TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 8, August 1997, Pages 3401–3408
S 0002-9947(97)01975-2

MATRIX EXTENSIONS AND EIGENVALUE COMPLETIONS, THE GENERIC CASE

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ABSTRACT. In this paper we provide new necessary and sufficient conditions for the so-called eigenvalue completion problem.

1. Introduction

Let $Mat_{n\times n}(\mathbb{C})$ be the set of all $n \times n$ matrices having complex entries. In the sequel we will identify $Mat_{n\times n}$ with the vector space $\mathbb{C}^{n^2}$. Let $A \in Mat_{n\times n}$ be a particular element and let $\mathcal{L} \subset Mat_{n\times n}$ be a **complex** linear subspace. Identify the set of monic polynomials of degree n in $\mathbb{C}[n]$ with the vector space $\mathbb{C}^n$. In this paper we present new conditions which guarantee that the characteristic map

$$\chi_A : \mathcal{L} \longrightarrow \mathbb{C}^n, \quad L \longmapsto \det(sI - A - L) = s^n - \sigma_1 s^{n-1} + \cdots + (-1)^n \sigma_n \tag{1.1}$$

is generically surjective, i.e. we will give conditions which guarantee that the image of χ_A contains a nontrivial Zariski open and therefore dense subset. (Recall that a set in $\mathbb{C}^n$ is called Zariski open if its complement is the set of zeros of some polynomials).

First we would like to remark that there are two obvious necessary conditions:

1. χ_A is almost onto only if $\dim \mathcal{L} \geq n$.
2. There must be at least one element $L \in \mathcal{L}$ whose trace $\mathrm{tr}(L) \neq 0$, i.e. $\mathcal{L} \not\subset sl_n$.

The main result of this paper states that if both those necessary conditions are satisfied then for a generic set of matrices in $Mat_{n\times n}$ the characteristic map χ is generically surjective.

There exists a large literature about so-called matrix completion problems, matrix extension problems and inverse eigenvalue problems in different areas of mathematics. We only mention the linear algebra literature [6], [12], [13], [15], the operator theory literature [1], [8], and the control literature [2], [3], [10].

For a treatment of many of these topics we recommend the recent book by Gohberg, Kaashoek and van Schagen [7], which concentrates to a large extent on eigenvalue completion problems. We now indicate the type of results which we consider interesting in connection with this paper.

As it turns out, our main result immediately implies classical theorems in a wide range of situations. We now list some of these.

Received by the editors September 8, 1995 and, in revised form, March 14, 1996.
1991 *Mathematics Subject Classification.* Primary 15A18; Secondary 93B60.
Key words and phrases. Eigenvalue completion, dominant morphism theorem, inverse eigenvalue problems.
J. Rosenthal is supported in part by NSF grant DMS-9400965, and X. Wang is supported in part by NSF grant DMS-9500594.

In the linear algebra literature probably one of the earliest results is due to Farahat and Ledermann [5]. The result states that for a matrix whose $(n-1)\times(n-1)$ top left-hand corner is non-derogatory, every characteristic polynomial can be achieved through the choice of entries in the last row and last column.

In 1977 S. Friedland derived an interesting result involving 'diagonal perturbations':

Theorem 1.1. *Let $A \in Mat_{n\times n}$ be arbitrary and let $\mathcal{L} = \mathcal{D}_n$ be the set of diagonal matrices. Then χ_A is surjective of mapping degree $n!$, i.e. when counted with multiplicity*

$$\#\{\chi_A^{-1}(D)\} = n! \quad \forall D \in \mathcal{D}_n.$$

Both the result of Farahat and Ledermann and that of Friedland belong to the class of matrix completion problems, i.e. one assumes that certain elements in a matrix are fixed and other elements can be freely chosen.

For the general problem at hand probably the strongest result is due to Byrnes and Wang [4], [14]. It covers the situation when $\mathcal{L} \subset Mat_{n\times n}$ is a Lie subalgebra:

Theorem 1.2. *Given a Lie algebra $\mathcal{L} \subset gl_n$. Then χ_A is onto for all A if and only if $\operatorname{rank}\mathcal{L} = n$ and some element of $\mathcal{L}$ has distinct eigenvalues.*

This result is basically saying that χ_A is onto for all A if and only if $\mathcal{D}_n \subset \mathcal{L}$. So the next natural question would be: When is χ_A onto (or almost onto) for a generic matrix A? That is the motivation for this paper.

In the control literature our theorem covers a wide range of so-called pole placement problems, and we would like to mention only the most prominent of them, namely, the *static output pole placement problem.* For this one considers a time invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad A \in Mat_{n\times n},\ B \in Mat_{n\times m},\ C \in Mat_{p\times n}.$$

It is the goal to construct for every monic polynomial $\phi \in \mathbb{C}[s]$ a so-called 'feedback compensator' $u = Fy, F \in Mat_{m\times p}$, such that the 'closed loop characteristic polynomial'

$$\det(sI - A - BFC) = \phi.$$

Note that in this situation $\mathcal{L} = \{BFC \mid F \in Mat_{m\times p}\}$ is a subspace (even a Lie subalgebra) of $Mat_{n\times n}$ of dimension at most mp. The main result of Brockett and Byrnes [2] states:

Theorem 1.3. *If $mp = n$, then for a generic set of matrices (A, B, C) the map*

$$\chi_A : \ \mathbb{C}^{mp} \longrightarrow \mathbb{C}^n, \quad F \longmapsto \det(sI - A - BFC)$$

is surjective, and there are

$$d(m,p) = \deg \operatorname{Grass}(m, m+p) = \frac{1!2!\cdots(p-1)!(mp)!}{m!(m+1)!\cdots(m+p-1)!} \tag{1.2}$$

solutions for each characteristic polynomial.

For a more extensive treatment of the class of inverse eigenvalue problems appearing in the control literature we refer to the survey article by Byrnes [3].

As a word of caution, note that since the dominant morphism theorem (see Lemma 2.1) is not true over $\mathbb{R}$, this paper does not address classical completion problems which involve real subspaces $\mathcal{L}$ of $Mat_{n\times n}(\mathbb{R})$. For example, completion to self-adjoint matrices is not covered here.

In the next section we will provide the main results of this paper. In Section 3 we will explain in geometric terms some of the main ingredients of the proof of Theorem 2.4, the major result of this article.

2. Dominant morphism theorem and the linearization of the characteristic map

Consider once more the characteristic map (1.1):

$$\chi_A:\ \mathcal{L}\longrightarrow\mathbb{C}^n,\quad L\longmapsto\left(-\sigma_1(A+L),\dots,(-1)^n\sigma_n(A+L)\right),$$

where $\sigma_i(A+L)$ denotes the ith elementary symmetric function of the eigenvalues $\lambda_1,\dots,\lambda_n$ of $A+L$. There are classical formulas which express the elementary symmetric functions $\sigma_i(A+L)$ uniquely as a polynomial in the power sum symmetric functions

$$p_i:=\lambda_1^i+\cdots+\lambda_n^i=\operatorname{tr}(A+L)^i.$$

To be precise, one has the formula (see e.g. [11])

$$\sigma_i(A+L)=\frac{1}{n!}\det\begin{pmatrix} p_1 & 1 & 0 & \dots & 0\\ p_2 & p_1 & 2 & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & p_1 & n-1\\ p_n & \dots & \dots & p_2 & p_1\end{pmatrix},$$

which induces an isomorphism $\mathbb{C}^n\to\mathbb{C}^n, (p_1,\dots,p_n)\mapsto(\sigma_1,\dots,\sigma_n)$. Based on this, we can equally well study the map

$$\psi:\ \mathcal{L}\longrightarrow\mathbb{C}^n,\quad L\longmapsto\left(\operatorname{tr}(A+L),\dots,\operatorname{tr}(A+L)^n\right),$$

which we call the *trace map for* A. The main ingredient of our proof will be a linearization of ψ. For this we compute the difference quotient

$$\lim_{\epsilon\mapsto 0}\frac{\operatorname{tr}(A+\epsilon L)^i-\operatorname{tr}A^i}{\epsilon}=i\cdot\operatorname{tr}(A^{i-1}L).$$

The linearization $d\psi_0$ around the origin is therefore given through

$$d\psi_0(L)=(\operatorname{tr}(L),2\operatorname{tr}(AL)\dots,n\cdot\operatorname{tr}(A^{n-1}L)).$$

Lemma 2.1. *If for some $A\in Mat_{n\times n}$ and some linear subspace $\mathcal{L}\subset Mat_{n\times n}$ the linearization $d\psi_0$ of the trace map of A is onto, then χ_A is generically surjective, i.e.* $\operatorname{Im}(\chi_A)$ *contains a Zariski open subset of* $\mathbb{C}^n$.

Proof. Direct consequence of the dominant morphism theorem (see e.g. [9]). □

In the sequel we will assume that $\dim\mathcal{L}=d\geq n$, and we will identify the set of all d-dimensional subspaces $\mathcal{L}\subset Mat_{n\times n}$ with the Grassmann variety $\operatorname{Grass}(d,\mathbb{C}^{n^2})$. Note that $\operatorname{Grass}(d,\mathbb{C}^{n^2})$ is an irreducible variety, and a subset $U\subset\operatorname{Grass}(d,\mathbb{C}^{n^2})$ is a generic set if U contains a non-empty Zariski open subset of $\operatorname{Grass}(d,\mathbb{C}^{n^2})$. With Lemma 2.1 we have:

Lemma 2.2. *If $d \geq n$, then for a generic subset of pairs*

$$(A, \mathcal{L}) \in \mathbb{C}^{n^2} \times \mathrm{Grass}(d, \mathbb{C}^{n^2})$$

χ_A *is almost onto.*

Proof. The set of pairs $(A, \mathcal{L})$ whose linearization $d\psi_0$ fails to be surjective is an algebraic subset in $\mathbb{C}^{n^2} \times \mathrm{Grass}(d, \mathbb{C}^{n^2})$. Since the complement is certainly non-empty, the result readily follows. □

In everyday language the lemma states that for almost all matrices and almost all linear subspaces $\mathcal{L} \in \mathrm{Grass}(d, \mathbb{C}^{n^2})$ almost all closed characteristic polynomials can be achieved. The following two theorems strengthen this result.

Theorem 2.3. *Let $A \in Mat_{n\times n}$ be an arbitrary matrix. Then for a generic set of subspaces in* $\mathrm{Grass}(n, \mathbb{C}^{n^2})$, χ_A *is almost onto.*

Proof. The set of subspaces $\mathcal{L}$ having the property that $d\psi_0 : \mathbb{C}^n \to \mathbb{C}^n$ is not surjective is an algebraic subset of $\mathrm{Grass}(n, \mathbb{C}^{n^2})$. By Friedland's result (Theorem 1.1) the result follows. □

The main result of the paper is now as follows:

Theorem 2.4. *Let $\mathcal{L} \subset Mat_{n\times n}$ be a linear subspace satisfying* $\dim \mathcal{L} \geq n$ *and* $\mathcal{L} \not\subset sl_n$. *Then for a generic set of matrices $A \in Mat_{n\times n}$ the characteristic map* χ_A *is almost onto.*

The proof of this theorem is not trivial and will require the rest of this section. In order to facilitate the reasoning we will divide the proof into several lemmas.

Let $\pi : Mat_{n\times n} \longrightarrow \mathbb{C}^n$ be the projection onto the diagonal elements. Let $sl_n \subset Mat_{n\times n}$ be the matrices having trace equal to zero and let $V := \pi(sl_n)$ be the hyperplane defined by

$$V = \{(x_1, \dots, x_n) \in \mathbb{C}^n | \sum_{i=1}^{n} x_i = 0\}. \tag{2.1}$$

If $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$, then V has a basis consisting of the vectors $\{e_1 - e_2, \dots, e_{n-1} - e_n\}$.

Let $L \in sl_n$ be an arbitrary nonzero matrix, and consider the associated polynomial map

$$\varphi^L : Gl_n \longrightarrow V, \qquad S \longmapsto \pi(SLS^{-1}).$$

One readily verifies that the Jacobian around the identity is given through

$$d\varphi^L_I : Mat_{n\times n} \longrightarrow V, \quad X \longmapsto \pi([X, L]) = \pi(XL - LX).$$

In the sequel we will derive an algebraic criterion which guarantees that the Jacobian $d\varphi^L_I$ is surjective. For this we will associate to a matrix $L = (l_{ij})$ a graph $\mathcal{G}(L)$ which consists of n vertices $v_1, \dots, v_n$ with edges between the ith and jth vertex whenever either l_{ij} or l_{ji} is nonzero. In terms of the matrix L we can identify the vertex v_i with the ith diagonal element, with the obvious meaning for the edges.

Recall that a graph with n vertices is called connected provided any two vertices can be joined by some path in the graph. If $\mathcal{G}$ is not connected, then there exist a permutation σ of n elements and r integers $1 \leq i_1 < \cdots < i_r \leq n$ with the property that $v_{\sigma(1)}, \dots, v_{\sigma(i_1)}$ represents the first connected component, $v_{\sigma(i_1+1)}, \dots, v_{\sigma(i_2)}$ represents the second connected component and so on.

Corresponding to the permutation σ there is a permutation matrix P_σ having the property that $P_\sigma L P_\sigma^{-1}$ is a block diagonal matrix $\operatorname{diag}(L_1, \dots, L_r)$ whose ith block L_i has an associated connected graph $\mathcal{G}(L_i)$.

Lemma 2.5. *For any $L \in Mat_{n\times n}$ the Jacobian $d\varphi_I^L$ is surjective if and only if the associated graph $\mathcal{G}(L)$ is connected.*

Proof. First assume that $\mathcal{G}(L)$ is not connected. Following the remark before the lemma, there is a permutation matrix P such that

$$\tilde{L} := PLP^{-1} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

is block diagonal, where we assume that L_1 is a $k \times k$ matrix with $1 \le k < n$.

Clearly the sum of the first k diagonal elements of the matrix $\left(X\tilde{L} - \tilde{L}X\right)$ is always zero. It follows that $d\varphi_I^{\tilde{L}}$ and therefore also $d\varphi_I^L$ are both not surjective.

In order to prove the other direction, assume that $\mathcal{G}(L)$ is connected. Let $E_{ij} \in Mat_{n\times n}$ be the matrix whose ijth entry is 1, and all the other entries are zero. Then

$$\pi(E_{ij}L - LE_{ij}) = \begin{cases} l_{ji}(e_i - e_j) & \text{if } i \ne j, \\ 0 & \text{if } i = j, \end{cases} \tag{2.2}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$ as introduced earlier. Since the vertex v_i is connected with the vertex v_{i+1} for $i = 1, \dots, n-1$, it follows that there is a path

$$v_i = v_{i_1} \leftrightarrow v_{i_2} \leftrightarrow \cdots \leftrightarrow v_{i_s} = v_{i+1}$$

connecting the vertices v_i and v_{i+1}. From the identity (2.2) it follows that the vectors $e_{i_1} - e_{i_2}$, $e_{i_2} - e_{i_3}$, ... , $e_{i_{s-1}} - e_{i_s}$ are all in the image of $d\varphi_I^L$. In particular the vector $e_i - e_{i+1}$ is also in the image of $d\varphi_I^L$. This completes the proof. ◻

Lemma 2.6. *Let $L \in Mat_{n\times n}$ be a nonzero matrix having the property that $\pi(L) = 0$. Then there exists a matrix X such that for all $\epsilon \ne 0$ the matrix*

$$\hat{L}(\epsilon) := (I_n + \epsilon X)L(I_n + \epsilon X)^{-1}$$

has the properties:

1. $d\varphi_I^{\hat{L}(\epsilon)}(Mat_{n\times n}) = V$,
2. $\pi(\hat{L}(\epsilon)) = 0$.

Proof. If the graph of L is connected the result is trivially fulfilled by using $X = 0$. If L is not connected there exists a permutation matrix P such that

$$\tilde{L} := PLP^{-1} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

has the properties that the entry $\tilde{l}_{2,1}$ of $\tilde{L}$ is nonzero and the graph of the $k \times k$ submatrix L_1 is connected. Let $\tilde{X} := E_{1,(k+1)} + \cdots + E_{1,n}$. Then

$$I + \epsilon\tilde{X} = \begin{pmatrix} I_k & \epsilon B \\ 0 & I_{n-k} \end{pmatrix},$$

where the first row of B is $(1, \dots, 1)$ and all the other rows are zero.

From the identity

$$(I_n + \epsilon\tilde{X})\tilde{L}(I_n + \epsilon\tilde{X})^{-1} = (I_n + \epsilon\tilde{X})\tilde{L}(I_n - \epsilon\tilde{X}) = \begin{pmatrix} L_1 & \epsilon(BL_2 - L_1B) \\ 0 & L_2 \end{pmatrix}$$

it follows that for all ϵ one has

$$\begin{aligned} 0 = \pi(\tilde{L}) &= \pi\left((I_n + \epsilon\tilde{X})\tilde{L}(I_n + \epsilon\tilde{X})^{-1}\right) \\ &= \pi\left((I_n + \epsilon P^{-1}\tilde{X}P)L(I_n + \epsilon P^{-1}\tilde{X}P)^{-1}\right). \end{aligned}$$

Let $X := P^{-1}\tilde{X}P$. By the last expression we have that $\pi(\hat{L}(\epsilon)) = \pi(L)$, and since the second row of $BL_2 - L_1B$ is $\tilde{l}_{2,1}(1, \dots, 1)$, the graph of $\hat{L}(\epsilon)$ is connected for all $\epsilon \neq 0$, i.e. by Lemma 2.5,

$$d\varphi_I^{\hat{L}(\epsilon)}(Mat_{n\times n}) = V$$

for all $\epsilon \neq 0$. □

Remark 2.7. It can be shown easily that the same is true for any $L \neq \lambda I$, but we will not need this result.

Lemma 2.8. *Let $\mathcal{L} \subset Mat_{n\times n}$ be a linear subspace of dimension n, $\mathcal{L} \not\subset sl_n$. (I.e., $\mathcal{L}$ contains an element with nonzero trace.) Then there exists an $S \in Gl_n$ such that $\pi\mid_{S\mathcal{L}S^{-1}}$ is one-one, i.e. the projection of $S\mathcal{L}S^{-1}$ onto the diagonal elements is one to one and onto.*

Proof. Let $\{L_1, L_2, \dots, L_n\}$ be a basis of $\mathcal{L}$ having the property that $L_1 \notin sl_n, L_i \in sl_n$ for $i = 2, 3, \dots, n$. Furthermore we will assume that $\pi(L_1), \dots, \pi(L_k)$ are linearly independent. If $k < n$ and if $\pi(L_{k+1})$ depends linearly on $\pi(L_1), \dots, \pi(L_k)$, we will show the existence of some $S \in Gl_n$ such that

$$\pi(SL_1S^{-1}), \dots, \pi(SL_kS^{-1}), \pi(SL_{k+1}S^{-1})$$

are linearly independent. The proof of the theorem will then follow from this claim by induction over k.

By assumption there are numbers $a_1, \dots, a_k$ having the property that

$$\pi(L_{k+1}) = a_1\pi(L_1) + \cdots + a_k\pi(L_k).$$

By possibly replacing L_{k+1} through $L_{k+1} - a_1L_1 - \cdots a_kL_k$ we can assume that $\pi(L_{k+1}) = 0$. Since L_{k+1} is not a diagonal matrix, it follows from Lemma 2.6 that there exist a matrix X_1 and a number ϵ_1 such that the matrices

$$\tilde{L}_i := (I_n + \epsilon_1X_1)L_i(I_n + \epsilon_1X_1)^{-1},\ i = 1, \dots, k+1$$

have the properties that $\pi(\tilde{L}_1), \dots, \pi(\tilde{L}_k)$ are linearly independent, $\pi(\tilde{L}_{k+1}) = 0$, and $d\varphi_I^{\tilde{L}_{k+1}}(Mat_{n\times n}) = V$. Since $d\varphi_I^{\tilde{L}_{k+1}}$ is surjective, there exists a matrix X such that

$$d\varphi_I^{\tilde{L}_{k+1}}(X) = \pi([X, \tilde{L}_{k+1}]) \notin \operatorname{span}\{\pi(\tilde{L}_1), \dots, \pi(\tilde{L}_k)\}.$$

Consider the Taylor series expansions

$$\pi((I_n + \epsilon X)\tilde{L}_i(I_n + \epsilon X)^{-1}) = \pi(\tilde{L}_i) + \beta_i(\epsilon), \quad i = 1, \dots, k,$$

and

$$\pi((I_n + \epsilon X)\tilde{L}_{k+1}(I_n + \epsilon X)^{-1}) = \epsilon\pi([X, \tilde{L}_{k+1}]) + \alpha_{k+1}(\epsilon),$$

where the vectors $\beta_i(\epsilon)$ and $\alpha_{k+1}(\epsilon)$ satisfy

$$\lim_{\epsilon\to 0}\beta_i(\epsilon)=0,\quad i=1,\dots,k,\quad \text{and}\quad \lim_{\epsilon\to 0}\frac{1}{\epsilon}\alpha_{k+1}(\epsilon)=0.$$

For a sufficiently small $\epsilon>0$,

$$\{\pi(\tilde{L}_1)+\beta_i(\epsilon),\dots,\pi(\tilde{L}_k)+\beta_i(\epsilon),\pi([X,\tilde{L}_{k+1}])+\frac{1}{\epsilon}\alpha_{k+1}(\epsilon)\}$$

are linearly independent, i.e.

$$\{\pi((I_n+\epsilon X)\tilde{L}_i(I_n+\epsilon X)^{-1})|i=1,\dots,k+1\}$$

are linearly independent. This completes the induction step and therefore the proof of the lemma. □

Proof of Theorem 2.4. Let $\mathcal{L}\subset Mat_{n\times n}$ be given. By possibly restricting to a subspace $\tilde{\mathcal{L}}\subset\mathcal{L}$ we will be able to assume that $\dim\mathcal{L}=n$. The set of matrices $A\in Mat_{n\times n}$ whose trace map has has surjective linearization $d\psi_0$ forms a Zariski open subset of $\mathbb{C}^{n^2}$. In order to prove the theorem it is therefore enough to show the existence of one matrix $\hat{A}$ whose trace map has surjective linearization.

By the last lemma there exists a $S\in Gl_n$ such that $\pi\mid_{S\mathcal{L}S^{-1}}$ is one to one and onto. Let D be the diagonal matrix

$$D:=\begin{pmatrix}1&&&\\&2&&\\&&\ddots&\\&&&n\end{pmatrix}.$$

Then

$$\mathcal{L}\longrightarrow\mathbb{C}^n,\quad L\longmapsto(\operatorname{tr}(SLS^{-1}),\operatorname{tr}(DSLS^{-1}),\dots,\operatorname{tr}(D^{n-1}SLS^{-1}))$$

is surjective. Let $\hat{A}:=S^{-1}DS$. Since $\operatorname{tr}(D^iSLS^{-1})=\operatorname{tr}(\hat{A}^iL)$ for $i=0,\dots,n-1$, it follows that

$$\mathcal{L}\longrightarrow\mathbb{C}^n,\quad L\longmapsto(\operatorname{tr}(L),\operatorname{tr}(\hat{A}L),\dots,\operatorname{tr}(\hat{A}^{n-1}L))$$

is surjective. This shows the existence of the desired matrix $\hat{A}$ and completes the proof. □

3. Some geometric remarks

The main technical result of this paper is Lemma 2.8. In this section we do some geometric interpretation of this result.

Consider once more the Grassmannian $\text{Grass}(n,\mathbb{C}^{n^2})$. The similarity transformation on $Mat_{n\times n}$ induces a group action

$$\begin{array}{rrcl}\phi: & Gl_n\times\text{Grass}(n,\mathbb{C}^{n^2}) & \longrightarrow & \text{Grass}(n,\mathbb{C}^{n^2}),\\ & (S,\mathcal{L}) & \longmapsto & S\mathcal{L}S^{-1}.\end{array}\tag{3.1}$$

Inside $\text{Grass}(n,\mathbb{C}^{n^2})$ there are two natural Schubert subvarieties:

$$\mathcal{S}_1 := \{\mathcal{L}\in\text{Grass}(n,\mathbb{C}^{n^2})\mid\pi|_{\mathcal{L}}\text{ is not surjective}\},\tag{3.2}$$

$$\mathcal{S}_2 := \{\mathcal{L}\in\text{Grass}(n,\mathbb{C}^{n^2})\mid\mathcal{L}\subset sl_n\}.\tag{3.3}$$

One readily verifies that $\mathcal{S}_2$ is isomorphic to a sub-Grassmann variety, $\mathcal{S}_2\subset\mathcal{S}_1$, and $\mathcal{S}_1$ is a Schubert hypersurface, i.e. a codimension one Schubert subvariety. Note that $\mathcal{S}_2$ is Gl_n invariant. Lemma 2.8 now states that if $X \in \mathcal{S}_1$, then the whole Gl_n orbit $Gl_n(X) \subset \mathcal{S}_1$ if and only if $X \in \mathcal{S}_2$.

Consider now the canonical Hermitian inner product on $\mathbb{C}^{n^2} \simeq Mat_{n\times n}$. This inner product induces an isomorphism between $\mathrm{Grass}(n, \mathbb{C}^{n^2})$ and $\mathrm{Grass}(n^2-n, \mathbb{C}^{n^2})$. Let $\mathcal{S}_1^\perp$ and $\mathcal{S}_2^\perp$ be the subvarieties in $\mathrm{Grass}(n^2-n, \mathbb{C}^{n^2})$ corresponding to $\mathcal{S}_1, \mathcal{S}_2$. One readily verifies that

$$\mathcal{S}_1^\perp := \{\mathcal{W} \in \mathrm{Grass}(n^2-n, \mathbb{C}^{n^2}) \mid \mathcal{D}_n \cap \mathcal{W} \neq \{0\}\}, \tag{3.4}$$

$$\mathcal{S}_2^\perp := \{\mathcal{W} \in \mathrm{Grass}(n^2-n, \mathbb{C}^{n^2}) \mid I_n \in \mathcal{W}\}. \tag{3.5}$$

Translating Lemma 2.8 into this dual version, we immediately have:

Lemma 3.1. *Let $\mathcal{W} \subset Mat_{n\times n}$ be a linear subspace of dimension $n^2 - n$ which does not contain the identity matrix I_n. Then there exists an $S \in Gl_n$ having the property that $S\mathcal{W}S^{-1}$ contains no diagonal matrix.*

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