A Smooth Compactification of the Space of Transfer Functions with Fixed McMillan Degree

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Abstract

It is a classical result of Clark that the space of all proper or strictly proper $p \times m$ transfer functions of a fixed McMillan degree d has in a natural way the structure of a non-compact, smooth manifold. There is a natural embedding of this space into the set of all $p \times (m+p)$ autoregressive systems of degree at most d. Extending the topology in a natural way we will show that this enlarged topological space is compact. Finally we describe a homogenization process which produces a smooth compactification.

1 Introduction

Let $G(s)$ be a proper $p \times m$ transfer function. As is well known, there exists a realization in the time domain given through

$$\sigma x = Ax + Bu, \quad y = Cx + Du. \tag{1.1}$$

Here σ denotes either the shift operator or the differentiation operator depending on whether one studies discrete time or continuous time problems.

If the realization 1.1 of the transfer function $G(s) = C(sI - A)^{-1}B + D$ has the property that the dimension d of the state vector x is minimal among all possible realizations one says that $G(s)$ has McMillan degree d. In this paper we will study topological properties of the space of all transfer functions with a fixed McMillan degree.

As shown by Clark [2] the set of all real (or complex) proper $p \times m$ transfer functions of fixed McMillan degree d has in a natural way the structure of a real (complex) manifold of dimension $d(m+p)+mp$, which we denote by $S^d_{p,m}$.

Many physical systems, which are linear in their nature, cannot be modeled by a dynamical system of the form 1.1. Due to this reason, recently there has been a great interest in the study of singular systems, i.e. systems described by

$$\sigma Ex = Ax + Bu, \quad y = Cx + Du, \tag{1.2}$$

where the square matrix E is not necessarily invertible. Examples of singular dynamical systems arise for example, in the theory of circuit systems or if one studies certain feedback configurations involving high gain compensators. Moreover as was already pointed out by Hazewinkel [5], it is possible for a system of type 1.1 to degenerate to a singular system of type 1.2 under parameter disturbances.

In the frequency domain, the class of singular systems corresponds to the class of improper transfer functions and more generally to the class of autoregressive systems of the form

$$R_1(\sigma)u + R_2(\sigma)y = 0. \tag{1.3}$$

For questions concerning the state space realization of systems of autoregressive equations and improper transfer functions we refer the interested reader to the recent paper of Kuijper and Schumacher [8] and to the dissertation of Glüsing-Lüerßen [4] where more references to the literature can be found.

Let Grass(p,$\mathbb{C}^k$) denote the Grassmann manifold consisting of all p-dimensional subspaces of the vector space $\mathbb{C}^k$. As shown by Hermann and Martin [11] every $p \times m$ transfer function $G(s)$ with entries in the field $\mathbb{C}(s)$ and McMillan degree d describes a holomorphic map of degree d from the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ into the Grassmannian Grass(p,$\mathbb{C}^{m+p}$).

Let $Rat_{d,p,m}$ denote the space of all base point preserving holomorphic maps from $\mathbb{P}^1(\mathbb{C})$ to Grass(p,$\mathbb{C}^{m+p}$) of degree d. Under the Hermann-Martin identification the space of strictly proper $p \times m$ transfer functions of degree d corresponds to $Rat_{d,p,m}$, a manifold that has been well studied in the literature. (See e.g. [10].) It is not difficult to show that the space $S^d_{p,m}$ of proper $p \times m$ transfer functions of McMillan degree d corresponds to the trivial bundle $\mathbb{C}^{mp} \times Rat_{d,p,m}$ and the space of all irreducible $p \times (m+p)$ autoregressive systems of degree d is a fibre bundle over Grass(p,$\mathbb{C}^{m+p}$) with fibres isomorphic to $Rat_{d,p,m}$. In particular, both those spaces are (non-compact) manifolds as well. (Compare with [4, 10, 14].)

Although homological properties of the above manifolds are important we believe that for many questions in systems theory it is even more important to have a good metric on those spaces and to have a good understanding of the boundary structure of those manifolds as well. Indeed, most control design questions can be viewed as an intersection problem in the space of possible compensators which is preferably a compact manifold. Due to this reason we were interested in a good compactification of the manifold $S^d_{p,m}$.

As we will explain in this paper, one can view the transition from the space of proper transfer functions to the class of improper transfer functions and eventually, to the class of autoregressive systems as a compactification process of the manifold $S^d_{p,m}$. We will show in this paper that the set of all autoregressive systems of McMillan degree at most d has in a natural way the structure of a compact topological space. Using a homogenization process we will construct a smooth manifold, which we denote by $\tilde{K}^d_{p,m}$ and which contains the manifold $S^d_{p,m}$ as a dense submanifold.

The compactification we present here was first constructed by Stromme in [16], in an attempt to understand maps of a fixed degree from the projective line into a Grassmannian. This compactification was also discovered by Lomadze in [9]. Lomadze's notation for our space was $\mathcal{S}^{co}_{d,m}$. Both Stromme and Lomadze use techniques from algebraic geometry and Grothendieck's construction of Quot schemes. In this paper we give, what we believe to be, a more elementary exposition of the construction of this space. However, we have not been able to avoid all use of algebraic geometry. In particular, our proof that $\tilde{K}^d_{p,m}$ is compact, uses some ideas from algebraic geometry, though we feel that even here our methods are more elementary. Stromme also shows that $\tilde{K}^d_{p,m}$ is smooth. Our proof of smoothness is new and completely elementary, though it is long. From a systems theoretic point of view however we believe that this proof is very appealing because it involves the construction of an explicit set of charts.

Stromme obtains more information on $\tilde{K}^d_{p,m}$, in particular on its cohomology groups. We intend to discuss the system theoretic implications of this in our subsequent work. We feel that [16] is a virtual gold mine of information on the space $\tilde{K}^d_{p,m}$.

The paper is structured as follows: The next section describes the main results of the paper. In section 3 we will prove those results and finally in the last section we will compare our compactification with other compactifications existing in the literature.

2 The topology of the space of autoregressive systems

Let $\mathbb{K}$ be an arbitrary field and let $\mathbb{K}[s]$ denote the polynomial ring in the indeterminate s. Consider a $p \times k$ matrix $P(s) = (f_{ij}(s))$ whose entries are elements of the ring $\mathbb{K}[s]$. Over the real or over the complex numbers $P(s)$ induces a system of autoregressive equations given by:

$$P(\frac{d}{dt})w(t) = 0. \tag{2.1}$$

Clearly a change in the row space of $P(s)$ does not change the solution set, so the behavior of the system 2.1 in the sense of Willems [17] remains the same. Moreover the behavior of two systems of autoregressive equations represented by $P(s), Q(s)$ is different if the matrices $P(s), Q(s)$ are not row equivalent.

Based on this observation we say $P(s)$ and $\tilde{P}(s)$ are externally equivalent or row equivalent if there is a unimodular $p \times p$ matrix $U(s)$ with $\tilde{P}(s) = U(s)P(s)$. Using this equivalence relation we define:

Definition 2.1 An equivalence class of $p \times k$ polynomial matrices is called an autoregressive system.

The set of autoregressive systems generalizes the set of transfer functions in the following way:

Assume $G(s)$ is an arbitrary proper or improper transfer function describing the input-output relation between an input u and an output y in the frequency domain through:

$$y = G(s)u. \tag{2.2}$$

If $D(s)^{-1}N(s) = G(s)$ is a left coprime factorization of the rational matrix $G(s)$ one can rewrite Equation 2.2 in form of a system of autoregressive equations given through:

$$(N(s)\ D(s)) \cdot \begin{pmatrix} u \\ -y \end{pmatrix}(s) \ = \ 0. \tag{2.3}$$

Moreover, if $\tilde{D}(s)^{-1}\tilde{N}(s) = G(s)$ is another left coprime factorization then the polynomial matrices $(N(s)\, D(s))$ and $(\tilde{N}(s)\, \tilde{D}(s))$ are row equivalent and define therefore the same autoregressive system. Note finally that the assignment

$$s \longmapsto \mathrm{rowsp}(N(s)\, D(s)) \tag{2.4}$$

describes exactly the Hermann-Martin map [11] associated to the transfer function $G(s)$.

The following definition extends the notion of McMillan degree to the class of autoregressive systems:

Definition 2.2 ([14]) The degree of an autoregressive system $P(s)$ is given by the maximal degree of the full size minors of $P(s)$.

Clearly row equivalent polynomial matrices have the same degree. Moreover if $D^{-1}(s)N(s)$ is a left coprime factorization of the transfer function $G(s)$ then the degree of $(N(s)\, D(s))$ coincides with the McMillan degree of $G(s)$ which itself is equal to the topological degree of the associated Hermann-Martin curve [11].

Without loss of generality assume in the following that $P(s)$ is row reduced with row indices equal to $d_1 \geq d_2 \geq \ldots \geq d_p$. Note that these row indices are different from the minimal indices defined by Forney [3] if the polynomial matrix $P(s)$ is not of full rank for some value $s \in \bar{\mathbb{K}}$. However if $P(s)$ has full rank for all $s \in \bar{\mathbb{K}}$, i.e. $P(s)$ is irreducible, then the two sets of indices coincide. Also, if $G(s) = D^{-1}(s)N(s)$ is a left coprime factorization of a transfer function $G(s)$, then $(N(s)\, D(s))$ is irreducible and the row indices of $(N(s)\, D(s))$ coincide with the observability indices of $G(s)$ [3].

Motivated by the fact that the Hermann-Martin map can be extended to ∞, i.e. is a map defined on the whole projective line $\mathbb{P}^1$ we homogenize the polynomial matrix $P(s)$, which we assume to have row indices $d_1 \geq d_2 \geq \ldots \geq d_p$, in the following way:

Denote by $f_i(s)$ the i-th row-vector of the polynomial matrix $P(s)$. In other words using earlier notation one has $f_i(s) = (f_{i1}(s), \ldots, f_{ik}(s))$ and the degree of $f_i(s)$ is given by $\max\{\deg f_{ij}(s) \mid j = 1, \ldots, k\} = d_i$. The homogenization of the i-th row-vector $f_i(s)$ is then defined by:

$$\hat{f}_i(s,t) := t^{d_i} f_i(\frac{s}{t}). \tag{2.5}$$

Using this homogenization process we can associate to each autoregressive system $P(s)$ the homogeneous system

$$P(s,t) := \begin{pmatrix} \hat{f}_{11}(s,t) & \hat{f}_{12}(s,t) & \ldots & \hat{f}_{1k}(s,t) \\ \hat{f}_{21}(s,t) & \hat{f}_{22}(s,t) & \ldots & \hat{f}_{2k}(s,t) \\ \vdots & \vdots & & \vdots \\ \hat{f}_{p1}(s,t) & \hat{f}_{p2}(s,t) & \ldots & \hat{f}_{pk}(s,t) \end{pmatrix}. \tag{2.6}$$

In the class of homogeneous systems we say two systems $P(s,t)$ and $\tilde{P}(s,t)$ are equivalent if they have the same row-degrees and if there is a unimodular matrix $U(s,t)$, whose entries are homogeneous polynomials, such that $\tilde{P} = UP$. A more precise formulation will be given in Definition 3.9. Again we call an equivalence class of homogeneous systems a homogeneous autoregressive system.

Note that $P(s,1) = P(s)$ and the matrix $P(1,0)$ describes the behavior at infinity. The degree of a homogeneous autoregressive system is now defined in the obvious way, namely through the sum of the row-degrees.

The following set will now be the main interest in our studies:

Definition 2.3 $\tilde{K}^d_{p,m}$ denotes the set of all homogeneous autoregressive systems of degree d.

We are now in a position to formulate one of the main results of this paper. The proof of this result will occupy all of the next section.

Theorem 2.4 *$\tilde{K}^d_{p,m}$ is a smooth projective variety containing the manifold $S^d_{p,m}$ as a Zariski dense subset.*

Note that this result establishes a smooth compactification of the variety $S^d_{p,m}$, a result sought after in systems theory for a long time. There are several other compactifications of this space published in the literature. We are aware of the following compactifications: [1, 5, 7, 9, 12, 13]. We will compare these compactifications in the last section of this paper.

We now explain the relation with the space of (inhomogeneous) autoregressive systems. Denote with $A^d_{p,m}$ the set of all $p \times (m+p)$ autoregressive systems of degree d and with

$$A^{\prec d}_{p,m} := \bigcup_{i=0}^{d} A^i_{p,m}. \tag{2.7}$$

This notation is the same as the one used in [14]. One has a natural projection

$$\begin{array}{rccc} \pi : & \tilde{K}^d_{p,m} & \longrightarrow & A^{\prec d}_{p,m} \\ & P(s,t) & \longmapsto & P(s,1) \end{array} \tag{2.8}$$

which is generically one-one. The map π induces on $A^{\prec d}_{p,m}$ a topology, namely the quotient topology. Using this topology the set $A^{\prec d}_{p,m}$ becomes a compact topological space. Moreover it is immediate that the set $S^d_{p,m}$ of proper transfer functions is dense in $A^{\prec d}_{p,m}$.

Before we go over to the proof of Theorem 2.4 in the next section we like to conclude this section with two examples.

Example 2.5 Consider the case of a "single output", i.e. $p = 1$. It is immediate that in this case $\tilde{K}^d_{1,m}$ is a projective space and the projection π is an isomorphism. To be precise one has

$$\tilde{K}^d_{1,m} \cong A^{\prec d}_{1,m} \cong \mathbb{P}(\mathbb{K}^{d+1} \otimes \mathbb{K}^{m+1}) = \mathbb{P}^{md+m+d} \tag{2.9}$$

We want to mention at this point that in the case $p = 1$, the compactification $\tilde{K}^d_{1,m}$ is the same as the one given in [1, 13], but different from the ones presented in [7, 12]

Example 2.6 The set of 2×2 homogeneous systems $\tilde{K}^1_{2,0}$ of degree 1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For this assume that $a, b, c, d, e, f \in \mathbb{K}$. Then an explicit isomorphism is given through:

$$\begin{array}{rccc} \varphi : & \tilde{K}^1_{2,0} & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ & \begin{pmatrix} as+bt & cs+dt \\ e & f \end{pmatrix} & \longmapsto & (e, f), (af - ce, de - bf) \end{array} \tag{2.10}$$

3 Proof of the main theorem and further results

Let d, m and p be fixed positive integers. Set $k = m + p$. S_d will denote the vector space of homogeneous polynomials of degree d in two variables s and t with coefficients in a field $\mathbb{K}$. We do not assume that $\mathbb{K}$ is algebraically closed or that it has characteristic zero. S_d is a $\mathbb{K}$-vector space of dimension $d + 1$. The standard ordered basis that we will use for S_d is $\{s^d, s^{d-1}t, \ldots, t^d\}$.

Definition 3.1 Let X be the set of all $p \times k$ matrices

$$A = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1k} \\ f_{21} & f_{22} & \cdots & f_{2k} \\ \vdots & \vdots & & \vdots \\ f_{p1} & f_{p2} & \cdots & f_{pk} \end{pmatrix} := \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix} \tag{3.1}$$

where $f_{ij} \in S_{d_i}$ is a homogeneous polynomial of degree d_i. We assume that $\sum_{i=1}^p d_i = d$ and that at least one of the maximal minors of the matrix is a nonzero polynomial, necessarily of degree d.

We allow the row-degrees $(d_1, \dots, d_p)$ to vary subject to the restriction that their sum is d. The condition on the non-vanishing of a maximal minor is equivalent to assuming that as a matrix of polynomials, A is "generically surjective". We define $\alpha = k(d+1)$ and $\beta = pd - d + p$. We shall now define a map ϕ from X to $\mathbb{M}_{\beta,\alpha}$, the set of all $\beta \times \alpha$ matrices with constant entries. With the notation for $A \in X$ as above, let

$$f_{ij} = \sum_{l=1}^{d_i+1} a_{ij}^l s^{d_i-l+1} t^{l-1}. \tag{3.2}$$

We set $a_{ij}^l = 0$ for $l > d_i$ and also for $l \leq 0$ and for all j. In order to describe the image of A we define first the matrices

$$A^j = \begin{pmatrix} a_{11}^j & \dots & a_{1k}^j \\ a_{11}^{j-1} & \dots & a_{1k}^{j-1} \\ \vdots & & \vdots \\ a_{11}^{j-d+d_1} & \dots & a_{1k}^{j-d+d_1} \\ a_{21}^j & \dots & a_{2k}^j \\ \vdots & & \vdots \\ a_{21}^{j-d+d_2} & \dots & a_{2k}^{j-d+d_2} \\ \vdots & & \vdots \\ a_{p1}^{j-d+d_p} & \dots & a_{pk}^{j-d+d_p} \end{pmatrix}. \tag{3.3}$$

Using this notation the image $\phi(A)$ is given as follows:

$$\phi(A) = (A^1 | A^2 | \dots | A^{d+1}) \tag{3.4}$$

So $\phi(A)$ is made up of $(d+1)$ vertical blocks of k columns each and m horizontal blocks. The i-th horizontal block has $d - d_i + 1$ rows.

Before proceeding any further, we wish to describe this map ϕ intrinsically . Let V_k and V_p be $\mathbb{K}$-vector spaces of dimension k and p respectively. Choose ordered bases $(u_1, u_2, \dots, u_k)$ and $(v_1, \dots, v_p)$ for V_k and V_m respectively. The matrix A defines a map, ρ_A from $W_p = \bigoplus_{i=1}^p S_{d-d_i} \cdot v_i$ to $W_k = \bigoplus_{i=1}^k S_d \cdot u_i$ as follows:

$$\rho_A(\sum_1^p g_i v_i) = (g_1, g_2, \dots, g_p) \cdot A. \tag{3.5}$$

Now, choose the ordered $\mathbb{K}$-basis $\{s^d \cdot u_1, s^d \cdot u_2, \dots s^d \cdot u_k, s^{d-1} t \cdot u_1, \dots t^d \cdot u_k\}$ for W_k and the ordered $\mathbb{K}$-basis $\{s^{d-d_1} \cdot v_1, s^{d-d_1-1} t \cdot v_1, \dots t^{d-d_1} \cdot v_1, \dots, s^{d-d_p} v_p, \dots, t^{d-d_p} v_p\}$ for W_p. Then $\phi(A)$ is the matrix of ρ_A as a map of vector spaces, with the bases for W_p and W_k chosen above.

Remark 3.2 Since our descriptions of ϕ depends on the choice of the bases for W_k and W_p, we wish to point out that there are two group actions on the image $\phi(X)$ in $\mathbb{M}_{\beta,\alpha}$:

1. The group PGL(2) acts on $\mathbb{P}^1$ by changing coordinates (s,t) to (s',t'). Each element of PGL(2) induces a bijective map between $\phi(X)$ in these two coordinate systems.

2. The group GL(V_k) acts on V_k in the natural way. If $g \in$GL(V_k), then there is an $\alpha \times \alpha$ matrix B, which is the matrix of the isomorphism induced by g on W_k with respect to the canonical bases chosen. Post multiplication by B sets up a bijection between the matrices $\phi(A)$ and $\phi'(A)$ defined with these different bases on V_k.

Definition 3.3 Fix positive integers $\beta_1 \leq \beta_2 \ldots \leq \beta_p$ such that $\sum \beta_i = \beta$. Let $\gamma_0 = 0$ and let $\gamma_i = \sum_1^i \beta_j$ for $i = 1, \ldots, p$. We say that a matrix $A \in \mathbb{M}_{\beta,\alpha}$ is in canonical $(\beta_1, \ldots, \beta_p)$ form, if $A = (A^1|A^2|\ldots|A^{d+1})$ and for $\beta_l < j \leq \beta_{l+1}$ we have A^j in the following form:

$$A^j = \begin{pmatrix} a^j_{11} & \cdots & a^j_{1l} & 0 & 0 & \ldots & 0 & a^j_{1,p+1} & \cdots & a^j_{1k} \\ a^j_{21} & \cdots & a^j_{2l} & 0 & 0 & \ldots & 0 & a^j_{2,p+1} & \cdots & a^j_{2k} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ a^j_{\gamma_l+j,1} & \cdots & a^j_{\gamma_l+j,l} & 1 & 0 & \ldots & 0 & a^j_{\gamma_l+j,p+1} & \cdots & a^j_{\gamma_l+j,k} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \\ a^j_{\gamma_{l+1}+j,1} & \cdots & a^j_{\gamma_{l+1}+j,l+1} & 0 & 1 & \ldots & 0 & a^j_{\gamma_{l+1}+j,p+1} & \cdots & a^j_{\gamma_{l+1}+j,k} \\ \vdots & & \vdots & \vdots & \vdots & & & \vdots & & \\ a^j_{\gamma_{l+p}+j,1} & \cdots & a^j_{\gamma_{l+p}+j,l+1} & 0 & 0 & \ldots & 1 & a^j_{\gamma_{l+p}+j,p+1} & \cdots & a^j_{\gamma_{l+p}+j,k} \\ a^j_{\beta 1} & \cdots & a^j_{\beta l} & 0 & 0 & \ldots & 0 & a^j_{\beta,p+1} & \cdots & a^j_{\beta k} \end{pmatrix}. \quad (3.6)$$

Proposition 3.4 *Let $A \in X$ have row-degrees $d_1 \geq d_2 \geq \ldots \geq d_p$. Let $\beta_i = d - d_i + 1$. Then after an appropriate change of coordinates on $\mathbb{P}^1$ and a change of basis for V_k, $\phi(A)$ is row equivalent to a matrix that is in canonical $(\beta_1, \ldots, \beta_p)$ form.*

Proof: Firstly, by changing basis on V_k, we can assume that the determinant of the first p columns of A is non-zero. Further, by changing coordinates on $\mathbb{P}^1$, we can assume that the coefficient of s^d in this determinant is non-zero. Let

$$A = \begin{pmatrix} a^1_{11}s^{d_1} & a^1_{12}s^{d_1} & \dots a^1_{1,k}s^{d_1} \\ a^1_{21}s^{d_2} & a^1_{22}s^{d_2} & \dots a^1_{2,k}s^{d_2} \\ \vdots & \vdots\vdots & \vdots \\ a^1_{p1}s^{d_p} & a^1_{p2}s^{d_p} & \dots a^1_{p,k}s^{d_p} \end{pmatrix} + \text{ terms of lower degree in } s. \tag{3.7}$$

Let

$$A_0 = \begin{pmatrix} a^1_{11} & a^1_{12} & \dots & a^1_{1,k} \\ a^1_{21} & a^1_{22} & \dots & a^1_{2,k} \\ \vdots & \vdots\vdots & \vdots & \\ a^1_{p1} & a^1_{p2} & \dots & a^1_{p,k} \end{pmatrix} \text{ and } A_{0p} = \begin{pmatrix} a^1_{11} & a^1_{12} & \dots & a^1_{1,p} \\ a^1_{21} & a^1_{22} & \dots & a^1_{2,p} \\ \vdots & \vdots\ddots & \vdots & \\ a^1_{p1} & a^1_{p2} & \dots & a^1_{p,p} \end{pmatrix} \tag{3.8}$$

By our choice of coordinates, A_{0p} is an invertible matrix. After a further reordering of the basis of V_k, we can assume that for $j = 1, \dots, p$, the matrix

$$\begin{pmatrix} a^1_{j1} & \dots & a^1_{jp} \\ \vdots & & \vdots \\ a^1_{p1} & \dots & a^1_{pp} \end{pmatrix} \text{ is row equivalent to } \begin{pmatrix} * & * & * & \dots & * \\ \vdots & \vdots & & I_{p-j+1} & \\ * & * & & & \end{pmatrix}. \tag{3.9}$$

Now

$$A^1 = \begin{pmatrix} a^1_{11} & \dots & a^1_{1k} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ a^1_{21} & \dots & a^1_{2k} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ a^1_{p1} & \dots & a^1_{pk} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}. \tag{3.10}$$

By our assumption on A_{0p}, we can row reduce the matrix A^1 to:

$$A^1 = \begin{pmatrix} 1 & 0 & \dots & 0 & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 1 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.11}$$

We perform these row operations on all of $\phi(A)$. We observe that these row operations do not effect a row l, if $l \neq \gamma_i$ for any i. Now, in A^2, the p rows γ_i+1, for $i=1,\ldots,p$ form the matrix A_{0p}. As in the first step, we can perform row operations on $\phi(A)$ to get A^2 in canonical $(\beta_1,\ldots,\beta_p)$ form. Again the row operations performed on $\phi(A)$ so far, do not affect row l, so long as $l \neq \gamma_i$ and $l \neq \gamma_i+1$ for any i.

Hence, the procedure above can be repeated, inductively, until A^{β_1}. For $\beta_1 < j \leq \beta_2$, the matrix A^j contains the last $(p-1)$ rows of A_{0p}. Again, we can do row operations on $\phi(A)$, that will row reduce it to a matrix, where A^j is in $(\beta_1,\ldots,\beta_p)$ canonical form. We continue this sequence of row operations, until we reach the block A^{β_p}. The resulting matrix $\phi(A)$ is in $(\beta_1,\ldots,\beta_p)$ canonical form. Q.e.d.

Corollary 3.5 *For each $A \in X$, the matrix $\phi(A)$ has full rank $\beta = pd-d+p$. Therefore, ϕ defines a map $\phi : X \to \mathrm{Grass}(\beta, W_k)$ obtained by mapping a matrix A in X to the row space of the matrix $\phi(A)$.*

Proof: By the previous Proposition, we know that if $A \in X$ then there exist invertible matrices C and D such that $CAD = B$ where B is in $(\beta_1,\ldots,\beta_p)$ canonical form. Therefore the rank of A is the same as the rank of B, which is clearly β. Q.e.d.

Remark 3.6 We would like to note for future reference that the group actions specified in Remark 3.2 on $\phi(X)$ arise from the action of a subgroup of $\mathrm{GL}(W_k)$ on the Grassmannian.

We now have a map from our space X to the Grassmannian of β-dimensional planes in $W_k = V_k \otimes S_d$. First of all we want to identify the image of X in the Grassmannian through this map. Secondly, we wish to show that the map ϕ is one-to-one on certain equivalence classes of matrices in X. To achieve the first objective, we introduce a map ψ from our Grassmannian into the space of $(2\beta)\times(\alpha+k)$ matrices with constant entries. Let $W \in \mathrm{Grass}(\beta, W_k)$ be a β plane. Recall that we have a canonical basis for W_k. Choose matrices

$$A^j = \begin{pmatrix} a^j_{11} & \ldots & a^j_{1k} \\ a^j_{21} & \ldots & a^j_{2k} \\ \vdots & & \vdots \\ a^j_{\beta 1} & \ldots & a^j_{\beta k} \end{pmatrix} \tag{3.12}$$

such that the $\beta \times \alpha$ matrix

$$A_W := (A^1|A^2|\ldots|A^{d+1}) \tag{3.13}$$

has row space equal to W. We set $a^0_{ij} = a^{d+2}_{ij} = 0$ for all i and j and define:

$$A^{(j,j+1)} := \begin{pmatrix} a^{j+1}_{11} & a^{j+1}_{12} & \ldots & a^{j+1}_{1k} \\ a^{j}_{11} & a^{j}_{12} & \ldots & a^{j}_{1k} \\ a^{j+1}_{21} & a^{j+1}_{22} & \ldots & a^{j+1}_{2k} \\ a^{j}_{21} & a^{j}_{22} & \ldots & a^{j}_{2k} \\ \vdots & \vdots & & \vdots \\ a^{j}_{\beta 1} & a^{j}_{\beta 2} & \ldots & a^{j}_{\beta k} \end{pmatrix}. \tag{3.14}$$

The image of the map ψ is now defined through:

$$\psi(A_W) := (A^{(0,1)}|A^{(1,2)}|\ldots|A^{(d+1,d+2)}). \tag{3.15}$$

The map ψ depends on the choice of the representative A_W for the plane W, but if we choose another representation B_W for W, then it is easy to see that $\psi(A_W)$ is row equivalent to $\psi(B_W)$. Therefore, the following definition makes sense.

Definition 3.7 Let

$$\tilde{K} = \{W \in \text{Grass}(\beta, W_k) \,|\, \text{rank of } \psi(A_W) = \beta + p\}. \tag{3.16}$$

This subset of the Grassmannian is the main object of study in this paper.

Theorem 3.8 *$\tilde{K} = \phi(X)$.*

Proof: To show that $\phi(X)$ is contained in $\tilde{K}$, it suffices to observe that each matrix A in X gives rise to a map

$$\bigoplus_{i=1}^{p} S_{d+1-d_i} \cdot v_i \xrightarrow{A_{d+1}} \bigoplus_{i=1}^{k} S_{d+1} \cdot u_i. \tag{3.17}$$

The row space of $\psi(\phi(A))$ is the image of the map A_{d+1}. As in 3.5, this map is injective. Therefore the rank of $\psi(\phi(A))$ is the sum of the dimensions of S_{d+1-d_i} which is $\beta + p$.

The other inclusion is somewhat harder and we will use some techniques from algebraic geometry in the proof. Let $W \in \tilde{K}$. So $W \subset V_k \otimes S_d$. Consider the subsheaf $\mathcal{L}$ of $V_k \otimes O_{\mathbb{P}^1}(d)$ defined as follows:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{M}_{\mathcal{W}} & \to & W \otimes O_{\mathbb{P}^1} & \to & \mathcal{L} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & (V_k \otimes S_d) \otimes O_{\mathbb{P}^1} & \to & V_k \otimes O_{\mathbb{P}^1}(d) & \to & 0 \end{array} \tag{3.18}$$

Here $\mathcal{M}_{\mathcal{W}}$ is just defined to be the kernel of the middle right map. Observe that both $\mathcal{L}$ and $\mathcal{M}_{\mathcal{W}}$ are locally free sheaves on $\mathbb{P}^1$. Therefore, $\mathcal{M}_{\mathcal{W}} = \bigoplus_{i=1}^{\beta-l} O_{\mathbb{P}^1}(-a_i)$ and $\mathcal{L} = \bigoplus_{i=1}^{l} O_{\mathbb{P}^1}(b_i)$ and $\sum a_i = \sum b_i = \delta$. Further, since $\mathcal{L}$ is generated by sections and since it is a subsheaf of $V_k \otimes O_{\mathbb{P}^1}(d)$, we have that $0 \leq b_i \leq d$. Taking cohomologies, on the top row of 3.18, we get

$$0 \to H^0(\mathbb{P}^1, \mathcal{M}_{\mathcal{W}}) \xrightarrow{\sigma} W \to H^0(\mathcal{L}) \to H^1(\mathbb{P}^1, \mathcal{M}_{\mathcal{W}}) \to 0 \tag{3.19}$$

The map σ is injective by construction. Therefore, $H^0(\mathbb{P}^1, \mathcal{M}_{\mathcal{W}}) = 0$. Hence $a_i \geq 1$. On twisting the middle row of 3.18 by $O_{\mathbb{P}^1}(1)$ and taking cohomologies again, one gets:

$$\begin{array}{ccccccc} 0 & \to & H^0(\mathbb{P}^1, \mathcal{M}_{\mathcal{W}}(1)) & \to & W \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) & \to & H^0(\mathbb{P}^1, \mathcal{L}(1)) \\ & & & & & & \downarrow \\ & & & & & & V_k \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(d+1)) \end{array} \tag{3.20}$$

The composition indicated is the map whose matrix is $\psi(A_W)$. Since W is in $\tilde{K}$, the image of ψ has rank $\beta+p$. Therefore the dimension of $H^0(\mathbb{P}^1, \mathcal{M}_{\mathcal{W}}(1)) = 2\beta - \beta - p = \beta - p$. Thus the rank of $\mathcal{M}_{\mathcal{W}} = \beta - l \geq \beta - p$ and $l \leq p$. Also, Dim $H^0(\mathbb{P}^1, \mathcal{L}) = l + \delta$. From the sequence 3.19 one has,

$$\begin{aligned} \operatorname{Dim} H^1(\mathbb{P}^1, \mathcal{M}_{\mathcal{W}}) &= \operatorname{Dim} H^0(\mathbb{P}^1, \mathcal{L}) - \beta \\ &= l + \delta - \beta \\ &= l + \delta - (pd - d + p) \\ &\leq 0 \text{ if } l \leq p. \end{aligned} \tag{3.21}$$

Therefore, $l = p$, $\delta = \beta - p$ and $\mathcal{L} \simeq \bigoplus_{i=1}^{p} O_{\mathbb{P}^1}(b_i)$. So we have:

$$0 \to \bigoplus_{i=1}^{p} O_{\mathbb{P}^1}(b_i - d) \xrightarrow{\tau} V_k \otimes O_{\mathbb{P}^1}. \tag{3.22}$$

The map τ is given by a $p \times k$ matrix A whose, i-th row consists of homogeneous polynomials of degree $d_i = d - b_i$, where $\sum d_i = d$. So $A \in X$ and the row space of $\phi(A)$ is $H^0(\mathbb{P}^1, \mathcal{L}) = W$. Thus each $W \in \tilde{K}$ arises as $\phi(A)$ for some $A \in X$. Q.e.d.

We now introduce an equivalence relation on the space of matrices X. Let $d_1 \geq d_2 \ldots \geq d_p$ be nonnegative integers and let

$$\begin{aligned} G \;=\; & \{U(s,t) = (p_{ij}(s,t)) \mid U(s,t) \text{ is unimodular i.e. } \det U(s,t) \in \mathbb{K}^* \\ & \text{and } p_{ij}(s,t) \text{ is homogeneous of degree } d_i - d_j\}. \end{aligned} \tag{3.23}$$

It is clear that G is a subgroup of the unimodular group acting transitively on the set of matrices with row-degrees $d_1 \geq d_2 \ldots \geq d_p$. Indeed this group was of crucial importance in the paper [14]. Using this group we define on X the following equivalence relation:

Definition 3.9 Two matrices A and A' in X are equivalent if after a possible reordering of the rows they have the same row-degrees $d_1 \geq d_2 \ldots \geq d_p$ and if there exists a $U(s,t) \in G$ with $A' = U(s,t)A$.

Note that the set of equivalence classes in X is exactly the space $\tilde{K}^d_{p,m}$ of homogeneous autoregressive systems as introduced in Definition 2.3.

It is easy to see from our description of the map ϕ, that if A and A' are equivalent, then $\phi(A)$ and $\phi(A')$ define the same plane in the Grassmannian.

Proposition 3.10 $\phi : X \to \text{Grass}(\beta, W_k)$ *gives a one-to-one map on equivalence classes of matrices in* X*. Therefore* $\phi : \tilde{K}^d_{p,m} \to \tilde{K} \subset \text{Grass}(\beta, W_k)$ *is a bijection.*

Proof: From the remarks preceding the statement, we only need to verify that if $\phi(A) = \phi(A') = W$, then A and A' are equivalent. From the proof of the Theorem 3.8, it is clear that $W = H^0(\mathbb{P}^1, \mathcal{L})$, where $\mathcal{L}$ is the subsheaf generated by W. Now, A and A' are two matrix representations of the inclusion map of sheaves $\mathcal{L} \hookrightarrow V_k \otimes O_{\mathbb{P}^1}$. Two such representations differ by an isomorphism of the sheaf $\bigoplus_{i=1}^p O_{\mathbb{P}^1}(b_i)$ to itself. All isomorphisms of this sheaf are given by matrices in the group G defined above. Thus A and A' are equivalent in X. Q.e.d.

Theorem 3.11 $\tilde{K}$ *is a smooth, connected, algebraic subvariety of* $\text{Grass}(\beta, W_k)$.

Corollary 3.12 $\tilde{K}^d_{p,m}$ *has the structure of a smooth, connected and compact manifold.*

Before we give the proof of Theorem 3.11 we reinterpret Example 2.6.

Example 3.13 The set $\tilde{K}^1_{2,0}$ is embedded through ϕ in $\mathrm{Grass}(3, \mathbb{K}^4) \cong \mathbb{P}^3$ as follows:

$$\begin{array}{ccc} \phi : \quad \tilde{K}^1_{2,0} & \longrightarrow & \mathrm{Grass}(3, \mathbb{K}^4) \\ \begin{pmatrix} as+bt & cs+dt \\ e & f \end{pmatrix} & \longmapsto & \begin{pmatrix} a & c & b & d \\ e & f & 0 & 0 \\ 0 & 0 & e & f \end{pmatrix} \end{array} \tag{3.24}$$

Using Plücker coordinates one verifies that the image of ϕ is $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding.

Proof of Theorem 3.11: By Theorem 3.8, each point $W \in \tilde{K}$ is $\phi(A)$ for some matrix $A \in X$. Further, by Proposition 3.4 and remark 3.6, there is an isomorphism $g \in \mathrm{GL}(W_k)$ of the Grassmannian, and there exist positive integers $\beta_1 \leq \ldots \leq \beta_p$ such that a matrix representation of $g(W)$ is in canonical $(\beta_1, \ldots, \beta_p)$ form. From here on we replace W by $g(W)$. Let $\mathcal{U} = \mathcal{U}_{(\beta_1,\ldots,\beta_p)}$ be the set of all matrices in $\mathbb{M}_{\beta\times\alpha}$ which are in canonical $(\beta_1, \ldots, \beta_p)$ form. $\mathcal{U}$ can be naturally identified with an affine open subset of the Grassmannian. We shall parametrize $\tilde{K} \cap \mathcal{U}$ and show that this is a smooth affine subset of $\mathcal{U}$.

Let $A \in \mathcal{U}$. $\psi(A) = (A^{(1,0)}|A^{(1,2)}|\ldots|A^{(d+1,d+2)})$ where for $\beta_l \leq j \leq \beta_{l+1}$ we have:

$$A^{(j,j+1)} = \begin{pmatrix} a^{j+1}_{11} & \ldots & a^{j+1}_{1l} & 0 & 0 & \ldots & 0 & a^{j+1}_{1,p+1} & \ldots & a^{j+1}_{1k} \\ a^{j}_{11} & \ldots & a^{j}_{1l} & 0 & 0 & \ldots & 0 & a^{j}_{1,p+1} & \ldots & a^{j}_{1k} \\ a^{j+1}_{21} & \ldots & a^{j+1}_{2l} & 0 & 0 & \ldots & 0 & a^{j+1}_{2,p+1} & \ldots & a^{j+1}_{2k} \\ a^{j}_{21} & \ldots & a^{j}_{2l} & 0 & 0 & \ldots & 0 & a^{j}_{2,p+1} & \ldots & a^{j}_{2k} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ a^{j+1}_{\gamma_l+j-1,1} & \ldots & a^{j+1}_{\gamma_l+j-1,l} & 1 & 0 & \ldots & 0 & a^{j+1}_{\gamma_l+j-1,p+1} & \ldots & a^{j+1}_{\gamma_l+j-1,k} \\ a^{j}_{\gamma_l+j,1} & \ldots & a^{j}_{\gamma_l+j,l} & 1 & 0 & \ldots & 0 & a^{j}_{\gamma_l+j,p+1} & \ldots & a^{j}_{\gamma_l+j,k} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \\ a^{j}_{\beta 1} & \ldots & a^{j}_{\beta l} & 0 & 0 & \ldots & 0 & a^{j}_{\beta,p+1} & \ldots & a^{j}_{\beta k} \end{pmatrix}. \tag{3.25}$$

So $\psi(A)$ is a $(2\beta) \times (\alpha + k)$ matrix. We perform the following row operations on $\psi(A)$: for $i = 1, \ldots, p$, and $1 \leq j \leq \beta_{i+1} - 1$, subtract the $(2\gamma_i + 2j)$ -row from the $(2\gamma_i + 2j + 1)$-row of $\psi(A)$.

After these row operations the matrix $\psi(A) = (A^{(1,0)}|A^{(1,2)}|\dots|A^{(d+1,d+2)})$ where for $\beta_l \leq j \leq \beta_{l+1}$, $A^{(j,j+1)}$ is given by:

$$\begin{pmatrix} a_{11}^{j+1} & \dots & a_{1l}^{j+1} & 0 & 0 & \dots & 0 & a_{1,p+1}^{j+1} & \dots & a_{1k}^{j+1} \\ a_{11}^{j} & \dots & a_{1l}^{j} & 0 & 0 & \dots & 0 & a_{1,p+1}^{j} & \dots & a_{1k}^{j} \\ a_{21}^{j+1} & \dots & a_{2l}^{j+1} & 0 & 0 & \dots & 0 & a_{2,p+1}^{j+1} & \dots & a_{2k}^{j+1} \\ a_{21}^{j}-a_{21}^{j+1} & \dots & a_{2l}^{j}-a_{2l}^{j+1} & 0 & 0 & \dots & 0 & a_{2,p+1}^{j}-a_{2,p+1}^{j+1} & \dots & a_{2k}^{j}-a_{2k}^{j+1} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ a_{\gamma_l+j-1,1}^{j+1} & \dots & a_{\gamma_l+j-1,l}^{j+1} & 1 & 0 & \dots & 0 & a_{\gamma_l+j-1,p+1}^{j+1} & \dots & a_{\gamma_l+j-1,k}^{j+1} \\ a_{\gamma_l+j,1}^{j}-a_{\gamma_l+j-1,1}^{j+1} & \dots & a_{\gamma_l+j,l}^{j}-a_{\gamma_l+j-1,l}^{j+1} & 0 & 0 & \dots & 0 & a_{\gamma_l+j,p+1}^{j}-a_{\gamma_l+j-1,p+1}^{j+1} & \dots & a_{\gamma_l+j,k}^{j}-a_{\gamma_l+j-1,k}^{j+1} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \\ a_{\beta 1}^{j} & \dots & a_{\beta l}^{j} & 0 & 0 & \dots & 0 & a_{\beta,p+1}^{j} & \dots & a_{\beta k}^{j} \end{pmatrix}. \tag{3.26}$$

This matrix has β standard unit vectors as columns. Let B be the submatrix of A obtained by deleting the rows that contain the non-zero entry in these standard column vectors. $B = (B^{(0,1)}|B^{(1,2)}|\dots|B^{(d+1,d+2)})$ where:

$$B^{(0,1)} = \begin{pmatrix} a_{2,p+1}^{1} & \dots & a_{2k}^{1} \\ a_{3,p+1}^{1} & \dots & a_{3,p+1}^{1} \\ \vdots & & \vdots \\ a_{\gamma_1,p+1}^{1} & \dots & a_{\gamma_1,k}^{1} \\ 0 & \dots & 0 \\ a_{\gamma_1+1,p+1}^{1} & \dots & \\ \vdots & & \vdots \\ 0 & \vdots & 0 \end{pmatrix}. \tag{3.27}$$

$$B^{(1,2)} = \begin{pmatrix} 0 & \dots & 0 & a_{2,p+1}^{2}-a_{1,p+1}^{1} & \dots & a_{2,k}^{2}-a_{1,k}^{1} \\ 0 & \dots & 0 & a_{3,p+1}^{2}-a_{2,p+1}^{1} & \dots & a_{3,p+1}^{2}-a_{2,k}^{1} \\ \vdots & & \vdots & & & \\ 0 & \dots & 0 & a_{\beta,p+1}^{1} & \dots & a_{\beta k}^{1} \end{pmatrix}. \tag{3.28}$$

For, $1 \leq i \leq p$,

$$B^{(\beta_i,\beta_i+1)} = \begin{pmatrix} a_{21}^{\beta_i+1} & 0 & \dots & 0 & a_{2,p+1}^{\beta_i+1}-a_{1,p+1}^{\beta_i} & \dots & a_{2k}^{\beta_i+1}-a_{1k}^{\beta_i} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{\gamma_1,1}^{\beta_i+1} & 0 & \dots & 0 & a_{\gamma_1,p+1}^{\beta_i+1}-a_{\gamma-1,p+1}^{\beta_i} & \dots & a_{\gamma_1 k}^{\beta_i+1}a_{\gamma_1-1k}^{\beta_i} \\ 1 & 0 & \dots & 0 & a_{\gamma_1,p+1}^{\beta_i} & \dots & a_{\gamma_1 k}^{\beta_i} \\ a_{\gamma_1+2,1}^{\beta_i+1} & 0 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & a_{\beta,p+1}^{\beta_i} & \dots & a_{\beta k}^{\beta_i} \end{pmatrix}. \tag{3.29}$$

We observe that the rank of $\psi(A)$ is β plus the rank of B. Consider the $p \times p$ submatrix of B obtained by choosing the rows $\gamma_1, \ldots, \gamma_p$ and the i-th column of $B^{(\beta_i, \beta_i+1)}$. This is a $p \times p$ identity matrix. Hence rank of B is at least p. Thus

$$\begin{aligned} \tilde{K} \cap \mathcal{U} &= \{A \in \mathcal{U} \quad | \quad \psi(A) \text{ has rank } \beta + p\} && (3.30) \\ &= \{A \in \mathcal{U} \quad | \quad \psi(A) \text{ has rank } \leq \beta + p\}. && (3.31) \end{aligned}$$

The second subset is defined by setting all the minors of order greater than p to be zero. Hence, $\tilde{K} \cap \mathcal{U}$ is clearly an algebraic subset of $\mathcal{U}$. Since each point of $\tilde{K}$ has a neighborhood $\mathcal{U}$ in G for which $\tilde{K} \cap \mathcal{U}$ is an algebraic subvariety, $\tilde{K}$ itself is an algebraic subvariety of Grass(β, W_k). Before starting the proof of the smoothness of $\tilde{K}$, we need to introduce some more notation.

$$S = \{a^j_{l,q} \quad | \quad \begin{array}{l} \text{either } \exists i \text{ s.t. } l = \gamma_i \text{ and } q \geq p+1 \text{ and } j \geq \beta_i + 1 \\ \text{or } q \leq p \text{ and } j = \beta_q + 1 \text{ and } l \geq \gamma_q + 1\}. \end{array} \tag{3.32}$$

Also, for $1 \leq i \leq p$ and $j \leq d+1$, we let

$$A^j_i = \{a^j_{st} \quad | \gamma_{i-1} + 1 \leq s \leq \gamma_i \text{ and } 1 \leq t \leq k\}. \tag{3.33}$$

Let $A \in \tilde{K} \cap \mathcal{U}$. Let $1 \leq l \leq \beta$ and $l \neq \gamma_i$ for any i. Let $p+1 \leq q \leq k$. Let $1 \leq i \leq p$. Consider the $(p+1) \times (p+1)$ submatrix of B obtained by adding the l-th row and the q-th column to the identity submatrix. This submatrix is of the form:

$$\begin{pmatrix} & & & * & \ldots & * \\ & I_i & & * & \ldots & * \\ 0 & \ldots & 0 & a^1_{l,q} & & \\ \vdots & & \vdots & * & I_{p-i} & \\ 0 & \ldots & 0 & * & & \end{pmatrix}. \tag{3.34}$$

Since $A \in \tilde{K} \cap \mathcal{U}$, $B^{(0,1)} = 0$. Further, for all $i \geq 1$ $a^1_{l,q} = 0$ for $p+1 \leq q \leq k$ and $l \neq \gamma_i + 1$.

For $0 \leq j \leq \beta_1 - 2$ and $r \leq j$, we assume by induction on j that $B^{(r,r+1)} = 0$ for all $r \leq j$ and that $a^{r+1}_{l,q} = 0$ for $\gamma_{i-1} + r + 1 \leq l \leq \gamma_i$. We can also assume that for $\gamma_i - r - 1 \leq l \leq \gamma_{i-1}$, $a^r_{l,q}$ is a function of $A^{(j+1)}_i$ for all $r \leq j$. We consider the $(p+1) \times (p+1)$ submatrix of B obtained by adding the l-th row and the q-th column of the $B^{(j+1,j+2)}$ block to the identity submatrix of B. This matrix is of the form:

$$\begin{pmatrix} & & & * & & \dots & * \\ & I_i & & * & & \dots & * \\ 0 & \dots & 0 & a^{(j+2)}_{l+1,q} - a^{(j+1)}_{l,q} & & & \\ \vdots & & \vdots & a^{(j+1)}_{\gamma_1,q} = 0 & & I_{p-i} & \\ 0 & \dots & 0 & a^{(j+1)}_{\gamma_p,q} = 0 & & & \end{pmatrix}. \tag{3.35}$$

Again on $\tilde{K} \cap \mathcal{U}$, we get that $B^{(j+1,j+2)} = 0$. Further, for $\gamma_{i-1}+j+2 \le l \le \gamma_i$, $a^{(j+1)}_{l,q} = 0$ and for $\gamma_i - (j+1) - 1 \le l \le \gamma_i - 1$, $a^{(j+1)}_{l,q}$ is a function of $A^{(j+2)}_i$.

So by induction on j, we have that $B^{(j,j+1)} = 0$ for all $j \le \beta_1 - 1$. Also A^j_i is either 0 or can be expressed as a function of $A^{\beta_1}_i$ and $a^{\beta_1}_{l,q} = 0$ for $\gamma_i + \beta_1 \le l \le \gamma_{i+1}$.

We now subtract $(a^{\beta_1+1}_{l,1})$ times the γ_1-row of B from the l-th row, for $l \ge \gamma_1 + 1$ and $l \ne \gamma_i$ for any i. We shall call the resulting matrix B as well. Thus for $j \ge \beta_1$ and l as above, we have

$$b^{(j,j+1)}_{l,q} = a^{j+1}_{l+1,q} - a^j_{l,q} - a^j_{\gamma_1,q} \cdot a^{\beta_1+1}_{l+1,1} \text{ for } q = 1, p+1, p+2, \dots, k. \tag{3.36}$$

$$= a^{j+1}_{l+1,q} - a^j_{l,q} + \text{ a function of the variables in } S. \tag{3.37}$$

Now, let $i \ge 2$ and let $q = 1$ or $p+1 \le q \le k$. Let $\beta_1 \le j \le \beta_2 - 1$. Consider the submatrix of B obtained by adding the l-th row of B and the q-th column of the $B^{(j,j+1)}$ to the identity submatrix. This submatrix is of the form:

$$\begin{pmatrix} 1 & * & * & \dots & * \\ 0 & b^{(j,j+1)}_{l,q} & * & \dots & * \\ 0 & * & & & \\ 0 & * & & & I_{p-1} \\ 0 & * & & & \end{pmatrix}. \tag{3.38}$$

Hence $B^{(j,j+1)}_i = 0$. We consider two ranges for l.

1. Let $\gamma_{i+1} - 1 \ge l \ge \gamma_i + j + 1$. By induction on j we can assume that we have $a^j_{l,q} = f^j_{l,q}(S)$ as a function of the variables in S. Thus, $b^{(j,j+1)}_{l,q} = 0$ implies that $a^{j+1}_{l+1,q} = f^{j+1}_{l+1,q}(S)$, for l in this range.

2. Let $\gamma_i + j \ge l \ge \gamma_{i-1} + 1$. Here, we proceed by descending induction on j. We can assume that $a^{j+1}_{l+1,q} = f^{j+1}_{l+1,q}(S; A^{\beta_2}_i)$. Now, $b^{(j,j+1)}_{l,q} = 0$ implies that $a^j_{l,q} = f^j_{l,q}(S; A^{\beta_2}_i)$.

Thus, for $i \geq 1$, $b_{l,q}^{(j,j+1)} = 0$ for $l \geq \gamma_i + 1$ and $1 \leq j \leq \beta_2 - 1$. Further,

$$a_{l,q}^{j} = \begin{cases} f_{l,q}^{j}(S) & \text{if } \gamma_{i+1} - 1 \geq l \geq \gamma_i + j + 1 \\ f(S; A_i^{\beta_2}) & \text{if } \gamma_i + j \geq l \geq \gamma_{i-1} + 1 \end{cases} \tag{3.39}$$

Also, $a_{l,q}^{\beta_2} = f_{l,q}^{\beta_2}(S)$ if $\gamma_i + \beta_2 \leq l \leq \gamma_{i+1} - 1$.

Inductive step for i: Let $1 \leq i \leq p-1$. Let $1 \leq q \leq i$ or $p+1 \leq q \leq k$. Let $t \geq i+1$.By induction on i, we can assume that $b_{l,q}^{(j,j+1)} = 0$ for $j \leq \beta_i - 1$ and $l \geq \gamma_i + 1$. Further, we can assume that, for $j \leq \beta_i$,

$$a_{l,q}^{j} = \begin{cases} f_{l,q}^{j}(S) & \text{if } \gamma_{t+1} - 1 \geq l \geq \gamma_t + j + 1 \\ f(S; A_t^{\beta_i}) & \text{if } \gamma_t + j \geq l \geq \gamma_{t-1} + 1 \end{cases} \tag{3.40}$$

Also, $a_{l,q}^{\beta_i} = f_{l,q}^{\beta_i}(S)$ if $\gamma_t + \beta_i \leq l \leq \gamma_{t+1} - 1$.

For, $l \geq \gamma_i + 1$ and $l \neq \gamma_t$ for any t, we subtract $(a_{l,i}^{\beta_i+1})$ times the row γ_i of B from the l-th row. We call this matrix B again. Note that

$$b_{l,q}^{j,j+1} = a_{l+1,q}^{j+1} - a_{lq}^{j} - \text{ some function of } S. \tag{3.41}$$

Let $\beta_i \leq j \leq \beta_{i+1} - 1$. Consider the submatrix of B obtained by adding the l-th row of B and the q-th column of the $B^{(j,j+1)}$. This submatrix is of the form:

$$\begin{pmatrix} & & & * & \dots & * \\ & I_i & & * & \dots & * \\ 0 & \dots & 0 & b_{lq}^{(j,j+1)} & & \\ \vdots & & \vdots & * & I_{p-i} & \\ 0 & \dots & 0 & * & & \end{pmatrix}. \tag{3.42}$$

Again, on $\tilde{K} \cap \mathcal{U}$, $b_{l,q}^{(j,j+1)} = 0$ for $l \geq \gamma_i + 1$ and $0 \leq j \leq \beta_{i+1} - 1$. As before, to determine the values of a, we consider two ranges for l.

1. Let $\gamma_t - 1 \geq l \geq \gamma_t + j + 1$. By an ascending induction on j, we can assume that $a_{l,q}^{j} = f_{l,q}^{j}(S)$. Now, $b_{l,q}^{j,j+1} = 0$ implies that $a_{l+1,q}^{j+1} = f_{l+1,q}^{j+1}(S)$ is a function of S.

2. Let $\gamma_t + j \geq l \geq \gamma_{t-1} + 1$. Proceeding by a downward induction on j, we can assume that $a_{l+1,q}^{j+1} = f_{l+1,q}^{j+1}(S; A_t^{\beta_{i+1}})$. So $b_{l,q}^{j,j+1} = 0$ implies that $a_{l,q}^{j} = f_{l,q}^{j}(S; A_t^{\beta_{i+1}})$.

Thus, one is eventually left with a matrix B such that $b^{(j,j+1)}_{l,q} = 0$ for $1 \leq p$ and $\gamma_{t-1}+1 \leq l \leq \gamma_t - 1$ and $j \leq \beta_i - 1$. Also, for j and l in this range, either $a^j_{l,q} = f^j_{l,q}(S)$ or $a^j_{l,q} = f^j_{l,q}(S; A_{\beta_i,i})$. Since this is true for all $t \geq i+1$, this finishes the proof for the inductive step for i.

We next perform the following row operations on B. For $i = 1, \ldots, p$ and $\gamma_{i-1}+1 \leq l \leq \gamma_i - 1$, subtract $(a^{\beta_i+1}_{l,i})$ times the row γ_i from the l-th row. As usual, we will call the resulting matrix also B. The i-th column of the (β_i, β_i+1) block of B is a standard unit vector with a 1 in the row γ_i. Hence if $A \in \tilde{K} \cap \mathcal{U}$, then $b^{(j,j+1)}_{l,q} = 0$ if $l \neq \gamma_i$ for any i between 1 and p. Also, for $\beta_i \leq j \leq d+1$ and $\gamma_{i-1}+1 \leq l \leq \gamma_i - 1$,

$$b^{(j,j+1)} = a^{j+1}_{l+1,q} - a^j_{l,q} - \sum_{r=1}^{p} a^{\beta_r+1}_{l+1,r} \cdot a^r_{\gamma_r,q}. \tag{3.43}$$

For $j = d+1$, $a^{(j+1)}_{l,q} = 0$. We now proceed by descending induction on j. We can assume that $a^{j+1}_{l,q} = f^{j+1}_{l,q}(S)$.Thus $b^{(j,j+1)} = 0$ implies that we can solve for $a^j_{l,q} = f^j_{l,q}(S)$. So we have that if $A \in \tilde{K} \cap \mathcal{U}$, then $a^j_{l,q} = f^j_{l,q}(S)$ for all indices j, l and q.

Conversely, consider a matrix $A \in \mathcal{U}$, given by $f^j_{l,q}(S)$, where these are the functions found above. Let B be the matrix constructed from $\psi(A)$ as before. Due to the choice of the functions $f^j_{l,q}$, we see that $B^{(j,j+1)} = 0$ for $j \leq \beta_1 - 1$.

Applying the same sequence of row operations as before, we see that B is row equivalent to a matrix C that has m standard unit column vectors and if $l \neq \gamma_i$, then all the entries in the row l are zero. So $\psi(A)$ has rank $\beta + p$ and $A \in \tilde{K}$. So each matrix $A \in \tilde{K} \cap \mathcal{U}$ can be parameterized as $A = (f^j_{l,q}(S))$. Thus $\tilde{K} \cap \mathcal{U}$ is isomorphic to $\mathbb{K}^s$, where s is the cardinality of the set S.

$$\begin{aligned} s &= \left(\textstyle\sum_{i=1}^{p} m(d+1-\beta_i)\right) + p(m+p)(d-\beta_p) + \textstyle\sum_{i=1}^{p}(p-i)(\beta_p - \beta_i)) \\ &= (m+p)d + mp. \end{aligned} \tag{3.44}$$

Notice that s is independent of the choice of the p-tuple $(\beta_1, \ldots, \beta_p)$. Thus each point $W \in \tilde{K}$ has a neighborhood $\mathcal{U} = g^{-1}(\mathcal{U}_{(\beta_1,\ldots,\beta_p)})$ in $\mathrm{Grass}(\beta, W_k)$ such that $\tilde{K} \cap \mathcal{U} \simeq \mathbb{K}^s$. Thus $\tilde{K}$ is a nonsingular variety, rational variety of dimension s.

To see that $\tilde{K}$ is connected, let W_1 and W_2 be two points in $\tilde{K}$. There exist affine open sets, $\mathcal{U}_i \ni W_i$ and $\tilde{K} \cap \mathcal{U}_i \simeq \mathbb{K}^s$. Now, $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \phi$. Let $W \in \tilde{K} \cap \mathcal{U}_1 \cap \mathcal{U}_2$. Then there is a path γ_1 from W_1 to W in $\mathcal{U}_1$ and a path γ_2 from W to W_2 in $\mathcal{U}_2$. Thus $\tilde{K}$ is connected. Q.e.d.

4 A comparative study of different compactifications

It was Hazewinkel [5, Theorem 2.22] who first showed that any sequence of time invariant linear systems of McMillan degree at most n naturally converges to a singular system of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + D(\frac{d}{dt})u. \tag{4.1}$$

Moreover if one defines the McMillan degree for 4.1 as the sum of the McMillan degrees of $C(sI-A)^{-1}B$ and $D(s^{-1})$ then this McMillan degree is necessarily at most n and any system of type 4.1 having McMillan degree at most n can be obtained as the limit of a sequence of time invariant linear systems of McMillan degree n.

Probably the first explicit compactification of the space of proper transfer functions was introduced by Byrnes [1] who introduced this space in order to study the dynamic pole placement problem. The compactification of Byrnes was done completely in the frequency domain. The idea behind the compactification of Byrnes is an embedding of the set of all transfer functions in a large dimensional Grassmann variety and the closure serves as a compactification.

We are aware of two compactifications which were derived in the time domain.

One compactification was derived by the second author as the categorical quotient obtained from an action of a reductive group on a projective variety and details are given in [12].

Recently Helmke [7] proposed a compactification, which was partially derived by geometric invariant theory as well and which is based on earlier results derived together with Shayman [6]. This compact space contains the class of controllable singular systems of the form $E\dot{x} = Ax + Bu$; $Fy = Cx + Du$ where the regularity conditions $\det(\lambda E - \mu A) \neq 0$ and $\operatorname{rank}(F, C, D) = p$ are satisfied. Helmke shows the surprising result that the categorical quotient of this extended class of systems under an extended group action is compact and smooth.

It is interesting to compare the above three compactifications in the case $m = p = d = 1$, i.e. in the case of one-input, one-output and McMillan degree 1. In this case the compactification of Byrnes [1] (as well as the compactifications we will describe in a moment [9, 13, 16]) are equal to $\mathbb{P}^3$ whereas the compactification in [12] in this case is equal to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the compactification in [7] has the same homotopy type as $\mathbb{P}^1 \times \mathbb{P}^2$. In particular they are all different.

In [9] Lomadze considers a more general class of linear systems. In terms of his definition [9, Definition 1, Section 2], our space $\tilde{K}^d_{p,m}$ coresponds to the space of all completely observable systems of McMillan degree d, input number m and output number p. He shows that this space is compact using some techniques from Grothendieck's construction of Quot schemes. The smoothness of the space is not shown in [9].

Closely related to the compactification constructed in this paper is the one obtained by the second author in [13], denoted by $K^d_{p,m}$ there. For the pole placement problem with dynamic compensators this compactification is of interest. On one hand one can identify this compact space with the set of all autoregressive systems of order at most n and size $p \times (m+p)$. On the other hand the pole placement map with dynamic compensators can be viewed as a central projection from this projective variety to the space of closed loop polynomials. In the following we establish the relation between the compactification $K^d_{p,m}$ and the compactification $\tilde{K}^d_{p,m}$ presented in this paper.

Proposition 4.1 *There is a surjective map $\pi : \tilde{K}^d_{p,m} \to K^d_{p,m}$ such that π is an isomorphism on an open subset of $K^d_{p,m}$. In other words, $\tilde{K}^d_{p,m}$ is desingularization of $K^d_{p,m}$.*

Proof: Let $x \in \tilde{K}^d_{p,m}$. So $x \in \mathrm{Grass}(\beta, W_k)$ corresponds to an equivalence class of matrices in X. Choose a representative matrix A for x. Let $\pi(A) = (f_1, \dots f_N)$ be the ordered maximal minors of A. So each $f_i \in S_d$. If A' is another matrix that represents x, then $\pi(A')$ is obtained by multiplying $\pi(A)$ by a scalar in $\mathbb{K}$. Thus π defines a map $\pi : \tilde{K}^d_{p,m} \to \mathbb{P}(\bigoplus_1^N S_d)$ where $\mathbb{P}(\bigoplus_1^N S_d)$ is the projective space of all lines in the $\mathbb{K}$ vector space $\bigoplus_1^N S_d$. The space $K^d_{p,m}$ is, by definition the closure of $\pi(X)$ in $\mathbb{P}(\bigoplus_1^N S_d)$. Since $\tilde{K}^d_{p,m}$ is compact, π is surjective.

To see that π is an isomorphism on an open subset, let X' be the subset of all matrices A in X such that, the minors $\pi(A) = (f_1, \dots f_N)$ do not have any common factors. Then as is shown in [13], the map π restricted to $\phi(X') \subset \tilde{K}^d_{p,m}$ is an isomorphism.

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