mathfrak{e}}}\,.$$

   Using Theorem 2.16, the transformed vector field $\mathrm{i}\mathcal{V}_{n+1}(t)$, $\mathcal{V}_{n+1}(t) = \mathrm{Op}\Big(v_{n+1}(t,x,\xi)\Big)$ has the symbol expansion

$$v_{n+1}(t,x,\xi) = \lambda(t)|\xi|^M + \mu_n(t,\xi) + w_n(t,x,\xi) - M\lambda(t)|\xi|^{M-2}\xi\partial_x g_n(t,x,\xi) + O(|\xi|^{M-(n+1)\bar{\mathfrak{e}}})$$

One then finds $g_n(t,x,\xi)$ so that $g_n = g_n^*$ (g_n^* is the symbol of the adjoint operator) and which solves the equation

$$w_n(t,x,\xi) - M\lambda(t)|\xi|^{M-2}\xi\partial_x g_n(t,x,\xi) = \langle w_n\rangle_x(t,\xi) + O(|\xi|^{M-(n+1)\bar{\mathfrak{e}}})$$

where $\langle w_n\rangle_x(t,\xi) := \frac{1}{2\pi}\int_{\mathbb{T}} w_n(t,x,\xi)\,dx$ (see Lemma 3.5). This implies that the transformed symbol has the form $v_{n+1}(t,x,\xi) = \lambda(t)|\xi|^M + \mu_{n+1}(t,\xi) + O(|\xi|^{M-(n+1)\bar{\mathfrak{e}}})$ with $\mu_{n+1} = \mu_n + \langle w_n\rangle_x$.

The paper is organized as follows: in Section 2 we provide some technical tools which are needed for the proof of Theorem 1.4. In Section 3 we develop the regularization procedure of the vector field that we use in Section 4 to deduce Theorem 1.4. Finally, in Section 5 we prove Theorem 1.3.

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2 Pseudo-differential operators

In this section, we recall some well-known definitions and results concerning pseudo differential operators on the torus $\mathbb{T}$. We always consider time dependent symbols $a(t,x,\xi)$ depending in a $\mathcal{C}^\infty$ way on the whole variables, see Definitions 1.1, 1.2. Actually the time t is only a parameter, hence all the classical results apply without any modification (we refer for instance to [20], [21]).
For the symbol class S^m given in the definition 1.1 and the operator class OPS^m given in the definition 1.2, the following standard inclusions hold:

$$S^m \subseteq S^{m'}, \quad OPS^m \subseteq OPS^{m'}, \quad \forall m \le m'. \tag{2.1}$$

We define the class of smoothing symbol and smoothing operators $S^{-\infty} := \cap_{m\in\mathbb{R}} S^m$, $OPS^{-\infty} := \cap_{m\in\mathbb{R}} OPS^m$

Theorem 2.1 **(Calderon-Vallancourt).** *Let $m \in \mathbb{R}$ and $A = a(t,x,D) \in OPS^m$. Then for any $s \in \mathbb{R}$, for any $\alpha \in \mathbb{N}$ the operator $\partial_t^\alpha A(t) \in \mathcal{B}(H^{s+m}(\mathbb{T}), H^s(\mathbb{T}))$ with $\sup_{t\in\mathbb{R}} \|\partial_t^\alpha A(t)\|_{\mathcal{B}(H^{s+m},H^s)} < +\infty$.*

Definition 2.2 **(Asymptotic expansion).** *Let $(m_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence of real numbers converging to $-\infty$ and $a_k \in S^{m_k}$ for any $k \in \mathbb{N}$. We say that $a \in S^{m_0}$ has the asymptotic expansion $\sum_{k\ge 0} a_k$, i.e.*

$$a \sim \sum_{k\ge 0} a_k$$

if for any $N \in \mathbb{N}$

$$a - \sum_{k=0}^{N} a_k \in S^{m_{N+1}}.$$

Given a symbol $a \in S^m$, we denote by $\widehat{a}$, the Fourier transform with respect to the variable x, i.e.

$$\widehat{a}(t,\eta,\xi) := \frac{1}{2\pi}\int_{\mathbb{T}} a(t,x,\xi) e^{-\mathrm{i}\eta x}\,dx, \quad (t,\eta,\xi) \in \mathbb{R}\times\mathbb{Z}\times\mathbb{R}. \tag{2.2}$$

Theorem 2.3 **(Composition).** *Let $m, m' \in \mathbb{R}$ and $A = a(t,x,D) \in OPS^m$, $B = b(t,x,D) \in OPS^{m'}$. Then the composition operator $AB := A\circ B = \sigma_{AB}(t,x,D)$ is a pseudo-differential operator in $OPS^{m+m'}$ with symbol*

$$\sigma_{AB}(t,x,\xi) = \sum_{\eta\in\mathbb{Z}} a(t,x,\xi+\eta)\widehat{b}(t,\eta,\xi)e^{\mathrm{i}\eta x}. \tag{2.3}$$

The symbol σ_{AB} has the following asymptotic expansion

$$\sigma_{AB}(t,x,\xi) \sim \sum_{\beta\ge 0} \frac{1}{\mathrm{i}^\beta \beta!}\partial_\xi^\beta a(t,x,\xi)\partial_x^\beta b(t,x,\xi), \tag{2.4}$$

that is, $\forall N \geq 1$,

$$\sigma_{AB}(t,x,\xi) = \sum_{\beta=0}^{N-1} \frac{1}{\beta! \mathrm{i}^\beta} \partial_\xi^\beta a(t,x,\xi)\, \partial_x^\beta b(t,x,\xi) + r_N(t,x,\xi) \qquad \text{where} \qquad r_N := r_{N,AB} \in S^{m+m'-N}\,. \tag{2.5}$$

The remainder r_N *has the explicit formula*

$$r_N(t,x,\xi) := \frac{1}{(N-1)!\,\mathrm{i}^N} \int_0^1 (1-\tau)^{N-1} \sum_{\eta\in\mathbb{Z}} (\partial_\xi^N a)(t,x,\xi+\tau\eta) \widehat{\partial_x^N b}(t,\eta,\xi) e^{\mathrm{i}\eta x}\, d\tau\,. \tag{2.6}$$

Corollary 2.4. *Let* $m, m' \in \mathbb{R}$ *and let* $A = \mathrm{Op}(a)$, $B = \mathrm{Op}(b)$. *Then the commutator* $[A,B] = \mathrm{Op}(a \star b)$, *with* $a \star b \in S^{m+m'-1}$ *having the following expansion:*

$$a \star b = -\mathrm{i}\{a,b\} + \mathtt{r}_2(a,b)\,, \quad \{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b \in S^{m+m'-1}\,, \quad \mathtt{r}_2(a,b) \in S^{m+m'-2}\,.$$

Theorem 2.5 (**Adjoint of a pseudo-differential operator**). *If* $A(t) = a(t,x,D) \in OPS^m$ *is a pseudo-differential operator with symbol* $a \in S^m$, *then its* L^2*-adjoint is the pseudo-differential operator* $A^* = \mathrm{Op}(a^*) \in OPS^m$ *defined by*

$$A^* = \mathrm{Op}(a^*) \qquad \text{with symbol} \qquad a^*(t,x,\xi) := \overline{\sum\nolimits_{\eta\in\mathbb{Z}} \widehat{a}(t,\eta,\xi-\eta) e^{\mathrm{i}\eta x}}\,. \tag{2.7}$$

The symbol $a^* \in S^m$ *admits the asymptotic expansion*

$$a^*(t,x,\xi) \sim \sum_{\alpha\in\mathbb{N}} \frac{(-\mathrm{i})^\alpha}{\alpha!} \overline{\partial_x^\alpha \partial_\xi^\alpha a(t,x,\xi)} \tag{2.8}$$

meaning that for any integer $N \geq 1$,

$$a^*(t,x,\xi) = \sum_{\alpha=0}^{N-1} \frac{(-\mathrm{i})^\alpha}{\alpha!} \overline{\partial_x^\alpha \partial_\xi^\alpha a(t,x,\xi)} + r_N^*(t,x,\xi) \qquad \text{where} \qquad r_N^* := r_{N,A}^* \in S^{m-N}\,.$$

The remainder r_N^* *has the explicit formula*

$$r_N^*(t,x,\xi) := \frac{(-1)^N}{(N-1)!} \int_0^1 (1-\tau)^{N-1} \overline{\sum_{\eta\in\mathbb{Z}} \widehat{\partial_\xi^N a}(t,\eta,\xi-t\eta) \eta^N e^{\mathrm{i}\eta x}}\, d\tau\,. \tag{2.9}$$

Note that if $a \in S^m$ is a symbol independent of x (Fourier multiplier) then

$$a^*(t,\xi) = \overline{a(t,\xi)}\,, \quad \forall (t,\xi) \in \mathbb{R}\times\mathbb{R}\,. \tag{2.10}$$

We now prove some useful lemmas which we apply in Section 3.

Lemma 2.6. *Let* $A = \mathrm{Op}(a) \in OPS^m$ *be self-adjoint, i.e.* $a(t,x,\xi) = a^*(t,x,\xi)$ *and let* $\varphi(t,\xi)$ *be a real Fourier multiplier of order* m'. *We define the symbol* $b(t,x,\xi) := \varphi(t,\xi) a(t,x,\xi) \in S^{m+m'}$. *Then*

$$b^*(t,x,\xi) - b(t,x,\xi) \in S^{m+m'-1}.$$

Proof. One has that

$$\mathrm{Op}(b^*) = \mathrm{Op}(b)^* = \mathrm{Op}(\varphi)^* \circ \mathrm{Op}(a)^*\,.$$

Since $\mathrm{Op}(a)$ is self-adjoint and since λ is real, one has that

$$\mathrm{Op}(b^*) = \mathrm{Op}(\varphi) \circ \mathrm{Op}(a)\,.$$

By applying Theorem 2.3, one gets that

$$\mathrm{Op}(b^*) = \mathrm{Op}\Big(\varphi(t,\xi) a(t,x,\xi) + r(t,x,\xi)\Big)\,, \qquad r \in S^{m+m'-1}$$

and then the lemma is proved. □

We define the operator ∂_x^{-1} by setting

$$\partial_x^{-1}[1] := 0\,, \quad \partial_x^{-1}[e^{\mathrm{i}xk}] := \frac{e^{\mathrm{i}xk}}{\mathrm{i}k}\,, \quad \forall k \in \mathbb{Z}\setminus\{0\}\,. \tag{2.11}$$

Furthermore, given a symbol $a \in S^m$, we define the averaged symbol $\langle a\rangle_x$ by

$$\langle a\rangle_x(t,\xi) := \frac{1}{2\pi}\int_{\mathbb{T}} a(t,x,\xi)\,dx\,, \quad \forall (t,\xi)\in\mathbb{R}\times\mathbb{R}\,. \tag{2.12}$$

The following elementary property holds:

$$a \in S^m \Longrightarrow \partial_x^{-1} a\,,\ \langle a\rangle_x \in S^m\,. \tag{2.13}$$

We now prove the following

Lemma 2.7. *Let $a \in S^m$. Then the following holds:*
(i) $\langle a^*\rangle_x = (\langle a\rangle_x)^* = \langle \overline{a}\rangle_x$,
(ii) $\partial_x^{-1}(a^*) = (\partial_x^{-1}a)^*$.

Proof. PROOF OF *(i)*. By Theorem 2.5 and since by the definition (2.12) the symbol $\langle a\rangle_x$ is x-independent, one has that

$$(\langle a\rangle_x)^*(t,\xi) = \langle\overline{a}\rangle_x(t,\xi)\,. \tag{2.14}$$

Moreover by (2.7), (2.12) one gets

$$\begin{aligned}\langle a^*\rangle_x(t,\xi) &= \frac{1}{2\pi}\int_{\mathbb{T}}\Big(\overline{\sum\nolimits_{\eta\in\mathbb{Z}}\widehat{a}(t,\eta,\xi-\eta)e^{\mathrm{i}\eta x}}\Big)\,dx = \overline{\widehat{a}(t,0,\xi)} \\ &\overset{(2.2)}{=} \overline{\frac{1}{2\pi}\int_{\mathbb{T}} a(t,x,\xi)\,dx} \overset{(2.12)}{=} \overline{\langle a\rangle_x(t,\xi)} \overset{(2.14)}{=} (\langle a\rangle_x)^*(t,\xi)\end{aligned}$$

hence the claimed statement follows.

PROOF OF *(ii)*. By (2.11), one has that

$$\widehat{\partial_x^{-1}a}(t,0,\xi) = 0\,, \quad \widehat{\partial_x^{-1}a}(t,\eta,\xi) = \frac{\widehat{a}(t,\eta,\xi)}{\mathrm{i}\eta}, \quad \eta\in\mathbb{Z}\setminus\{0\},$$

hence by formula (2.7)

$$\begin{aligned}(\partial_x^{-1}a)^*(t,x,\xi) &= \overline{\sum\nolimits_{\eta\in\mathbb{Z}\setminus\{0\}}\frac{\widehat{a}(t,\eta,\xi-\eta)}{\mathrm{i}\eta}e^{\mathrm{i}\eta x}} = \overline{\sum\nolimits_{\eta\in\mathbb{Z}}\widehat{a}(t,\eta,\xi-\eta)\partial_x^{-1}(e^{\mathrm{i}\eta x})} \\ &= \partial_x^{-1}\Big(\overline{\sum\nolimits_{\eta\in\mathbb{Z}}\widehat{a}(t,\eta,\xi-\eta)e^{\mathrm{i}\eta x}}\Big) = \partial_x^{-1}(a^*)(t,x,\xi)\end{aligned} \tag{2.15}$$

which proves item *(ii)*. □

For any $\alpha\in\mathbb{R}$, the operator $|D|^\alpha$, acting on 2π-periodic functions $u(x) = \sum_{\xi\in\mathbb{Z}}\widehat{u}(\xi)e^{\mathrm{i}x\xi}$ is defined by

$$|D|^\alpha u(x) := \sum_{\xi\in\mathbb{Z}\setminus\{0\}} |\xi|^\alpha\widehat{u}(\xi)e^{\mathrm{i}x\xi}\,.$$

We shall identify the operator $|D|^\alpha$ with the operator associated to a Fourier multiplier $|\xi|^\alpha\chi(\xi)$ in S^α where $\chi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is an even cut-off function satisfying

$$\chi(\xi) := \begin{cases} 1 & \text{if } \ |\xi|\geq 1 \\ 0 & \text{if } \ |\xi|\leq\frac{1}{2}\,. \end{cases} \tag{2.16}$$

Then, for any $\alpha\in\mathbb{R}$,

$$|D|^\alpha \equiv \mathrm{Op}\big(|\xi|^\alpha\chi(\xi)\big) \tag{2.17}$$

since the action of the two operators on 2π-periodic functions $u\in L^2(\mathbb{T})$ coincides.

We conclude this section by stating an interpolation theorem, which is an immediate consequence of the classical Riesz-Thorin interpolation theorem in Sobolev spaces.

Theorem 2.8. *Let $0 \le s_0 < s_1$ and let $A \in \mathcal{B}(H^{s_0}) \cap \mathcal{B}(H^{s_1})$. Then for any $s_0 \le s \le s_1$ the operator $A \in \mathcal{B}(H^s)$ and*

$$\|A\|_{\mathcal{B}(H^s)} \le \|A\|^{\lambda}_{\mathcal{B}(H^{s_0})} \|A\|^{1-\lambda}_{\mathcal{B}(H^{s_1})}, \quad \lambda := \frac{s_1 - s}{s_1 - s_0}.$$

2.1 Well posedness of some linear PDEs

In this section we study the properties of the flow of some linear pseudo-PDEs. We start with the following lemma.

Lemma 2.9. *Let $\mathcal{A}(\tau; t) := \mathrm{Op}\Big(a(\tau; t, x, \xi)\Big)$, $\tau \in [0,1]$ be a smooth τ-dependent family of pseudo differential operators in OPS^1. Assume that $\mathcal{A}(\tau; t) + \mathcal{A}(\tau; t)^* \in OPS^0$. Then the following holds.*
(i) Let $s \ge 0$, $u_0 \in H^s(\mathbb{T})$, $\tau_0 \in [0,1]$. Then there exists a unique solution $u \in \mathcal{C}^0_b\Big([0,1], H^s(\mathbb{T})\Big)$ of the Cauchy problem

$$\begin{cases} \partial_\tau u = \mathcal{A}(\tau; t)[u] \\ u(\tau_0; x) = u_0(x) \end{cases} \tag{2.18}$$

satisfying the estimate

$$\|u\|_{\mathcal{C}^0_b([0,1], H^s)} \lesssim_s \|u_0\|_{H^s}.$$

As a consequence, for any $\tau_0, \tau \in [0,1]$, the flow map $\Phi(\tau_0, \tau; t)$, which maps the initial datum $u(\tau_0) = u_0$ into the solution $u(\tau)$ of (2.18) at the time τ, is in $\mathcal{B}(H^s)$ with $\sup_{\substack{\tau_0, \tau \in [0,1] \\ t \in \mathbb{R}}} \|\Phi(\tau_0, \tau; t)\|_{\mathcal{B}(H^s)} < +\infty$ for any $s \ge 0$. Moreover, the operator $\Phi(\tau_0, \tau; t)$ is invertible with inverse $\Phi(\tau_0, \tau; t)^{-1} = \Phi(\tau, \tau_0; t)$.
(ii) For any $\tau_0, \tau \in [0,1]$, the flow map $t \mapsto \Phi(\tau_0, \tau; t)$ is differentiable and

$$\sup_{\substack{\tau_0, \tau \in [0,1] \\ t \in \mathbb{R}}} \|\partial_t^k \Phi(\tau_0, \tau; t)\|_{\mathcal{B}(H^{s+k}, H^s)} < +\infty, \quad \forall k \in \mathbb{N}, \quad s \ge 0. \tag{2.19}$$

Proof. Proof of (i). The proof of item (i) is classical. We refer for instance to [21], Section 0.8.
Proof of (ii). For any $\tau_0 \in [0,1]$, the flow map $\Phi(\tau_0, \tau; t)$ solves

$$\begin{cases} \partial_\tau \Phi(\tau_0, \tau; t) = \mathcal{A}(\tau; t) \Phi(\tau_0, \tau; t) \\ \Phi(\tau_0, \tau_0; t) = \mathrm{Id}. \end{cases} \tag{2.20}$$

By differentiating (2.20) with respect to t, one gets that $\partial_t \Phi(\tau_0, \tau; t)$ solves

$$\begin{cases} \partial_\tau \Big(\partial_t \Phi(\tau_0, \tau; t)\Big) = \mathcal{A}(\tau; t)\Big(\partial_t \Phi(\tau_0, \tau; t)\Big) + \Big(\partial_t \mathcal{A}(\tau; t)\Big)\Phi(\tau_0, \tau; t) \\ \partial_t \Phi(\tau_0, \tau_0; t) = 0. \end{cases}$$

By Duhamel principle, we then get

$$\partial_t \Phi(\tau_0, \tau; t) = \int_{\tau_0}^{\tau} \Phi(\tau_0, \tau; t) \Phi(\zeta, \tau_0; t) \partial_t \mathcal{A}(\zeta; t) \Phi(\tau_0, \zeta; t)\, d\zeta.$$

By item (i) and by Theorem 2.1 (using that $\partial_t \mathcal{A} \in OPS^1$) one gets that $\partial_t \Phi(\tau_0, \tau; t) \in \mathcal{B}(H^{s+1}, H^s)$ with estimates which are uniform with respect to $\tau_0, \tau \in [0,1]$ and $t \in \mathbb{R}$. Hence (2.19) has been proved for $k = 1$. Iterating the above argument, one can prove the estimate (2.19) for any positive integer k. □

In the next lemma we prove the global well-posedness for a class of Schrödinger type equations. Let $\varphi(t, \xi) \in OPS^m$ be a real Fourier multiplier, i.e.

$$\varphi \in S^m, \quad \varphi(t, \xi) = \overline{\varphi(t, \xi)}, \qquad \forall (t, \xi) \in \mathbb{R} \times \mathbb{R}. \tag{2.21}$$

Moreover, let us consider a time dependent linear operator $t \mapsto \mathcal{R}(t)$ satisfying

$$\mathcal{R} \in \mathcal{C}^0_b(\mathbb{R}, \mathcal{B}(H^s)), \qquad \forall s \ge 0. \tag{2.22}$$

The following lemma holds:

Lemma 2.10. *Let $s \geq 0$, $u_0 \in H^s(\mathbb{T})$, $t_0 \in \mathbb{R}$. Then there exists a unique global solution $u \in \mathcal{C}^0(\mathbb{R}, H^s)$ of the Cauchy problem*

$$\begin{cases} \partial_t u + \mathrm{i}\varphi(t,D)u + \mathcal{R}(t)[u] = 0 \\ u(t_0,x) = u_0(x) \,. \end{cases} \tag{2.23}$$

Proof. The local existence follows by a fixed point argument applied to the map

$$\mathcal{F}(u) := \exp\Big(- \mathrm{i}\Phi(t,D)\Big)[u_0] + \int_{t_0}^{t} \exp\Big(- \mathrm{i}\Big(\Phi(t,D) - \Phi(\tau,D)\Big)\Big)\mathcal{R}(\tau)[u(\tau,\cdot)]\, d\tau$$

where

$$\Phi(t,D) := \mathrm{Op}(\Phi(t,\xi))\,, \qquad \Phi(t,\xi) := \int_{t_0}^{t} \varphi(\zeta,\xi)\, d\zeta \,.$$

Since $\varphi(t,\xi)$ is real, then also $\Phi(t,\xi)$ is real, implying that the propagator $\exp\Big(- \mathrm{i}\Phi(t,D)\Big)$ is unitary on Sobolev spaces. Choosing

$$R := 2\|u_0\|_{H^s}\,, \qquad T := \frac{1}{2\|\mathcal{R}\|_{\mathcal{C}^0_b(\mathbb{R},\mathcal{B}(H^s))}}$$

and defining

$$\mathcal{B}_{R,T}(s) := \Big\{ u \in \mathcal{C}^0_b([t_0 - T, t_0 + T], H^s) : \|u\|_{\mathcal{C}^0_b([t_0-T,t_0+T],H^s)} \leq R \Big\}$$

one can prove that

$$\mathcal{F} : \mathcal{B}_{R,T}(s) \to \mathcal{B}_{R,T}(s)$$

is a contraction. The global well posedness follows from the fact that the solution is bounded on any bounded interval and then it can be extended to the whole real line. □

2.2 Some Egorov-type theorems

In this section we collect some abstract egorov type theorems, namely we study how a pseudo differential operator transforms under the action of the flow of a first order hyperbolic PDE. Let $\alpha : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function with all the derivatives bounded, satisfying

$$\alpha \in \mathcal{C}^\infty_b(\mathbb{R}\times\mathbb{T},\mathbb{R})\,, \qquad \inf_{(t,x)\in\mathbb{R}\times\mathbb{T}} \big(1 + \alpha_x(t,x)\big) > 0 \,. \tag{2.24}$$

We then consider the non-autonomous transport equation

$$\partial_\tau u = \mathcal{A}(\tau;t,x,D)u\,, \qquad \mathcal{A}(\tau;t) = \mathcal{A}(\tau;t,x,D) := b_\alpha(\tau;t,x)\partial_x + \frac{(\partial_x b_\alpha)(\tau;t,x)}{2}\,, \tag{2.25}$$

$$b_\alpha(\tau;t,x) := -\frac{\alpha(t,x)}{1+\tau\alpha_x(t,x)}\,, \qquad \tau \in [0,1]\,. \tag{2.26}$$

Note that the condition (3.1) implies that

$$\inf_{\substack{\tau\in[0,1] \\ (t,x)\in\mathbb{R}\times\mathbb{T}}} \big(1 + \tau\alpha_x(t,x)\big) > 0\,,$$

hence the function $b \in \mathcal{C}^\infty_b([0,1]\times\mathbb{R}\times\mathbb{T})$. Then $\mathcal{A}(\tau;\cdot) \in OPS^1$, $\tau \in [0,1]$ is a smooth family of pseudo-differential operators and it is straightforward to see that $\mathcal{A}(\tau;t) + \mathcal{A}(\tau;t)^* = 0$. Therefore, the hypotheses of Lemma 2.9 are verified, implying that, for any $\tau \in [0,1]$, the flow $\Phi(\tau;t) \equiv \Phi(0,\tau;t)$, $\tau \in [0,1]$ of the equation (3.2), i.e.

$$\begin{cases} \partial_\tau \Phi(\tau;t) = \mathcal{A}(\tau;t)\Phi(\tau;t) \\ \Phi(0;t) = \mathrm{Id} \end{cases} \tag{2.27}$$

is a well defined map and satisfies all the properties stated in the items (i), (ii) of Lemma 2.9. Furthermore, $\mathcal{A}(\tau;t)$ is a Hamiltonian vector field. Indeed

$$\begin{aligned} &\mathcal{A}(\tau;t) = \mathrm{i}\widetilde{\mathcal{A}}(\tau;t)\,, \quad \widetilde{\mathcal{A}}(\tau;t) := -\mathrm{i}\Big(b_\alpha(\tau;t,x)\partial_x + \frac{(\partial_x b_\alpha)(\tau;t,x)}{2}\Big) \\ &\text{and} \quad \widetilde{\mathcal{A}}(\tau;t) = \widetilde{\mathcal{A}}(\tau;t)^* \end{aligned} \tag{2.28}$$

implying that the map $\Phi(\tau;t)$ is symplectic. We then have the following

Lemma 2.11. *The flow $\Phi(\tau;t)$ given by (3.3) is a symplectic, invertible map satisfying*

$$\sup_{\substack{\tau\in[0,1]\\ t\in\mathbb{R}}} \|\partial_t^k \Phi(\tau;t)^{\pm 1}\|_{\mathcal{B}(H^{s+k},H^s)} < +\infty\,, \quad \forall k\in\mathbb{N}, \quad s\geq 0\,.$$

In order to state Theorem 2.14 of this section, we need some preliminary results.

Lemma 2.12. *Let $\alpha \in \mathcal{C}_b^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$ satisfy the condition (3.1). Then for any $t\in\mathbb{R}$, the map*

$$\varphi_t : \mathbb{T}\to\mathbb{T}\,, \quad x\mapsto x+\alpha(t,x)$$

is a diffeomorphism of the torus whose inverse has the form

$$\varphi_t^{-1} : \mathbb{T}\to\mathbb{T}, \quad y\mapsto y+\widetilde{\alpha}(t,y)\,, \tag{2.29}$$

with $\widetilde{\alpha} : \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ satisfying

$$\widetilde{\alpha}\in\mathcal{C}_b^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})\,, \qquad \inf_{(t,y)\in\mathbb{R}\times\mathbb{T}} \big(1+\widetilde{\alpha}_y(t,y)\big) > 0\,. \tag{2.30}$$

Furthermore, the following identities hold:

$$1+\alpha_x(t,x) = \frac{1}{1+\widetilde{\alpha}_y\big(t,x+\alpha(t,x)\big)}\,, \quad 1+\widetilde{\alpha}_y(t,y) = \frac{1}{1+\alpha_x\big(t,y+\widetilde{\alpha}(t,y)\big)} \tag{2.31}$$

Proof. The condition (3.1) and the inverse function theorem imply that for any $t\in\mathbb{R}$, the map $\varphi_t:\mathbb{R}\to\mathbb{R}$ is a $\mathcal{C}^\infty$ diffeomorphism with a $\mathcal{C}^\infty$ inverse given by $\varphi_t^{-1}:\mathbb{R}\to\mathbb{R}$. Since α is 2π-periodic in x one verifies easily that $\varphi_t(x+2\pi)=\varphi_t(x)+2\pi$, implying that $\varphi_t:\mathbb{T}\to\mathbb{T}$ is a diffeomorphism of the torus. We now verify that φ_t^{-1} has the form (2.29). In order to see this, it is enough to show that $\widetilde{\alpha}(t,y):=\varphi_t^{-1}(y)-y$ is 2π-periodic in y. Let $y=\varphi_t(x)$. Applying φ_t^{-1} to both sides of the equality $\varphi_t(x+2\pi)=\varphi_t(x)+2\pi$, one gets that $x+2\pi=\varphi_t^{-1}(y+2\pi)$, i.e. $\varphi_t^{-1}(y)+2\pi=\varphi_t^{-1}(y+2\pi)$. This implies that

$$\widetilde{\alpha}(t,y+2\pi)=\varphi_t^{-1}(y+2\pi)-y-2\pi=\varphi_t^{-1}(y)+2\pi-y-2\pi=\widetilde{\alpha}(t,y)$$

and then $\widetilde{\alpha}$ is 2π-periodic in y. Since

$$y=x+\alpha(t,x) \iff x=y+\widetilde{\alpha}(t,y)$$

one has

$$\widetilde{\alpha}(t,y)+\alpha(t,y+\widetilde{\alpha}(t,y))=0\,, \quad \forall (t,y)\in\mathbb{R}\times\mathbb{T}\,. \tag{2.32}$$

It follows by the standard implicit function theorem that $\widetilde{\alpha}$ is $\mathcal{C}^1$ with derivatives

$$\partial_y\widetilde{\alpha}(t,y) = -\frac{\alpha_x(t,y+\widetilde{\alpha}(t,y))}{1+\alpha_x(t,y+\widetilde{\alpha}(t,y))}\,, \qquad \partial_t\widetilde{\alpha}(t,y) = -\frac{\alpha_t(t,y+\widetilde{\alpha}(t,y))}{1+\alpha_x(t,y+\widetilde{\alpha}(t,y))}\,. \tag{2.33}$$

By induction, it can be proved that $\widetilde{\alpha}$ is $\mathcal{C}^\infty$ with all the derivatives bounded, namely $\widetilde{\alpha}\in\mathcal{C}_b^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$. The identities (2.31) follow easily by (2.33) and then also (2.30) holds. The proof of the lemma is then concluded. □

In the next we study the flow of the ODE

$$\begin{cases} \dot{x}(\tau) = -b_\alpha(\tau; t, x(\tau)) \\ \dot{\xi}(\tau) = \partial_x b_\alpha(\tau; t, x(\tau))\xi(\tau) \end{cases} \tag{2.34}$$

where $b_\alpha(\tau; t, x)$ is defined in (2.26). Given $\tau_0, \tau_1 \in [0, 1]$, we denote by $\gamma^{\tau_0,\tau_1}(t, x, \xi) = (\gamma_1^{\tau_0,\tau_1}(t, x), \gamma_2^{\tau_0,\tau_1}(t, x, \xi))$ the flow of the ODE (2.34) with initial time τ_0 and final time τ_1. We point out that the first equation in (2.34) is independent of ξ, hence the first component of the flow is independent of ξ too. We now prove the following lemma concerning the characteristic equation (2.34).

Lemma 2.13. *For any $\tau_0, \tau \in [0, 1]$, $\gamma_1^{\tau_0,\tau} \in \mathcal{C}_b^\infty(\mathbb{R} \times \mathbb{T}, \mathbb{R})$ and $\gamma_2^{\tau_0,\tau} \in S^1$. Furthermore, for any $\tau_0 \in [0, 1]$ one has that*

$$\gamma_1^{\tau_0,0}(t, x) = x + \tau_0\alpha(t, x)\,, \qquad \gamma_2^{\tau_0,0}(t, x, \xi) = (1 + \tau_0\alpha_x(t, x))^{-1}\xi\,.$$

Proof. Given $\tau_0 \in [0, 1]$, we consider the Cauchy problem

$$\begin{cases} \dot{x}(\tau) = -b_\alpha(\tau; t, x(\tau))\,, \qquad x(\tau_0) = x \\ \dot{\xi}(\tau) = \partial_x b_\alpha(\tau; t, x(\tau))\xi(\tau)\,, \qquad \xi(\tau_0) = \xi\,. \end{cases} \tag{2.35}$$

Let $x(\tau) = \gamma_1^{\tau_0,\tau}(t, x)$, $\xi(\tau) = \gamma_2^{\tau_0,\tau}(t, x, \xi)$ be the unique solution of (2.35). The second equation can be integrated explicitly, leading to

$$\xi(\tau) = \gamma_2^{\tau_0,\tau}(t, x, \xi) = \exp\Big(\int_{\tau_0}^{\tau} \partial_x b_\alpha\big(\zeta; \gamma_1^{\tau_0,\zeta}(t, x)\big)\, d\zeta\Big)\xi\,, \qquad \forall \tau \in [0, 1]\,. \tag{2.36}$$

Note that, since b_α is $\mathcal{C}^\infty$ with respect to all its variables and all its derivatives are bounded, by the smooth dependence of the flow on the initial data (x, ξ) and on the parameter t, one has that $\gamma_1^{\tau_0,\tau}$ is $\mathcal{C}^\infty$ w.r. to (t, x) with all bounded derivatives. Hence by (2.36) one gets that $\gamma_2^{\tau_0,\tau} \in S^1$. By differentiating with respect to the initial datum x the first equation in (2.35) one gets that

$$\partial_\tau\big(\partial_x\gamma_1^{\tau_0,\tau}(x)\big) = -\partial_x b_\alpha\big(\tau; \gamma_1^{\tau_0,\tau}(x)\big)\partial_x\gamma_1^{\tau_0,\tau}(x)\,, \quad \partial_x\gamma_1^{\tau_0,\tau_0}(x) = 1$$

whose solution is given by

$$\partial_x\gamma_1^{\tau_0,\tau}(x) = \exp\Big(-\int_{\tau_0}^{\tau} \partial_x b_\alpha\big(\zeta; \gamma_1^{\tau_0,\zeta}(x)\big)\, d\zeta\Big)\,. \tag{2.37}$$

By formulae (2.36), (2.37), one then obtains that

$$\gamma_2^{\tau_0,\tau}(x, \xi) = \Big(\partial_x\gamma_1^{\tau_0,\tau}(x)\Big)^{-1}\xi\,. \tag{2.38}$$

Note that by the definition of b_α given in (3.2) and by the first equation in (2.35), one has that

$$\frac{d}{d\tau}\Big(x(\tau) + \tau\alpha(x(\tau))\Big) = \alpha(x(\tau)) + \big(1 + \tau\alpha_x(x(\tau))\big)\dot{x}(\tau) = 0$$

implying that

$$x(\tau) + \tau\alpha(x(\tau)) = x + \tau_0\alpha(x)\,, \quad \forall \tau, \tau_0 \in [0, 1]\,.$$

In particular, for $\tau = 0$, one gets that

$$\gamma_1^{\tau_0,0}(x) = x(0) = x + \tau_0\alpha(x)$$

and therefore by (2.38) we obtain

$$\gamma_2^{\tau_0,0}(x, \xi) = (1 + \tau_0\alpha_x(x))^{-1}\xi\,,$$

which proves the claimed statement. $\square$

Now, we are ready to state the Egorov Theorem.

Theorem 2.14. *Let $m \in \mathbb{R}$, $\mathcal{V}(t) = \mathrm{Op}\big(v(t,x,\xi)\big)$ be in the class S^m and $\Phi(\tau;t)$, $\tau \in [0,1]$ be the flow map of the PDE* (3.3)*. Then $\mathcal{P}(\tau;t) := \Phi(\tau;t)\mathcal{V}(t)\Phi(\tau;t)^{-1}$ is a pseudo differential operator in the class OPS^m, i.e. $\mathcal{P}(\tau;t) = \Phi(\tau;t)\mathcal{V}(t)\Phi(\tau;t)^{-1} = \mathrm{Op}\big(p(\tau;t,x,\xi)\big)$ with $p(\tau,\cdot,\cdot,\cdot) \in S^m$, $\tau \in [0,1]$. Furthermore $p(\tau;t,x,\xi)$ admits the expansion*

$$p(\tau;t,x,\xi) = p_0(\tau;t,x,\xi) + p_{\geq 1}(\tau;t,x,\xi)\,, \qquad p_0(\tau,\cdot,\cdot,\cdot) \in S^m\,, \quad p_{\geq 1}(\tau;\cdot,\cdot,\cdot) \in S^{m-1}$$

and the principal symbol p_0 has the form

$$\begin{aligned} &p_0(\tau;t,x,\xi) := v\Big(t, x + \tau\alpha(t,x), (1+\tau\alpha_x(t,x))^{-1}\xi\Big)\,, \\ &\forall (t,x,\xi) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}\,, \quad \forall \tau \in [0,1]\,. \end{aligned}$$

Proof. We closely follow Theorem A.0.9 in [21]. A direct calculation shows that $\mathcal{P}(\tau;t)$ solves the Heisenberg equation

$$\begin{cases} \partial_\tau \mathcal{P}(\tau;t) = [\mathcal{A}(\tau;t), \mathcal{P}(\tau;t)] \\ \mathcal{P}(0;t) = \mathcal{V}(t)\,. \end{cases} \tag{2.39}$$

We then look for a solution $\mathcal{P}(\tau;t) = \mathrm{Op}\Big(p(\tau;t,x,\xi)\Big) \in OPS^m$ with

$$p(\tau;t,x,\xi) \sim \sum_{n\geq 0} p_n(\tau;t,x,\xi)\,, \qquad p_n \in S^{m-n}\,, \quad \forall n \geq 0\,.$$

We show how to compute the asymptotic expansion of the symbol p. The operator $\mathcal{A}(\tau;t)$ in (3.2) has symbol

$$\begin{aligned} &a = \mathrm{i}a_1 + a_0\,, \\ &a_1(\tau;t,x,\xi) := b_\alpha(\tau;t,x)\xi \in S^1\,, \quad a_0(\tau;t,x,\xi) := \frac{(\partial_x b_\alpha)(\tau;t,x)}{2} \in S^0\,. \end{aligned} \tag{2.40}$$

The symbol of the commutator $[\mathcal{A}(\tau;t), \mathcal{P}(\tau;t)] = \mathrm{Op}(a \star p)$ has the asymptotic expansion

$$g \star p \sim \sum_{n\geq 0} a \star p_n\,. \tag{2.41}$$

Note that if $p_n \in S^{m-n}$, by (2.40) and Corollary 2.4, one has that

$$\begin{aligned} &\mathrm{i}a_1 \star p_n = \{a_1, p_n\} + \mathrm{i}\mathtt{r}_2(a_1,p_n), \quad \{a_1,p_n\} \in S^{m-n}\,, \quad \mathtt{r}_2(a_1,p_n) \in S^{m-n-1} \\ &a_0 \star p_n \in S^{m-n-1}\,. \end{aligned} \tag{2.42}$$

This implies that

$$a \star p_n = \{a_1, p_n\} + q_n\,, \quad q_n := \mathrm{i}\mathtt{r}_2(a_1,p_n) + a_0 \star p_n \in S^{m-n-1}\,.$$

We then solve iteratively

$$\begin{cases} \partial_\tau p_0 = \{a_1, p_0\} \\ p_0(0;t,x,\xi) = v(t,x,\xi) \end{cases} \tag{2.43}$$

and

$$\begin{cases} \partial_\tau p_n(\tau;t,x,\xi) = \{a_1, p_n\} + q_{n-1} \\ p_n(0;t,x,\xi) = 0\,, \end{cases} \qquad \forall n \geq 1\,. \tag{2.44}$$

Using the characteristic method, the solutions of (2.43), (2.44) are given by

$$p_0(\tau;t,x,\xi) := v(t, \gamma_1^{\tau,0}(t,x), \gamma_2^{\tau,0}(t,x,\xi))\,, \quad \forall \tau \in [0,1] \tag{2.45}$$

and

$$p_n(\tau;t,x,\xi) = \int_0^\tau q_{n-1}(\zeta;t,\gamma_1^{\tau,\zeta}(t,x), \gamma_2^{\tau,\zeta}(t,x,\xi))\,d\zeta\,, \qquad \forall n \geq 1\,, \tag{2.46}$$

where for any $\tau,\zeta \in [0,1]$, $(\gamma_1^{\tau,\zeta}, \gamma_2^{\tau,\zeta})$ is the flow of the ODE (2.34). The claimed statement then follows by applying Lemma 2.13 and by setting $p_{\geq 1} \sim \sum_{n\geq 1} p_n \in OPS^{m-1}$. ☐

In the following we will also need to analyse the operator $\Phi(\tau;t)\partial_t(\Phi(\tau;t)^{-1})$. The following lemma holds:

Theorem 2.15. *The operator* $\Psi(\tau;t) = \Phi(\tau;t)\partial_t(\Phi(\tau;t)^{-1})$, $\tau \in [0,1]$ *is a pseudo differential operator in the class* OPS^1.

Proof. First we compute $\partial_\tau \Psi(\tau;t)$. One has

$$\begin{aligned}
\partial_\tau \Psi(\tau;t) &= \partial_\tau \Phi(\tau;t)\partial_t\big(\Phi(\tau;t)^{-1}\big) + \Phi(\tau;t)\partial_t\partial_\tau\big(\Phi(\tau;t)^{-1}\big) \\
&= \partial_\tau \Phi(\tau;t)\partial_t\big(\Phi(\tau;t)^{-1}\big) - \Phi(\tau;t)\partial_t\Big(\Phi(\tau;t)^{-1}\partial_\tau\Phi(\tau;t)\Phi(\tau;t)^{-1}\Big) \\
&\overset{(3.3)}{=} \mathcal{A}(\tau;t)\Phi(\tau;t)\partial_t\Phi(\tau;t)^{-1} - \Phi(\tau;t)\partial_t\Big(\Phi(\tau;t)^{-1}\mathcal{A}(\tau;t)\Big) \\
&= [\mathcal{A}(\tau;t), \Psi(\tau;t)] - \partial_t\mathcal{A}(\tau;t)\,.
\end{aligned} \tag{2.47}$$

Therefore $\Psi(\tau;t)$ solves

$$\begin{cases} \partial_\tau \Psi(\tau;t) = [\mathcal{A}(\tau;t), \Psi(\tau;t)] - \partial_t\mathcal{A}(\tau;t) \\ \Psi(0;t) = 0\,. \end{cases} \tag{2.48}$$

Arguing as in Theorem 2.14, we find that $\Psi(\tau;t) = \mathrm{Op}\Big(\psi(\tau;t,x,\xi)\Big) \in OPS^1$, by solving (2.48) in decreasing orders and by determing an asymptotic expansion of the symbol ψ of the form

$$\psi \sim \sum_{n\geq 0} \psi_n\,, \quad \psi_n \in S^{1-n}\,, \quad \forall n \geq 0\,.$$

□

We also state another *semplified* version of the Egorov theorem in which we conjugate a symbol by means of the flow of a vector field which is a pseudo differential operator of order strictly smaller than one. We consider a pseudo differential operator $\mathcal{G}(t) = \mathrm{Op}(g(t,x,\xi))$, with $g \in S^\eta$, $\mathcal{G}(t) = \mathcal{G}(t)^*$, $\eta < 1$ and for any $\tau \in [0,1]$, let $\Phi_{\mathcal{G}}(\tau;t)$ be the flow of the pseudo-PDE

$$\partial_\tau u = \mathrm{i}\mathcal{G}(t)u\,, \tag{2.49}$$

which is a well-defined invertible map by Lemma 2.9. Then $\Phi_{\mathcal{G}}(\tau;t)$ solves

$$\begin{cases} \partial_\tau \Phi_{\mathcal{G}}(\tau;t) = \mathrm{i}\mathcal{G}(t)\Phi_{\mathcal{G}}(\tau;t) \\ \Phi_{\mathcal{G}}(0;t) = \mathrm{Id}\,. \end{cases} \tag{2.50}$$

The following theorem holds.

Theorem 2.16. *Let* $m \in \mathbb{R}$, $\mathcal{V}(t) = \mathrm{Op}\big(v(t,x,\xi)\big) \in OPS^m$ *and* $\mathcal{G}(t) = \mathrm{Op}(g(t,x,\xi))$, *with* $g \in S^\eta$, $\eta < 1$. *Then for any* $\tau \in [0,1]$, *the operator* $\mathcal{P}(\tau;t) := \Phi_{\mathcal{G}}(\tau;t)\mathcal{V}(t)\Phi_{\mathcal{G}}(\tau;t)^{-1}$ *is a pseudo differential operator of order* m *with symbol* $p(\tau;\cdot,\cdot,\cdot) \in S^m$. *The symbol* $p(\tau;t,x,\xi)$ *admits the expansion*

$$p(\tau;t,x,\xi) = v(t,x,\xi) + \tau\{g,v\}(t,x,\xi) + p_{\geq 2}(\tau;t,x,\xi)\,, \quad p_{\geq 2}(\tau;t,x,\xi) \in S^{m-2(1-\eta)}\,. \tag{2.51}$$

Proof. We show how to compute the asymptotic expansion of the operator $\mathcal{P}(\tau;t)$ by taking advantage from the fact that the order of $\mathcal{G}(t)$ is strictly smaller than 1. A direct calculation shows that $\mathcal{P}(\tau;t)$ solves the Heisenberg equation

$$\begin{cases} \partial_\tau \mathcal{P}(\tau;t) = \mathrm{i}[\mathcal{G}(t), \mathcal{P}(\tau;t)] \\ \mathcal{P}(0;t) = \mathcal{V}(t)\,. \end{cases} \tag{2.52}$$

We then look for $\mathcal{P}(\tau;t) = \mathrm{Op}\Big(p(\tau;t,x,\xi)\Big) \in OPS^m$ with

$$p(\tau;t,x,\xi) \sim \sum_{n\geq 0} p_n(\tau;t,x,\xi)\,, \qquad p_n(\tau;t,x,\xi) \in S^{m-n(1-\eta)}\,, \quad \forall n \geq 0\,.$$

The symbol of the commutator $[\mathcal{G}(t), \mathcal{P}(\tau; t)] = \mathrm{Op}(g \star p)$ has the asymptotic expansion

$$g \star p \sim \sum_{n \geq 0} g \star p_n \tag{2.53}$$

Note that if $p_n \in S^{m-n(1-\eta)}$, by Corollary 2.4, one has that $g \star p_n \in S^{m-(n+1)(1-\eta)}$. We then solve iteratively

$$\begin{cases} \partial_\tau p_0(\tau; t, x, \xi) = 0 \\ p_0(0; t, x, \xi) = v(t, x, \xi) \end{cases} \tag{2.54}$$

and

$$\begin{cases} \partial_\tau p_n(\tau; t, x, \xi) = \mathrm{i} g \star p_{n-1} \\ p_n(0; t, x, \xi) = 0 \,, \end{cases} \qquad \forall n \geq 1 \,. \tag{2.55}$$

The solutions of (2.54), (2.55) are then given by

$$p_0(\tau; t, x, \xi) := v(t, x, \xi) \,, \quad \forall \tau \in [0, 1] \tag{2.56}$$

and

$$p_n(\tau; t, x, \xi) = \mathrm{i} \int_0^\tau g \star p_{n-1}(\zeta; t, x, \xi) \, d\zeta \,, \qquad \forall n \geq 1 \,. \tag{2.57}$$

In order to determine the expansion (2.51), we analyze the symbol p_1. By (2.56), (2.57), one gets

$$p_1(\tau; t, x, \xi) = \mathrm{i} \tau g \star v(t, x, \xi) \overset{Corollary\ 2.4}{=} \tau \{g, v\} + \mathrm{i} \tau \mathtt{r}_2(g, v)$$

and

$$\mathtt{r}_2(g, v) \in S^{m+\eta-2} \overset{(2.1)}{\subseteq} S^{m-2(1-\eta)}$$

therefore the expansion (2.51) is determined by taking

$$p_{\geq 2}(\tau; t, x, \xi) \sim \mathrm{i} \tau \mathtt{r}_2(g, v) + \sum_{n \geq 2} p_n(\tau; t, x, \xi) \,.$$

□

3 Regularization of the vector field $\mathcal{V}(t)$

In this section we develop the regularization procedure on the vector field $\mathrm{i}\mathcal{V}(t) = \mathrm{i}\big(V(t,x)|D|^M + \mathcal{W}(t)\big)$, see (1.2), which is needed to prove Theorem 1.4. In Section 3.1 we reduce to constant coefficients the highest order $V(t,x)|D|^M$, see Proposition 3.1. Then, in Section 3.2, we perform the reduction of the lower order terms up to arbitrarily regularizing remainders, see Proposition 3.3.

3.1 Reduction of the highest order

Our first aim is to eliminate the x-dependence from the highest order of the vector field $\mathrm{i}\mathcal{V}(t)$, namely we want to eliminate the x-dependence from the term $V(t,x)|D|^M$. To this aim, let us consider a $\mathcal{C}^\infty$ function $\alpha : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ (that will be fixed later) satisfying the following ansatz:

$$\alpha \in \mathcal{C}^\infty_b(\mathbb{R} \times \mathbb{T}, \mathbb{R}) \,, \qquad \inf_{(t,x) \in \mathbb{R} \times \mathbb{T}} \big(1 + \alpha_x(t,x)\big) > 0 \,. \tag{3.1}$$

Then, we consider the non-autonomous transport equation

$$\begin{aligned} &\partial_\tau u = \mathcal{A}(\tau; t, x, D)[u] \,, \\ &\mathcal{A}(\tau; t) = \mathcal{A}(\tau; t, x, D) := b_\alpha(\tau; t, x) \partial_x + \frac{(\partial_x b_\alpha)(\tau; t, x)}{2} \,, \\ &b_\alpha(\tau; t, x) := -\frac{\alpha(t,x)}{1 + \tau \alpha_x(t,x)} \,, \qquad \tau \in [0, 1] \,. \end{aligned} \tag{3.2}$$

By Lemma 2.11, the flow $\Phi(\tau;t)$, $\tau\in[0,1]$ of the equation (3.2), i.e.

$$\begin{cases}\partial_\tau\Phi(\tau;t)=\mathcal{A}(\tau;t)\Phi(\tau;t)\\ \Phi(0;t)=\mathrm{Id}\end{cases}\tag{3.3}$$

is a well-defined, symplectic, invertible map $H^s\to H^s$ for any $s\in\mathbb{R}$. We define $\Phi(t):=\Phi(1;t)$. In order to state the Proposition below, we introduce the constant

$$\bar{\mathfrak{e}}:=M-\max\{M-1,1,M-\mathfrak{e}\}\,.\tag{3.4}$$

Note that, by the above definition and using that $M>1$, it follows easily that

$$\bar{\mathfrak{e}}>0\,,\quad M-\bar{\mathfrak{e}}\geq M-1\,,\,1\,,\,M-\mathfrak{e}\,.\tag{3.5}$$

Proposition 3.1. *The symplectic invertible map $\Phi(t)=\Phi(1;t)$, given by* (3.3)*, satisfies*

$$\sup_{t\in\mathbb{R}}\|\Phi(t)^{\pm1}\|_{\mathcal{B}(H^s)}+\sup_{t\in\mathbb{R}}\|\partial_t\Phi(t)^{\pm1}\|_{\mathcal{B}(H^{s+1},H^s)}<+\infty\,,\quad\forall s\geq 0\,.\tag{3.6}$$

There exist a function $\lambda\in\mathcal{C}^\infty_b(\mathbb{R},\mathbb{R})$ satisfying

$$\inf_{t\in\mathbb{R}}\lambda(t)>0\tag{3.7}$$

and an operator

$$\mathcal{W}_1(t)=\mathrm{Op}\Big(w_1(t,x,\xi)\Big),\quad w_1\in S^{M-\bar{\mathfrak{e}}}$$

with $\mathcal{W}_1(t)=\mathcal{W}_1(t)^*$ for any $t\in\mathbb{R}$, such that

$$(\Phi^{-1})_*(\mathrm{i}\mathcal{V})(t)=\mathrm{i}\mathcal{V}_1(t)\quad\textit{with}\quad\mathcal{V}_1(t):=\lambda(t)|D|^M+\mathcal{W}_1(t)\,.\tag{3.8}$$

All the rest of this section is devoted to the proof of the proposition stated above. The property (3.6) follows by applying Lemma 2.9, using that $\mathcal{A}(\tau;t)\in OPS^1$ and using that, by a direct calculation, $\mathcal{A}(\tau;t)+\mathcal{A}(\tau;t)^*=0$. The push-forward of the vector field $\mathrm{i}\mathcal{V}(t)$ by means of the map $\Phi(t)^{-1}$ is then given by $\mathrm{i}\mathcal{V}_1(t)$ with

$$\mathcal{V}_1(t)=\Phi(t)\mathcal{V}(t)\Phi(t)^{-1}+\mathrm{i}\Phi(t)\partial_t\big(\Phi(t)^{-1}\big)\,.\tag{3.9}$$

By applying Lemmata 2.14, 2.15, one has that $\mathcal{V}_1(t)=\mathrm{Op}(v_1(t,x,\xi))\in OPS^M$ with

$$v_1(t,x,\xi)=p_0(t,x,\xi)+p_{\geq1}(t,x,\xi)\,,\tag{3.10}$$

with

$$p_0(t,x,\xi):=v\Big(t,x+\alpha(t,x),(1+\alpha_x(t,x))^{-1}\xi\Big)\,,\quad p_{\geq1}\in S^{\max\{M-1,1\}}\overset{(2.1),(3.5)}{\subseteq}S^{M-\bar{\mathfrak{e}}}\,.\tag{3.11}$$

In the next lemma we compute the expansion of the symbol $v_1(t,x,\xi)$ of the operator $\mathcal{V}_1(t)$ defined in (3.9).

Lemma 3.2. *The symbol $v_1(t,x,\xi)$ has the form*

$$v_1(t,x,\xi)=\Big[V(t,y)\big(1+\widetilde{\alpha}_y(t,y)\big)^M\Big]_{y=x+\alpha(t,x)}|\xi|^M\chi(\xi)+w_1(t,x,\xi)\,,\quad w_1\in S^{M-\bar{\mathfrak{e}}}\tag{3.12}$$

where we recall the definitions (2.16)*,* (2.17) *and $y\mapsto y+\widetilde{\alpha}(t,y)$ is the inverse diffeomorphism of $x\mapsto x+\alpha(t,x)$.*

Proof. Using (3.9)-(3.11), one obtains

$$\begin{aligned} v_1(t,x,\xi) &= p_0(t,x,\xi) + p_{\geq 1}(t,x,\xi) \\ &= v\Big(t, x+\alpha(t,x), (1+\alpha_x(t,x))^{-1}\xi\Big) + p_{\geq 1}(t,x,\xi) \end{aligned} \tag{3.13}$$

By (1.2), the symbol of the operator $\mathcal{V}(t)$ has the form

$$v(t,x,\xi) = V(t,x)|\xi|^M \chi(\xi) + w(t,x,\xi)\,, \quad w \in S^{M-\mathfrak{e}}\,,$$

hence, by (3.11), one has

$$\begin{aligned} p_0(t,x,\xi) &= v\Big(t, x+\alpha(t,x), (1+\alpha_x(t,x))^{-1}\xi\Big) \\ &= V(t, x+\alpha(t,x))(1+\alpha_x(t,x))^{-M}|\xi|^M \chi\Big((1+\alpha_x(t,x))^{-1}\xi\Big) \\ &\quad + w_{p_0}(t,x,\xi) \end{aligned} \tag{3.14}$$

where

$$w_{p_0}(t,x,\xi) := w\Big(t, x+\alpha(t,x), (1+\alpha_x(t,x))^{-1}\xi\Big) \in S^{M-\mathfrak{e}} \overset{(3.5)}{\subseteq} S^{M-\bar{\mathfrak{e}}}. \tag{3.15}$$

By using the mean value theorem, one writes

$$\chi\Big((1+\alpha_x(t,x))^{-1}\xi\Big) = \chi(\xi) + w_\chi(t,x,\xi)\,,$$

$$w_\chi(t,x,\xi) := -\frac{\alpha_x(t,x)\xi}{1+\alpha_x(t,x)} \int_0^1 \partial_\xi \chi\Big(\zeta(1+\alpha_x(t,x))^{-1}\xi + (1-\zeta)\xi\Big)\, d\zeta\,.$$

Since $\partial_\xi \chi(\xi) = 0$ for any $|\xi| \geq 1$ (see (2.16)) one has that the symbol $w_\chi \in OPS^{-\infty}$, hence by (3.13), (3.14) one obtains that

$$v_1(t,x,\xi) = V(t, x+\alpha(t,x))(1+\alpha_x(t,x))^{-M}|\xi|^M \chi(\xi) + w_1(t,x,\xi)$$

where

$$\begin{aligned} w_1(t,x,\xi) &:= p_{\geq 1}(t,x,\xi) + w_{p_0}(t,x,\xi) \\ &\quad + V(t, x+\alpha(t,x))(1+\alpha_x(t,x))^{-M}|\xi|^M w_\chi(t,x,\xi)\,. \end{aligned}$$

Recalling (3.11), (3.15) and that $w_\chi \in S^{-\infty}$, one obtains that $w_1 \in S^{M-\bar{\mathfrak{e}}}$. Since α satisfies (3.1), we can apply lemma 2.12, obtaining that the diffeomorphism of the torus $x \mapsto x + \alpha(t,x)$ is invertible with inverse $y \mapsto y + \widetilde{\alpha}(t,y)$ and $\widetilde{\alpha} \in \mathcal{C}^\infty_b(\mathbb{R}\times\mathbb{T},\mathbb{R})$ satisfies (2.30). Using the identity (2.31), we then have

$$V(t, x+\alpha(x))(1+\alpha_x(x))^{-M} = \Big[V(t,y)\big(1+\widetilde{\alpha}_y(y)\big)^M\Big]_{y=x+\alpha(t,x)}$$

and the lemma is proved. □

We now determine the function $\widetilde{\alpha}(t,y)$ so that

$$V(t,y)\big(1+\widetilde{\alpha}_y(t,y)\big)^M = \lambda(t)\,, \tag{3.16}$$

for some bounded and real-valued $\mathcal{C}^\infty$ function λ, to be determined. The equation (3.16) is equivalent to the equation

$$\widetilde{\alpha}_y(t,y) = \frac{\lambda(t)^{\frac{1}{M}}}{V(t,y)^{\frac{1}{M}}} - 1\,. \tag{3.17}$$

Notice that, by the assumption **(H2)**, $V(t,y)$ does never vanish. We choose $\lambda(t)$ so that the average of the right hand side of the equation (3.17) is 0, hence we set

$$\lambda(t) := \Big(\frac{1}{2\pi}\int_{\mathbb{T}} V(t,y)^{-\frac{1}{M}}\,dy\Big)^{-M}. \tag{3.18}$$

Therefore, we solve (3.17) by defining

$$\widetilde{\alpha}(t,y) := \partial_y^{-1}\Big[\frac{\lambda_1(t)^{\frac{1}{M}}}{V(t,y)^{\frac{1}{M}}} - 1\Big] \tag{3.19}$$

(recall the definition (2.11)). Note that by the hypothesis **(H2)** on V and by the definitions (3.18), (3.19), one has $\lambda \in \mathcal{C}_b^\infty(\mathbb{R},\mathbb{R})$, $\widetilde{\alpha} \in \mathcal{C}_b^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$ and

$$\inf_{t\in\mathbb{R}} \lambda(t) > 0\,, \qquad \inf_{(t,y)\in\mathbb{R}\times\mathbb{T}} \Big(1 + \widetilde{\alpha}_y(t,y)\Big) > 0$$

which verifies (3.7). Then by applying lemma 2.12 one gets that the function α satisfies the ansatz (3.1) since $x \mapsto x + \alpha(t,x)$ is the inverse diffeomorphism of $y \mapsto y + \widetilde{\alpha}(t,y)$.
Finally, by lemma 3.2 and since $\widetilde{\alpha}$ and λ solve the equation (3.16), we obtain that $\mathcal{V}_1(t)$ is given by

$$\mathcal{V}_1(t) = \lambda(t)|D|^M + \mathcal{W}_1(t)\,, \quad \mathcal{W}_1(t) = \mathrm{Op}\big(w_1(t,x,\xi)\big) \in OPS^{M-\bar{\mathfrak{e}}}\,. \tag{3.20}$$

Since $\Phi(t)$ is symplectic, the vector field $\mathrm{i}\mathcal{V}_1(t)$ is Hamiltonian, i.e. $\mathcal{V}_1(t)$ is L^2 self-adjoint. Since $\lambda(t)|D|^M$ is selfadjoint, then $\mathcal{W}_1(t) = \mathcal{V}_1(t) - \lambda(t)|D|^M$ is self-adjoint too, hence the proof of Proposition 3.1 is concluded.

3.2 Reduction of the lower order terms

In this Section we transform the vector field $\mathrm{i}\mathcal{V}_1(t)$, obtained in Proposition 3.1, into another one which is an arbitrarily regularizing perturbation of a *space-diagonal* operator. This is done in the following

Proposition 3.3. *Let $N \in \mathbb{N}$. For any $n = 1,\dots,N$ there exists a linear Hamiltonian vector field $\mathrm{i}\mathcal{V}_n(t)$ of the form*

$$\mathcal{V}_n(t) := \lambda(t)|D|^M + \mu_n(t,D) + \mathcal{W}_n(t)\,, \tag{3.21}$$

where

$$\mu_n(t,D) := \mathrm{Op}\Big(\mu_n(t,\xi)\Big)\,, \qquad \mu_n \in S^{M-\bar{\mathfrak{e}}}\,, \tag{3.22}$$

$$\mathcal{W}_n(t) := \mathrm{Op}\Big(w_n(t,x,\xi)\Big)\,, \qquad w_n \in S^{M-n\bar{\mathfrak{e}}}\,, \tag{3.23}$$

with $\mu_n(t,\xi)$ real and $\mathcal{W}_n(t)$ L^2 self-adjoint, i.e. $w_n = w_n^*$ (see Theorem 2.5).
For any $n \in \{1,\dots,N-1\}$, there exists a symplectic invertible map $\Phi_n(t)$ satisfying

$$\sup_{t\in\mathbb{R}} \|\Phi_n(t)^{\pm 1}\|_{\mathcal{B}(H^s)} + \sup_{t\in\mathbb{R}} \|\partial_t\Phi_n(t)^{\pm 1}\|_{\mathcal{B}(H^{s+1},H^s)} < +\infty\,, \quad \forall s \geq 0 \tag{3.24}$$

and

$$\mathrm{i}\mathcal{V}_{n+1}(t) = (\Phi_n^{-1})_*(\mathrm{i}\mathcal{V}_n)(t)\,, \qquad \forall n \in \{1,\dots,N-1\}\,. \tag{3.25}$$

The rest of the section is devoted to the proof of the above Proposition. It is proved arguing by induction. Let us describe the induction step. At the n-th step , we deal with a Hamiltonian vector field of the form $\mathrm{i}\mathcal{V}_n(t)$ which satisfies the properties (3.21)-(3.23). We look for an operator $\mathcal{G}_n(t)$ of the form

$$\mathcal{G}_n(t) := \mathrm{Op}\big(g_n(t,x,\xi)\big) \in OPS^{1-n\bar{\mathfrak{e}}} \quad \text{with} \quad \mathcal{G}_n(t) = \mathcal{G}_n(t)^* \tag{3.26}$$

and we consider the flow $\Phi_{\mathcal{G}_n}(\tau;t)$ of the pseudo PDE

$$\partial_\tau u = \mathrm{i}\mathcal{G}_n(t)[u]\,. \tag{3.27}$$

The flow map $\Phi_{\mathcal{G}_n}(\tau;t)$ solves

$$\begin{cases} \partial_\tau \Phi_{\mathcal{G}_n}(\tau;t) = \mathrm{i}\mathcal{G}_n(t)\Phi_{\mathcal{G}_n}(\tau;t) \\ \Phi_{\mathcal{G}_n}(0;t) = \mathrm{Id}\,. \end{cases} \tag{3.28}$$

Note that, since $\mathcal{G}_n(t)$ is self-adjoint, $\mathrm{i}\mathcal{G}_n(t)$ is a Hamiltonian vector field, implying that $\Phi_{\mathcal{G}_n}(\tau;t)$ is symplectic for any $\tau \in [0,1]$, $t \in \mathbb{R}$. Since $\mathcal{G}_n(t) \in OPS^{1-n\bar{\mathfrak{e}}} \overset{(2.1)}{\subseteq} OPS^1$ and $(\mathrm{i}\mathcal{G}_n(t)) + (\mathrm{i}\mathcal{G}_n(t))^* = \mathrm{i}\big(\mathcal{G}_n(t) - \mathcal{G}_n(t)^*\big) = 0$, by Lemma 2.9, the maps $\Phi_{\mathcal{G}_n}(\tau;t)^{\pm 1}$ satisfy the property (3.24). Note that, since the vector field $\mathcal{G}_n(t)$ does not depend on τ, one has $\Phi_{\mathcal{G}_n}(\tau;t)^{-1} = \Phi_{\mathcal{G}_n}(-\tau;t)$. We set $\Phi_n(t) := \Phi_{\mathcal{G}_n}(1;t)$. The transformed vector field is given by $(\Phi_n^{-1})_*(\mathrm{i}\mathcal{V}_n)(t) = \mathrm{i}\mathcal{V}_{n+1}(t)$, where

$$\mathcal{V}_{n+1}(t) := \Phi_n(t)\mathcal{V}_n(t)\Phi_n(t)^{-1} + \mathrm{i}\Phi_n(t)\partial_t\big(\Phi_n(t)^{-1}\big)\,. \tag{3.29}$$

Since $\mathcal{G}_n(t)$ is a pseudo-differential operator of order strictly smaller than 1, we can apply Theorem 2.16, obtaining that $\mathcal{P}_n(t) = \mathrm{Op}\big(p_n(t,x,\xi)\big) := \Phi_n(t)\mathcal{V}_n(t)\Phi_n(t)^{-1} \in OPS^M$ with

$$p_n = v_n + \{g_n, v_n\} + p_{n,\geq 2}\,, \quad p_{n,\geq 2} \in S^{M-2n\bar{\mathfrak{e}}} \overset{(2.1)}{\subseteq} S^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.30}$$

Furthermore, defining $\Psi_n(\tau;t) := \mathrm{i}\Phi_{\mathcal{G}_n}(\tau;t)\partial_t\big(\Phi_{\mathcal{G}_n}(\tau;t)^{-1}\big)$, a direct calculation shows that

$$\Psi_n(\tau;t) = \mathrm{i}\int_0^\tau \mathcal{S}_{\mathcal{G}_n}(\zeta;t)\,d\zeta\,, \quad \mathcal{S}_{\mathcal{G}_n}(\zeta;t) := \Phi_{\mathcal{G}_n}(\zeta;t)\partial_t\mathcal{G}_n(t)\Phi_{\mathcal{G}_n}(\zeta;t)^{-1}.$$

Since $\partial_t\mathcal{G}_n(t) \in OPS^{1-n\bar{\mathfrak{e}}}$, by Theorem 2.16

$$\Psi_n(t) \equiv \Psi_n(1;t) = \mathrm{i}\Phi_n(t)\partial_t\big(\Phi_n(t)^{-1}\big) = \mathrm{Op}\Big(\psi_n(t,x,\xi)\Big) \in OPS^{1-n\bar{\mathfrak{e}}}\,.$$

Using that $M - \bar{\mathfrak{e}} \geq 1$ (see (3.4)), one gets that

$$\Psi_n(t) = \mathrm{Op}\big(\psi_n(t,x,\xi)\big) \in OPS^{1-n\bar{\mathfrak{e}}} \overset{(2.1)}{\subseteq} OPS^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.31}$$

In the next lemma, we provide an expansion of the symbol $v_{n+1}(t,x,\xi)$ of the operator $\mathcal{V}_{n+1}(t)$ given in (3.29).

Lemma 3.4. *The operator $\mathcal{V}_{n+1}(t) = \mathrm{Op}\big(v_{n+1}(t,x,\xi)\big) \in OPS^M$ admits the expansion*

$$\begin{aligned} v_{n+1}(t,x,\xi) = {} & \lambda(t)|\xi|^M\chi(\xi) + \mu_n(t,\xi) + w_n(t,x,\xi) \\ & - M\lambda(t)|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n)(t,x,\xi) + r_{v_n}(t,x,\xi) \end{aligned} \tag{3.32}$$

where $r_{v_n} \in S^{M-(n+1)\bar{\mathfrak{e}}}$.

Proof. By (3.29)-(3.31), one has

$$v_{n+1} = v_n + \{g_n, v_n\} + p_{n,\geq 2} + \psi_n\,. \tag{3.33}$$

Since, by the induction hypothesis,

$$v_n(t,x,\xi) = \lambda(t)|\xi|^M\chi(\xi) + \mu_n(t,\xi) + w_n(t,x,\xi)\,, \quad \mu_n \in S^{M-\bar{\mathfrak{e}}}\,, \quad w_n \in S^{M-n\bar{\mathfrak{e}}}$$

one has that

$$\begin{aligned} \{g_n, v_n\} &= \{g_n, \lambda(t)|\xi|^M\chi(\xi)\} + \{g_n, \mu_n\} + \{g_n, w_n\} \\ &= -\lambda(t)\partial_\xi\Big(|\xi|^M\chi(\xi)\Big)(\partial_x g_n) + \{g_n, \mu_n\} + \{g_n, w_n\} \\ &= -M\lambda(t)|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n) - \lambda(t)|\xi|^M(\partial_\xi\chi(\xi))(\partial_x g_n) \\ &\quad + \{g_n, \mu_n\} + \{g_n, w_n\}\,. \end{aligned} \tag{3.34}$$

Using that $\partial_\xi \chi(\xi) = 0$ for $|\xi| \geq 1$, since $g_n \in S^{1-n\bar{\mathfrak{e}}}$, $\mu_n \in S^{M-\bar{\mathfrak{e}}}$, $w_n \in S^{M-n\bar{\mathfrak{e}}}$, by Corollary 2.4 one gets

$$\lambda(t)|\xi|^M (\partial_\xi \chi(\xi))(\partial_x g_n) \in S^{-\infty}\,, \quad \{g_n, \mu_n\} \in S^{M-(n+1)\bar{\mathfrak{e}}}\,, \tag{3.35}$$

$$\{g_n, w_n\} \in S^{M-2n\bar{\mathfrak{e}}} \overset{(2.1)}{\subseteq} S^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.36}$$

Thus, (3.33), (3.34) imply the claimed expansion with

$$r_{v_n} := -\lambda(t)|\xi|^M (\partial_\xi \chi(\xi))(\partial_x g_n) + \{g_n, \mu_n\} + \{g_n, w_n\} + p_{n,\geq 2} + \psi_n\,.$$

Finally, (3.30), (3.31), (3.35), (3.36) imply that $r_{v_n} \in S^{M-(n+1)\bar{\mathfrak{e}}}$. $\square$

Choice of the symbol g_n. In the next lemma, we show that the symbol g_n can be chosen in order to eliminate the x-dependence from the term of order $M - n\bar{\mathfrak{e}}$ in the expansion (3.32).

Lemma 3.5. *There exists a symbol $g_n \in S^{1-n\bar{\mathfrak{e}}}$, $g_n = g_n^*$, such that*

$$-\lambda(t)M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n)(t,x,\xi) + w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi) \in S^{M-(n+1)\bar{\mathfrak{e}}} \tag{3.37}$$

(recall the definition (2.12)).

Proof. Let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ be a cut-off function satisfying

$$\begin{aligned} &\chi_0(\xi) = 1\,, \quad \forall|\xi| \geq 2\,, \\ &\chi_0(\xi) = 0\,, \quad \forall|\xi| \leq 1\,. \end{aligned} \tag{3.38}$$

Writing $1 = \chi_0 + 1 - \chi_0$, one gets that

$$\begin{aligned} &-\lambda(t)M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n)(t,x,\xi) + w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi) \\ &= -\lambda(t)M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n)(t,x,\xi) + \chi_0(\xi)\big(w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi)\big) \\ &\quad + \big(1-\chi_0(\xi)\big)\big(w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi)\big)\,. \end{aligned} \tag{3.39}$$

By the definition of χ_0 given in (3.38), one easily gets that

$$\big(1-\chi_0(\xi)\big)\big(w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi)\big) \in S^{-\infty}\,, \tag{3.40}$$

therefore we look for a solution g_n of the equation

$$-\lambda(t)M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n)(t,x,\xi) + \chi_0(\xi)\big(w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi)\big) \in S^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.41}$$

Since we require that $\mathcal{G}_n = \mathrm{Op}(g_n)$ is self-adjoint, we look for a symbol of the form

$$g_n(t,x,\xi) = \sigma_n(t,x,\xi) + \sigma_n^*(t,x,\xi) \in S^{1-n\bar{\mathfrak{e}}} \tag{3.42}$$

with the property that

$$\sigma_n^*(t,x,\xi) = \sigma_n(t,x,\xi) + r_n(t,x,\xi), \qquad r_n \in S^{-n\bar{\mathfrak{e}}}. \tag{3.43}$$

Plugging the ansatz (3.42) into the equation (3.41), using (3.43) and since

$$-\lambda(t)M|\xi|^{M-2}\xi\chi(\xi)(\partial_x r_n)(t,x,\xi) \in S^{M-1-n\bar{\mathfrak{e}}} \subseteq S^{M-(n+1)\bar{\mathfrak{e}}}, \tag{3.44}$$

we are led to solve the equation

$$-2\lambda(t)M|\xi|^{M-2}\xi\chi(\xi)(\partial_x \sigma_n)(t,x,\xi) + \chi_0(\xi)\big(w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi)\big) = 0 \tag{3.45}$$

whose solution is given by

$$\sigma_n(t,x,\xi) := \frac{\chi_0(\xi)\partial_x^{-1}\Big[w_n(t,x,\xi) - \langle w_n\rangle_x(t,\xi)\Big]}{2\lambda(t)M|\xi|^{M-2}\xi}\,. \tag{3.46}$$

Since $w_n, \langle w_n \rangle_x \in S^{M-n\bar{\mathfrak{e}}}$, using that $M > 1$ and recalling the definition of the cut-off function χ_0 in (3.38), one gets that $\sigma_n \in S^{1-n\bar{\mathfrak{e}}}$ and hence also $g_n = \sigma_n + \sigma_n^* \in S^{1-n\bar{\mathfrak{e}}}$. We now use Lemma 2.6 with $\varphi(t,\xi) = \frac{\chi_0(\xi)}{2\lambda(t)M|\xi|^{M-2}\xi}$, $a(t,x,\xi) = \partial_x^{-1}\Big[w_n(t,x,\xi) - \langle w_n \rangle_x(t,\xi)\Big]$. Recalling that $w_n = w_n^*$, by Lemma 2.7 we have that $a = a^*$, hence we can apply Lemma 2.6, obtaining that the ansatz (3.43) is satisfied. By (3.39), (3.42), (3.43), (3.45) one then gets

$$
\begin{aligned}
&- \lambda(t)M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n)(t,x,\xi) + w_n(t,x,\xi) - \langle w_n \rangle_x(t,\xi) \\
&= \big(1 - \chi_0(\xi)\big)\big(w_n(t,x,\xi) - \langle w_n \rangle_x(t,\xi)\big) - \lambda(t)M|\xi|^{M-2}\xi\chi(\xi)(\partial_x r_n)(t,x,\xi)
\end{aligned}
$$

and recalling (3.40), (3.44) one then gets (3.37). □

By Lemmata 3.4, 3.5, the operator $\mathcal{V}_{n+1}(t)$ has the form

$$
\mathcal{V}_{n+1}(t) = \lambda(t)|D|^M + \mu_{n+1}(t,D) + \mathcal{W}_{n+1}(t)\,, \tag{3.47}
$$

where

$$
\mu_{n+1}(t,\xi) := \mu_n(t,\xi) + \langle w_n \rangle_x(t,\xi) \in S^{M-\bar{\mathfrak{e}}}\,, \tag{3.48}
$$

$$
\begin{aligned}
\mathcal{W}_{n+1}(t) := \mathrm{Op}\Big(& r_{v_n}(t,x,\xi) - \lambda(t)M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n)(t,x,\xi) \\
& + w_n(t,x,\xi) - \langle w_n \rangle_x(t,\xi)\Big) \in OPS^{M-(n+1)\bar{\mathfrak{e}}}\,.
\end{aligned} \tag{3.49}
$$

Since by the induction hypothesis $w_n = w_n^*$ and $\mu_n(t,\xi)$ is real, by (2.10) and Lemma 2.7-(i), one has that $\langle w_n \rangle_x(t,\xi)$ is real and therefore $\mu_{n+1}(t,\xi)$ is real. Furthermore, since Φ_n is symplectic and $\mathrm{i}\mathcal{V}_n$ is a Hamiltonian vector field, one has that $\mathrm{i}\mathcal{V}_{n+1}$ is still a Hamiltonian vector field, meaning that $\mathcal{V}_{n+1}$ is self-adjoint. Using that $\mu_{n+1}(t,\xi)$ is a real Fourier multiplier, one has that $\lambda(t)|D|^M + \mu_{n+1}(t,D)$ is a self-adjoint operator, implying that

$$
\mathcal{W}_{n+1}(t) = \mathcal{V}_{n+1}(t) - \lambda(t)|D|^M - \mu_{n+1}(t,D)
$$

is self-adjoint too. Then, the proof of Proposition 3.3 is concluded.

4 Proof of Theorem 1.4

Let $K \in \mathbb{N}$ and let us fix a positive integer $N_K \in \mathbb{N}$ as

$$
N_K := \Big[\frac{M+K}{\bar{\mathfrak{e}}}\Big] + 1 \tag{4.1}
$$

so that $M - N_K\bar{\mathfrak{e}} < -K$ (for any $x \in \mathbb{R}$, we denote by $[x]$ its integer part). Then we define

$$
\begin{aligned}
&\mathcal{T}_K(t) := \Phi(t)^{-1} \circ \Phi_1(t)^{-1} \circ \ldots \circ \Phi_{N_K-1}(t)^{-1}\,, \\
&\mathcal{W}_K(t) := \mathcal{W}_{N_K}(t)\,, \quad \lambda_K(t,D) := \lambda(t)|D|^M + \mu_{N_K}(t,D)
\end{aligned} \tag{4.2}
$$

where $\Phi(t) = \Phi(1;t)$ is given by (3.3), $\lambda(t)$ is defined in (3.18) and for any $n \in \{1,\ldots,N_K-1\}$, $\Phi_n(t)$, $\mathcal{W}_n(t)$, $\mu_n(t,D)$ are given in Theorem 3.3. By (3.6), (3.24), using the product rule, one gets that $\mathcal{T}_K$ satisfies the property (1.14). Furthermore, by (3.8), (3.21), (3.25) one obtains (1.15), with $\lambda_K(t,D)$, $\mathcal{W}_K(t)$ defined in (4.2), hence the proof of Theorem 1.4 is concluded.

5 Proof of Theorem 1.3.

Let $s > 0$, $t_0 \in \mathbb{R}$, $u_0 \in H^s(\mathbb{T})$. We fix the constant $K \in \mathbb{N}$, appearing in Theorem 1.4, as

$$
K = K_s := [s] + 1 \tag{5.1}
$$

so that $K > s$. By applying Theorem 1.4, one has that $u(t)$ is a solution of the Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{i}\mathcal{V}(t)[u] = 0 \\ u(t_0) = u_0 \end{cases} \tag{5.2}$$

if and only if $v(t) := \mathcal{T}_{K_s}^{-1}(t)u(t)$ is a solution of the Cauchy problem

$$\begin{cases} \partial_t v + \mathrm{i}\lambda_{K_s}(t,D)v + \mathrm{i}\mathcal{W}_{K_s}(t)[v] = 0 \\ v(t_0) = v_0\,, \end{cases} \qquad v_0 := \mathcal{T}_{K_s}^{-1}(t_0)[u_0] \tag{5.3}$$

with $\lambda_{K_s}(t,D) = \mathrm{Op}(\lambda_{K_s}(t,\xi)) \in OPS^M$ with $\lambda_{K_s}(t,\xi) = \overline{\lambda_{K_s}(t,\xi)}$. Since the symbol $\lambda_{K_s}(t,\xi)$ is real, we have

$$\lambda_{K_s}(t,D) = \lambda_{K_s}(t,D)^*\,. \tag{5.4}$$

Moreover, since $K_s > s > 0$, by (2.1), one has

$$\mathcal{W}_{K_s}(t) \in OPS^{-K_s} \subset OPS^{-s} \subset OPS^0\,, \quad \mathcal{W}_{K_s}(t) = \mathcal{W}_{K_s}(t)^*\,. \tag{5.5}$$

By applying Lemma 2.23 one gets that there exists a unique global solution $v \in \mathcal{C}^0(\mathbb{R}, H^s)$ of the Cauchy problem (5.3), therefore $u \in \mathcal{C}^0(\mathbb{R}, H^s)$ is the unique solution of the Cauchy problem (5.2). In order to conclude the proof, it remains only to prove the bound (1.13).

Estimate of $v(t)$. By a standard energy estimate, using (5.4), (5.5), one gets easily that

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2}\,, \qquad \forall t \in \mathbb{R}\,. \tag{5.6}$$

Writing the Duhamel formula for the Cauchy problem (5.3), one obtains

$$v(t) = e^{\mathrm{i}\Lambda_{K_s}(t,D)}v_0 + \int_{t_0}^{t} e^{\mathrm{i}\left(\Lambda_{K_s}(t,D) - \Lambda_{K_s}(\tau,D)\right)}\mathcal{W}_{K_s}(\tau)[v(\tau)]\,d\tau \tag{5.7}$$

where

$$\Lambda_{K_s}(t,D) := \mathrm{Op}\Big(\Lambda_{K_s}(t,\xi)\Big)\,, \qquad \Lambda_{K_s}(t,\xi) := \int_{t_0}^{t} \lambda_{K_s}(\tau,\xi)\,d\tau\,.$$

Since $\lambda_{K_s}(t,\xi)$ is real, $\Lambda_{K_s}(t,\xi)$ is real too, and therefore the propagator $e^{\mathrm{i}\Lambda_{K_s}(t,D)}$ is unitary on $H^s(\mathbb{T})$. Hence, one has

$$\begin{aligned} \|v(t)\|_{H^s} &\leq \|v_0\|_{H^s} + \Big|\int_{t_0}^{t} \|\mathcal{W}_{K_s}(\tau)[v(\tau)]\|_{H^s}\,d\tau\Big| \\ &\overset{(5.5),\,Theorem\,2.1,}{\lesssim_s} \|v_0\|_{H^s} + \Big|\int_{t_0}^{t} \|v(\tau)\|_{L^2}\,d\tau\Big| \\ &\overset{(5.6)}{\lesssim_s} \|v_0\|_{H^s} + |t-t_0|\|v_0\|_{L^2}\,. \end{aligned} \tag{5.8}$$

Estimate of $u(t)$. Since $u(t) = \mathcal{T}_{K_s}(t)[v(t)]$ and $v_0 = \mathcal{T}_{K_s}(t_0)^{-1}[u_0]$ and $\mathcal{T}_{K_s}(t)^{\pm 1}$ satisfy (1.14), one gets that

$$\|u(t)\|_{H^s} \simeq_s \|v(t)\|_{H^s}\,, \quad \|u_0\|_{H^s} \simeq_s \|v_0\|_{H^s}\,, \quad \|v_0\|_{L^2} \simeq \|u_0\|_{L^2}\,,$$

therefore, by (5.8) one deduce that

$$\|u(t)\|_{H^s} \lesssim_s \|u_0\|_{H^s} + |t-t_0|\|u_0\|_{L^2}\,. \tag{5.9}$$

Proof of (1.13). The estimate (5.9) proves that the propagator $\mathcal{U}(t_0,t)$ of the PDE $\partial_t u = \mathrm{i}\mathcal{V}(t)[u]$, i.e.

$$\begin{cases} \partial_t \mathcal{U}(t_0,t) = \mathrm{i}\mathcal{V}(t)\mathcal{U}(t_0,t) \\ \mathcal{U}(t_0,t_0) = \mathrm{Id} \end{cases}$$

satisfies

$$\|\mathcal{U}(t_0,t)\|_{\mathcal{B}(H^s)} \lesssim_s 1+|t-t_0|\,, \quad \forall s>0\,, \quad \forall t,t_0\in\mathbb{R}\,. \tag{5.10}$$

Furthermore, since $\mathcal{V}(t)$ is self-adjoint, the L^2 of the solutions is constant, namely

$$\|\mathcal{U}(t_0,t)\|_{\mathcal{B}(L^2)} = 1\,, \quad \forall t,t_0\in\mathbb{R}\,. \tag{5.11}$$

Hence, for any $0<s<S$, by applying Theorem 2.8, one gets that

$$\|\mathcal{U}(t_0,t)\|_{\mathcal{B}(H^s)} \le \|\mathcal{U}(t_0,t)\|_{\mathcal{B}(L^2)}^{\frac{S-s}{S}} \|\mathcal{U}(t_0,t)\|_{\mathcal{B}(H^S)}^{\frac{s}{S}} \overset{(5.10),(5.11)}{\lesssim_S} (1+|t-t_0|)^{\frac{s}{S}}\,. \tag{5.12}$$

Then, for any $\varepsilon>0$, choosing S large enough so that $s/S\le\varepsilon$, the estimate (1.13) follows. This concludes the proof of Theorem 1.3.

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