On Decentralized Dynamic Pole Placement and Feedback Stabilization

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***Abstract*— In this paper we study the feedback control problem using an r-channel decentralized dynamic feedback control scheme. We will develop the theory in the behavioral framework. Using this framework we introduce an algebraic parameterization of the space of all possible feedback compensators having a bounded McMillan degree, and we show that this parameterization has the structure of an algebraic variety. We define the pole-placement map for this problem, and we give exact conditions when this map is onto, and almost onto. Finally we provide new necessary and sufficient conditions which guarantee that the set of stabilizable plants is a generic set.**

I. Introduction

FOR many large scale systems like electric power systems, transportation systems, and whole economic systems, it is desirable to decentralize the control task. This is in particular preferable if the measurements have been taken on decentralized local channels and the controls can be applied on local channels only. There are two major results about decentralized stabilization and pole assignment. Wang and Davison [16] proved that decentralized stabilization using local dynamic compensators is possible if and only if the fixed modes are stable. Corfmat and Morse [3] proved that a strongly connected system can be made controllable and observable through a single channel by local static feedback if and only if the set of fixed modes is empty. Thus the poles of such a system can be assigned freely. The control strategy of [3] is to apply local static feedback to all but one channel to make the resulting single channel system controllable. They identify a property of systems, called completeness, that is a necessary and sufficient condition for their control strategy to work. It is not clear to us that this property of completeness is a necessary condition for pole assignability, if one does not constrain all but one channel to have static feedback.

The problem we are interested in is to find minimal order decentralized dynamic compensators to stabilize or to assign the poles of a given system. Recent developments on centralized dynamic and decentralized static pole assignment [18], [12] indicate that the current estimates on the order of the dynamic compensators are too conservative. In [18], e.g., Wang proved that if the dimension of the set of all local compensators is greater than the McMillan degree, then almost all r-channel systems having the same number of inputs or outputs on the local channels are arbitrarily pole assignable by decentralized static output feedback. For example, almost all systems of McMillan degree 7 with two two-input two-output local channels are arbitrarily pole assignable by two static local controllers. This result compares very favorably to the earlier results in the literature and indeed even the strong results by Corfmat and Morse [3] require in this situation a static compensator and a dynamic compensator of McMillan degree at least three.

In this paper we will study the decentralized dynamic pole placement and stabilization problem. Crucial for this study is the associated pole-placement map (see Section III), which is a map from a parameterization of the set of all compensators to a parameterization of the set of closed-loop characteristic polynomials. Using an algebraic geometric framework we arrive at new general results describing exact conditions when the associated pole-placement map is onto, and almost onto. In addition we will report new necessary and sufficient conditions which guarantee that the set of stabilizable plants is a generic set. In special instances (e.g., if one only considers the centralized problem or if one restricts to static compensators), these results incorporate the ones reported in [1], [2], [10], [12], and [18].

First let us say a word about our methods. We use some ideas and results from algebraic geometry. Even though these are classical in mathematics, they have not been used very often in control theory. We have therefore made an attempt to provide examples that illustrate these ideas. In these illustrations we also show that for problems in low dimensions, one needs to solve a few linear and polynomial equations only. For readers who would like to learn more of algebraic geometry we highly recommend the graduate text book by Harris [5] which contains all the tools used in this paper.

The paper is organized as follows: In Section II we formulate the problem as it can be found, e.g., in [3] and [16]. We also establish the connection to the behavioral framework [20], since we believe that a general theory should incorporate singular systems and more general autoregressive (AR) systems in the sense of [10] and [14]. In a concluding paragraph we formulate our results in an intuitive, nontechnical form and discuss the nature of our results.

In Section III we introduce a new parameterization of the space of all possible feedback compensators which we will denote by $K^{\vec{q}}_{\vec{m},\vec{p}}$. This parameterization might be of

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independent interest for the study of questions like robustness, for instance. But we will use the set $K^{\vec{q}}_{\vec{m},\vec{p}}$ in this paper for our investigation on pole placement and stabilization only. As we will show in this section, $K^{\vec{q}}_{\vec{m},\vec{p}}$ is a projective variety and the pole-placement map from $K^{\vec{q}}_{\vec{m},\vec{p}}$ to the set of closed-loop characteristic polynomials is a central projection. From the properties of a central projection we deduce (Theorem 3.13 and Theorem 3.15) that the pole-placement map is onto over $\mathbb{C}$ if $\dim K^{\vec{q}}_{\vec{m},\vec{p}}$ is at least the degree of the desired closed-loop polynomial and if the center of this map intersects the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$ properly. If the degree of the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$ is in addition odd, we show that the pole-placement map is even onto over the reals $\mathbb{R}$. We work out an illustrative example of a two-channel system of McMillan degree three and show that it can be pole assigned by two decentralized feedback compensators, one static and the other of McMillan degree one. We conclude this section with a closed formula [formula (3.12)] of the topological degree of the pole-placement map in the critical dimension ($\dim K^{\vec{q}}_{\vec{m},\vec{p}}$ equals the degree of the closed-loop polynomial). By Theorem 3.13 this degree is equal to the degree of the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$.

In Section IV we study the important technical concept of $\vec{q}$-nondegeneracy (Lemma 4.1). Using this concept, we deduce (Corollary 4.2 and Corollary 4.5) that all the results derived in Section III hold generically.

In Section V we consider the problem of generic stabilization using decentralized controllers. The main result in this section is Theorem 5.1 which states that when $\dim K^{\vec{q}}_{\vec{m},\vec{p}}$ is strictly less than the degree of the closed-loop polynomial, then the set of systems that can be stabilized by real (or even complex) decentralized dynamic compensators is not a generic set. This result establishes a generalization of a result reported in [10] from the centralized (one-channel) to the decentralized (multichannel) situation. We conclude this section with an illustrative example which connects our framework with the concept of decentralized fixed modes as originally introduced by Wang and Davison [16].

Finally in the Appendix, the more technical proofs of this paper can be found.

II. Problem Formulation

Consider an r-channel linear system

$$\sigma\colon\ \dot{x} = Ax + \sum_{i=1}^{r} B_i u_i, \quad y_i = C_i x, \quad i = 1, 2, \cdots, r \tag{2.1}$$

where x, u_i, y_i are n, m_i, p_i vectors over $\mathbb{R}$, respectively, and u_i and y_i are the input and output of the ith channel. If dynamic compensators of order q_i

$$\dot{w}_i = F_i w_i + E_i y_i, \quad u_i = H_i w_i + K_i y_i, \quad i = 1, 2, \cdots, r \tag{2.2}$$

are applied to the local channels, respectively, the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ w \end{bmatrix} = \begin{bmatrix} A + BKC & BH \\ EC & F \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}, \qquad y = Cx \tag{2.3}$$

where

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_r \end{bmatrix},$$
$$B = [B_1, \cdots, B_r]$$

and K, H, E, F are the block diagonal matrices with $\{K_i\}$, $\{H_i\}$, $\{E_i\}$, $\{F_i\}$ on the diagonals, respectively. The closed-loop characteristic polynomial is

$$\begin{aligned} \phi(s) &= \det \begin{bmatrix} sI - A - BKC & -BH \\ -EC & sI - F \end{bmatrix} \\ &= \det(sI - A)\det(I - G(s)T(s))\det(sI - F) \end{aligned}$$

where $G(s) = C(sI - A)^{-1}B$ and $T(s) = K + H(sI - F)^{-1}E$ are the transfer functions of the system and compensator, respectively. Let $G(s) = D(s)^{-1}N(s)$ and $T(s) = T_d^{-1}(s)T_n(s)$ be left coprime factorizations over the polynomial ring. If representations (2.1) and (2.2) are minimal and if the coprime factorizations have in addition the property that $\det(sI - A) = \det D(s)$ and $\det(sI - F) = \det T_d(s)$ (note $\det(sI - A) = c\det D(s)$ for some nonzero number c generally), then it is well known that

$$\phi(s) = \det \begin{bmatrix} D(s) & N(s) \\ T_n(s) & T_d(s) \end{bmatrix}. \tag{2.4}$$

Remark 2.1: Since we are only interested in the zeros of $\phi(s)$, we can identify $\phi(s)$ with $c\phi(s)$ for any nonzero constant c. In this way, (2.4) will be determined uniquely by the system and compensators and not depend on particular coprime factorizations. Using a terminology from algebraic geometry we say that

$$\phi(s) = a_0 + a_1 + \cdots + a_{n+q}s^{n+q} \tag{2.5}$$

is a point in the projective $(n + q)$-space $\mathbb{P}^{n+q}$ with homogeneous coordinates

$$(a_0, a_1, \cdots, a_{n+q}). \tag{2.6}$$

Note that $T(s)$ is a block diagonal matrix. So we can take

$$T_n(s) = \text{block diag}(T_{n1}(s), \cdots, T_{nr}(s)),$$
$$T_d(s) = \text{block diag}(T_{d1}(s), \cdots, T_{dr}(s))$$

where $T_{di}^{-1}(s)T_{ni}(s) = K_i + H_i(sI - F_i)^{-1}E_i$. Let $Q_i(s) = [T_{ni}(s), T_{di}(s)]$. Then by exchanging the columns of (2.4) we get

$$\phi(s) = \det \begin{bmatrix} & P(s) & & \\ \hline Q_1(s) & 0 & \cdots & 0 \\ 0 & Q_2(s) & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & Q_r(s) \end{bmatrix}. \tag{2.7}$$

Such a formulation can be extended to include improper and singular systems. For this let us use the behavioral framework of Willems [20]. Let the plant P be described through a system of autoregressive equations (compare with [20])

$$P\left(\frac{d}{dt}\right)v(t) = 0. \tag{2.8}$$

Here we assume that $P(s) \in \mathbb{R}^{p\times(m+p)}[s]$ describes an autoregressive system as defined in [10] and [14] of size $p \times (m+p)$ and McMillan degree n.

It is our goal to stabilize or pole assign the plant $P(s)$ with r decentralized dynamic compensators $Q_i(s)$, $i = 1, \cdots, r$ of combined McMillan degree $q := q_1 + \cdots + q_r$. To make this precise, define vectors $\vec{m}$, $\vec{p}$, $\vec{q} \in \mathbb{N}^r$ which have the property that their components $m_1, \cdots, m_r, p_1, \cdots, p_r, q_1, \cdots, q_r$ add up to m, p, q, respectively.

Corresponding to this partitioning we assume we have r local controllers $Q_i(s)$ each of size $m_i \times (m_i + p_i)$ and of McMillan degree q_i. The combined controllers and the plant restrict the behavior through the combined system of autoregressive equations

$$\begin{bmatrix} & P(\frac{d}{dt}) & & \\ \hline Q_1(\frac{d}{dt}) & 0 & \cdots & 0 \\ 0 & Q_2(\frac{d}{dt}) & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & Q_r(\frac{d}{dt}) \end{bmatrix} v(t) = 0. \tag{2.9}$$

Note that $Q_1(s)$ acts only on the first $m_1 + p_1$ components of v, $Q_2(s)$ acts on the components $m_1 + p_1 + 1, \cdots, m_2 + p_2$, and so on. In addition, the total McMillan degree of the compensator is q. The closed-loop characteristic polynomial under this formulation is also (2.7).

We say a set of decentralized compensators $\{Q_i(s)\}$ is admissible if the closed-loop characteristic polynomial $\phi(s)$ is nonzero. In this case (2.9) represents an autonomous system, and the dynamics are uniquely determined through the initial conditions. Moreover if the roots of $\phi(s)$, i.e., the poles of the closed-loop system (2.3), are all in the open left-half plane then it is well known that the origin is globally asymptotically stable for system (2.9).

In this way we say that a particular plant $P(s)$ is stabilizable in the class of $m_i \times (m_i + p_i)$ decentralized controllers of McMillan degree q_i if there are admissible controllers $\{Q_i(s)\}$ of the required size and McMillan degree such that all the roots of the closed-loop characteristic polynomial $\phi(s)$ are in the open left-half plane. Similarly we say that $P(s)$ is pole assignable, if given any polynomial of degree $n + q$, there exist admissible controllers such that the resulting closed-loop characteristic polynomial is the specified polynomial.

Since the technical details of our methods are somewhat involved, we present our results here in an intuitive form.

The first step is to construct a space, that we call, $K^{\vec{q}}_{\vec{m},\vec{p}}$. "Most" points of this space correspond to r-tuples of regular systems. The rest of the points correspond to singular systems, which are added to make the space compact. We also consider the space of all nonzero polynomials of degree $\leq n + q$, denoted by $\mathbb{P}^{n+q}$. Now, if the plant $P(s)$ is fixed, we get a map, called the pole-placement map ρ_P, from the subset of $K^{\vec{q}}_{\vec{m},\vec{p}}$ consisting of the admissible plants, to the space of polynomials. Clearly, system $P(s)$ is pole assignable if, and only if, the map ρ_P is onto. Since this map is algebraic, it is easy to see that a necessary condition for this map to be onto is that the dimension of the target, which is $n + q$, be greater than or equal to the dimension of $K^{\vec{q}}_{\vec{m},\vec{p}}$, which we compute to be $\sum_{i=1}^r q_i(m_i + p_i) + m_i p_i$. Our main result is that this necessary condition is in fact sufficient to ensure that a "generic" plant $P(s)$ is pole assignable over $\mathbb{C}$, and over $\mathbb{R}$ if the degree of $K^{\vec{q}}_{\vec{m},\vec{p}}$ is also an odd number. The degree of $K^{\vec{q}}_{\vec{m},\vec{p}}$ given by (3.12) is an integer that depends on the number of inputs, outputs, and the order of dynamic compensator of each local channel. For a precise statement of these results see Theorem 3.13, Corollaries 4.2 and 4.3, and Theorem 4.6. We also show that the necessary and sufficient condition given above for a "generic" plant to be pole assignable is in fact also a necessary and sufficient condition for a "generic" plant to be stabilizable. This result is formulated in Corollary 5.3.

Our methods raise various questions:

1) Given a plant $P(s)$ is it possible to decide if it is "generic" enough for our result to hold? The answer to this question is yes, in theory. Given a plant $P(s)$, it is possible to write down a list of polynomials in a large number of variables, such that the dimension of the common zero set of these polynomials decides if P is "generic." The dimension computation can be done by many computer algebra systems. While this can be done in principle, we do not expect this to be feasible in most practical situations.
2) Given a plant $P(s)$ and a desired closed-loop polynomial, is there an algorithm to find the compensators? Again, the answer to this question is yes, in principle. As in the preceding question, one can write down a list of polynomials and find their common solutions. In practice, we do not expect this method to be a useful method of designing compensators.

III. The Pole Placement Map and a Parameterization of the Compensator Space

In this section we will develop a framework which will then enable us to derive new conditions for stabilizability and pole assignability. Crucial in our investigation will be the so-called pole-placement map which is a map having as domain the set of all compensators and having as range the set of closed-loop characteristic polynomials. Once we have parameterized both the domain and the range of this map in a suitable algebraic way, it will turn out that the pole-placement map is a linear map restricted to an algebraic variety.

We first give some definitions. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Recall that the projective n-space $\mathbb{P}^n$ is the set of all lines through the origin in $\mathbb{K}^{n+1}$. (Compare with [5, Lecture 1].) A point in $\mathbb{P}^n$ (i.e., a line through the origin in $\mathbb{K}^{n+1}$) can be represented by homogeneous coordinates

$$(z_0, z_1, \cdots, z_n)$$

with the properties that at least one of the $z_i \neq 0$ where $(z_0, z_1, \cdots, z_n)$ is not distinguished from $(cz_0, cz_1, \cdots, cz_n)$ for any nonzero number c. The Euclidean topology on $\mathbb{K}^{n+1}$, induces a topology on $\mathbb{P}^n$, that we will refer to as the classical topology. There is another topology on $\mathbb{P}^n$, called the Zariski topology, where the closed sets are zero sets of homogeneous polynomials in $z_0, \cdots, z_n$. In particular a Zariski open set is the complement of a zero set of polynomials. A Zariski open set is also open in the classical topology, but the converse is not in general true.

Definition 3.1: (Compare with [5, Lecture 6].) The Grassmannian Grass $(m, m+p)$ is the set of all equivalence classes of $m\times(m+p)$ full rank matrices over $\mathbb{K}$ under the equivalence relation

$$Z' \equiv QZ$$

for any $m \times m$ invertible matrix Q.

For any $m\times(m+p)$ full rank matrix Z and any multi-index

$$\underline{i} = (i_1, \cdots, i_m), \quad 1 \le i_1 < i_2 < \cdots < i_m \le m+p$$

let z_i be the $m \times m$ minor of Z consisting of the i_1th through i_mth columns. As homogeneous coordinate, the set of all $z_{\underline{i}}$

$$z = (z_{\underline{i}})$$

depends only on the equivalence class of Z. The map $Z \mapsto z$ induces an embedding

$$\text{Grass}\,(m, m+p) \subset \mathbb{P}^{\binom{m+p}{m}-1}$$

which is called the Plücker embedding, and z is called the Plücker coordinate of Z. The Grass $(m, m+p)$ in $\mathbb{P}^N$, where $N = \binom{m+p}{m} - 1$ is defined by a set of quadratic equations (see [5, pp. 65–66], for example) and is a variety of dimension mp.

Remark 3.2: The Grass $(m, m+p)$ can also be thought of as the space of all m-dimensional subspaces of $\mathbb{K}^{m+p}$, where the Plücker embedding is induced by the map $L \to e_1\wedge\cdots\wedge e_m$ for any basis $\{e_1, \cdots, e_m\}$ of L.

Example 3.3: Let $\mathbb{P}^5$ be the projective space with homogeneous coordinates $(z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34})$. Grass $(2, 4)$ is embedded in $\mathbb{P}^5$ by the Plücker embedding. For example, the class of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

is sent to the point $(1, -2, 1, -2, -1, 4)$ in $\mathbb{P}^5$. Given any 2×4 matrix A, let z_{ij} be the 2×2 minor of A formed by the columns i and j. Then by performing cofactor expansion along the first two rows one has

$$0 = \det\begin{pmatrix} A \\ A \end{pmatrix} = 2(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23})$$

i.e.,

$$z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0.$$

In fact, one can show that the image of Grass $(2, 4)$ is exactly the set of points $(z_{ij}) \in \mathbb{P}^5$ that satisfy the above equation.

Definition 3.4 [12]: Let $K^q_{m,p}$ be the set of all equivalence classes of $m \times (m+p)$ full rank matrices over $\mathbb{K}[s]$ whose maximum degree of $m \times m$ minors is at most q, under the equivalence relation

$$P(s) \equiv Q(s) \Leftrightarrow P(s) = H(s)Q(s) \tag{3.1}$$

for some $m \times m$ rational matrix $H(s)$ whose determinant is a nonzero constant.

$K^q_{m,p}$ is a projective variety in $\mathbb{P}^{(q+1)\binom{m+p}{m}-1}$ of dimension $q(m+p)+mp$ defined by a set of quadratic equations: For any multi-index

$$\underline{i} = (i_1, \cdots, i_m), \ 1 \le i_1 \le i_2 < \cdots < i_m \le m+p$$

let

$$p_{\underline{i}}(s) = z_{(\underline{i};\,0)} + z_{(\underline{i};\,1)}s + \cdots + z_{(\underline{i};\,q)}s^q$$

be the $m \times m$ minor of $P(s)$ consisting of the i_1 through i_m columns. Then $\{p_i(s_0)\}$ must satisfy the set of quadratic equations defining Grass $(m, m+p)$ for each $s_0 \in \mathbb{C}$. So one gets a set of quadratic equations satisfied by $\{z_{(\underline{i};\,d)}\}$ (see (3.6) for details). The projective variety in $\mathbb{P}^{(q+1)\binom{m+p}{m}-1}$ defined by this set of polynomial equations is the variety $K^q_{m,p}$ (see [12] for detail).

Example 3.5: Consider $K^q_{1,p}$. Every element $P(s) \in K^q_{1,p}$ is described through an equivalence class of $1\times(p+1)$ nonzero matrices of the form

$$P(s) = (f_0(s), f_1(s), \cdots, f_p(s)).$$

Here $f_i(s)$ are arbitrary polynomials of degree at most q and the equivalence relation (3.1) reduces to

$$P(s) \equiv Q(s) \Leftrightarrow P(s) = cQ(s)$$

for some nonzero constant $c \in \mathbb{K}$.

Note that in this situation there are no Plücker relations present and every equivalence class simply describes a line in the vector space $\mathbb{K}^{q+1} \otimes \mathbb{K}^{p+1} = \mathbb{K}^{pq+p+q+1}$, i.e., a point in the projective space $\mathbb{P}^{pq+p+q}$. The variety $K^q_{1,p}$ is therefore simply the whole projective space $\mathbb{P}^{pq+p+q} = \mathbb{P}^{(q+1)\binom{p+1}{1}-1}$.

Finally note that if $f_0(s) \neq 0$ and if $\{f_i(s) | i = 0, \cdots, p\}$ are coprime then the equivalence class of $P(s)$ defines in a unique way the transfer function

$$G(s) := \left(\frac{f_1(s)}{f_0(s)}, \cdots, \frac{f_p(s)}{f_0(s)}\right).$$

Example 3.6: Consider $K^1_{2,2}$. Let $p_{ij}(s) = z_{(ij;\,0)} + z_{(ij;\,1)}s$ be the 2×2 minor consisting of the ith and jth columns. Then $\{p_{ij}(s)\}$ has to satisfy one quadratic equation coming from the defining equation of Grass $(2, 4)$ (see Example 3.3)

$$\begin{aligned} p_{12}(s)p_{34}(s) &- p_{13}(s)p_{24}(s) + p_{14}(s)p_{23}(s) \\ &= z_{(12;0)}z_{(34;0)} - z_{(13;0)}z_{(24;0)} + z_{(14;0)}z_{(23;0)} \\ &\quad + \big(z_{(12;0)}z_{(34;1)} - z_{(13;0)}z_{(24;1)} + z_{(14;0)}z_{(23;1)} \\ &\quad + z_{(12;1)}z_{(34;0)} - z_{(13;1)}z_{(24;0)} + z_{(14;1)}z_{(23;0)}\big)s \\ &\quad + \big(z_{(12;1)}z_{(34;1)} - z_{(13;1)}z_{(24;1)} + z_{(14;1)}z_{(23;1)}\big)s^2 \\ &= 0. \end{aligned}$$

Equating coefficients we arrive at three quadratic equations

$$\begin{aligned} z_{(12;\,0)}z_{(34;\,0)} - z_{(13;\,0)}z_{(24;\,0)} + z_{(14;\,0)}z_{(23;\,0)} &= 0 \\ z_{(12;\,0)}z_{(34;\,1)} - z_{(13;\,0)}z_{(24;\,1)} + z_{(14;\,0)}z_{(23;\,1)} & \\ + z_{(12;\,1)}z_{(34;\,0)} - z_{(13;\,1)}z_{(24;\,0)} + z_{(14;\,1)}z_{(23;\,0)} &= 0 \\ z_{(12;\,1)}z_{(34;\,1)} - z_{(13;\,1)}z_{(24;\,1)} + z_{(14;\,1)}z_{(23;\,1)} &= 0 \end{aligned}$$

which define the variety $K^1_{2,2}$ in $\mathbb{P}^{11}$.

We call an autoregressive system irreducible or controllable (compare with [10] and [14]) if the full size minors of one and therefore any polynomial representation $P(s)$ are relatively prime, i.e., $P(s)$ has full rank for all $s \in \mathbb{C}$. As shown by Rosenthal in [12], two irreducible polynomial matrices $P(s)$ and $Q(s)$ are equivalent under the equivalence relation (3.1), if and only if $P(s)$ and $Q(s)$ are unimodular row equivalent; so each irreducible autoregressive system of McMillan degree at most q corresponds to one and only one point in $K^q_{m,p}$. In particular $K^q_{m,p}$, contains the set of $m \times p$ proper transfer functions of McMillan degree at most q as a Zariski dense subset.

Definition 3.7: (Compare with [5, p. 25].) Let $\mathbb{P}^n$ and $\mathbb{P}^m$ be two projective spaces. For any points $x = (x_0, \cdots, x_n) \in \mathbb{P}^n$ and $y = (y_0, \cdots, y_m) \in \mathbb{P}^m$, the map $(x, y) \mapsto x^t y$ induces an embedding $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{mn+m+n}$ where the homogeneous coordinates of $\mathbb{P}^{mn+m+n}$ are given by the entries of $(n+1) \times (m+1)$ matrix $W = x^t y$. This embedding is called the Segre embedding and $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{mn+m+n}$ is defined by rank $W = 1$; i.e., all the 2×2 minors are equal to zero. Similarly for a set of points $x_i = (x_{i0}, x_{i1}, \cdots, x_{in_i}) \in \mathbb{P}^{n_i}$, $i = 1, 2, \cdots, r$ the Segre embedding $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{N_r} \subset \mathbb{P}^N$, $N = \prod_{i=1}^r (n_i + 1) - 1$ is defined through the assignment

$$(x_1, \cdots, x_r) \mapsto \left\{ \prod_{l=1}^r x_{li_l} | i_l = 0, \cdots, n_l \right\}.$$

Finally we give a definition of central projections.

Definition 3.8: (Compare with [5, Lecture 3].) Let M be an $(n+1) \times (N+1)$ full rank matrix, $z = (z_0, z_1, \cdots, z_N)^t \in \mathbb{P}^N$ and

$$E = \{z \in \mathbb{P}^N | Mz = 0\}.$$

E is a linear subspace of dimension $N - n - 1$. Then the projection with center E is the map ρ: $\mathbb{P}^N - E \to \mathbb{P}^n$ defined by

$$\rho(z) = Mz$$

i.e., a central projection is a linear map restricted on an open set of a projective variety.

A central projection can be geometrically interpreted as follows: Take any projective n-subspace $H \subset \mathbb{P}^N$ disjoint from E. If one identifies H with $\mathbb{P}^n$, then the map ρ is given by the following geometric construction. Through any point $z \in \mathbb{P}^N - E$ and E, there passes a unique projective $(N-n)$-subspace L_z. L_z intersects H in a unique point, namely $\rho(z)$. For example, let E be a point. To find the image of any other point z one can draw a line through E and z, then the intersection of the line with $\mathbb{P}^n$ will be the image of z.

Proposition 3.9: Let ρ: $\mathbb{P}^N - E \to \mathbb{P}^n$ be a central projection with

$$\dim E = N - n - 1$$

and $X \subset \mathbb{P}^N - E$ be a projective variety of dimension n. Then ρ: $X \to \mathbb{P}^n$ is onto over $\mathbb{C}$ and there are $\deg X$ complex solutions (counted with multiplicity) of $\rho(z) = b$ in X for each $b \in \mathbb{P}^n$.

Proof: (Compare also with [5, p. 235] and [9, Corollary (2.29), Corollary (5.6)].) Let $b \in \mathbb{P}^n$. Then $z \in \rho^{-1}(b) \cap X$ if and only if the $(N-n)$ linear subspace L_b of $\mathbb{P}^N$ spanned by E and b intersects X at z. Note that X and L_b intersect properly, i.e., $\dim X \cap L_b = 0$, because otherwise $X \cap E$, which is the intersection of $X \cap L_b$ and a hyperplane would be nonempty by the projection dimension theorem (see, e.g., [6, p. 48]). Now, by Bézout's theorem (see, e.g., [5, p. 227]), every $(N-n)$ dimensional linear subspace of $\mathbb{P}^N$ intersects X in $\deg X$ points counted with multiplicity. □

It is our first goal to parameterize all compensators appearing in (2.9) in a suitable way. To deal with the general r-channel problem consider the variety

$$K^{\vec{q}}_{\vec{m},\vec{p}} := K^{q_1}_{m_1,p_1} \times \cdots \times K^{q_r}_{m_r,p_r}. \tag{3.2}$$

As it will turn out $K^{\vec{q}}_{\vec{m},\vec{p}}$ serves as a natural parameterization in the general decentralized situation. Since each set $K^{q_i}_{m_i,p_i}$, $i = 1, \cdots, r$ is a projective variety we conclude that $K^{\vec{q}}_{\vec{m},\vec{p}}$ is a projective variety as well. Moreover using the Segre embedding we can view it as variety in $\mathbb{P}^N$ where

$$N = \prod_{i=1}^r (q_i + 1) \binom{m_i + p_i}{m_i} - 1. \tag{3.3}$$

Note also that the dimension is given through

$$\dim K^{\vec{q}}_{\vec{m},\vec{p}} = \sum_{i=1}^r \dim K^{q_i}_{m_i,p_i} = \sum_{i=1}^r q_i(m_i+p_i) + m_i p_i. \tag{3.4}$$

Next we will define the pole-placement map on the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$, and we will show that this map has the structure of a central projection. For this we will expand the determinant $\phi(s)$ introduced in (2.7) along the block defined through the first p rows. Such an expansion is classical (see, e.g., [8, Theorem 7.7]) and goes back to Laplace.[1]

To be precise we expand the characteristic polynomial $\phi(s)$ of the closed-system (2.9) through

$$\begin{aligned}\phi(s) &= \det \begin{bmatrix} \multicolumn{4}{c}{P(s)} \\ \hline Q_1(s) & 0 & \cdots & 0 \\ 0 & Q_2(s) & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & Q_r(s) \end{bmatrix} \\ &= \sum_{\underline{i}} p_{\underline{i}}(s) q_{\underline{i}}(s) \end{aligned} \tag{3.5}$$

where for each multi-index $\underline{i} = (i_1, \cdots, i_m)$ $q_{\underline{i}}(s)$ is the $m \times m$ minor of

$$Q(s) := \begin{bmatrix} Q_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_r(s) \end{bmatrix} \tag{3.6}$$

consisting of the i_1th through i_mth columns, and $p_{\underline{i}}(s)$ is the cofactor of $q_{\underline{i}}(s)$ in determinant (3.5). Due to the special

[1] Laplace: Mém. Acad. Sci. Paris 1772.

structure of the compensator $Q(s)$, many of its full size minors $q_{\underline{i}}(s)$ are zero. Note that $q_{\underline{i}}(s)$ is nonzero only if it is a product of full size minors of $Q_1(s), Q_2(s), \cdots, Q_r(s)$, with one minor from each $Q_i(s)$. Note also that the coefficients of such products are exactly the coordinates of the projective space $\mathbb{P}^N$, where $K^{\vec{q}}_{\vec{m},\vec{p}}$ is embedded through the Segre embedding. Therefore if we let $z = (z_0, \cdots, z_N)$ be the coordinates of $\mathbb{P}^N$, we can represent the characteristic polynomial through $\phi(s) = \sum_{i=0}^N f_i(s)z_i$, where $f_i(s)$ are induced by the expansion (3.5), i.e.,

$$\{f_i(s)\} = \{s^d p_{\underline{i}}(s) | 0 \le d \le q,\ q_{\underline{i}}(s) \ne 0\}.$$

Remark 3.10: The greatest common divisor of

$$\{p_{\underline{i}}(s) | q_{\underline{i}}(s) \ne 0\}$$

(and therefore the greatest common divisor of $\{f_i\}$) is the decentralized fixed polynomial of the system, and its zeros are the fixed modes [15].

To define the pole-placement map let B_P be the linear subspace of $\mathbb{P}^N$ defined through $B_P := \{z \in \mathbb{P}^N \mid \sum_{i=0}^N f_i(s)z_i = 0\}$. Identify a nonzero polynomial $\phi(s) = a_0 + a_1 s + \cdots + a_{n+q}s^{n+q}$ with a point in the projective space $\mathbb{P}^{n+q}$ (compare with [12]). Then the decentralized pole-placement map ρ_P for a plant $P(s)$ is defined through

$$\rho_P : K^{\vec{q}}_{\vec{m},\vec{p}} - B_P \to \mathbb{P}^{n+q}, \quad z \mapsto \sum_{i=0}^N f_i(s)z_i. \tag{3.7}$$

Due to the fact that ρ_P is induced by a linear map defined on all $\mathbb{P}^N$, namely the map $z \mapsto \sum_{i=0}^N f_i(s)z_i$, we immediately have the following result.

Theorem 3.11: The pole-placement map ρ_P is a central projection with center B_P.

We would like to remark that for $q = 0$ (static situation) this result reduces to the one obtained by Wang [18], if $r = 1$ (centralized situation) the result was obtained by Rosenthal [12], and if $q = 0$ and $r = 1$ we reduce to a result of Brockett and Byrnes [1]. The following definition generalizes the important technical concepts defined in [1], [12], and [18].

Definition 3.12: A particular plant $P(s)$ is called $\vec{q}$ nondegenerate in the class of $\vec{m}$, $\vec{p}$, $\vec{q}$ controllers if $K^{\vec{q}}_{\vec{m},\vec{p}} \cap B_P = \emptyset$.

In other words $P(s)$ is $\vec{q}$ nondegenerate if every controller $Q(s)$ in the class $\vec{m}$, $\vec{p}$, $\vec{q}$ is admissible (compare with [12], [20]), i.e., the pole-placement map is well defined for all compensators $Q(s) \in K^{\vec{q}}_{\vec{m},\vec{p}}$. We need to define one other concept from algebraic geometry: If $V \subset \mathbb{P}$ is a projective variety of dimension n, then a generic linear space in $\mathbb{P}$ of codimension n, intersects V in a finite number of points and the number of points of intersection is independent of the linear space and is called the degree of the variety V ([5, 18.1]). We are now ready to state one of the main results of this paper.

Theorem 3.13: For a $\vec{q}$ nondegenerate system of McMillan degree $n = \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$ the pole-placement map ρ_P is onto over $\mathbb{C}$ and there are $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ (counted with multiplicity) complex decentralized dynamic compensators in $K^{\vec{q}}_{\vec{m},\vec{p}}$ which solve the pole-placement problem. In particular, if the coefficients of the desired closed-loop polynomial are real and $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd, then a real solution always exists.

Remark 3.14: In the above theorem, the number of compensators refers to the number of equivalence classes, in the space $K^{\vec{q}}_{\vec{m},\vec{p}}$ as defined in 3.4. Also:

a) $n + q \le \dim K^{\vec{q}}_{\vec{m},\vec{p}}$ is a necessary condition for ρ_P to be onto (or almost onto) over either $\mathbb{C}$ or $\mathbb{R}$ by a dimension argument.
b) $n + q \ge \dim K^{\vec{q}}_{\vec{m},\vec{p}}$ is a necessary condition for the existence of a $\vec{q}$ nondegenerate system, since otherwise $K^{\vec{q}}_{\vec{m},\vec{p}} \cap B_P \ne \emptyset$ by the projective dimension theorem [9, Corollary (3.30)].
c) If $n + q = \dim K^{\vec{q}}_{\vec{m},\vec{p}}$ and the system is $\vec{q}$ nondegenerate, then $\operatorname{codim} B_P = n + q + 1$ (see the proof of Theorem 3.13 in the Appendix).

When $n < \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$, we have the following result.

Theorem 3.15: If for some given system, $n < \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$ and $\dim K^{\vec{q}}_{\vec{m},\vec{p}} \cap B_P = \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q - n - 1$, then the pole-placement map ρ_P is onto over $\mathbb{C}$. If in addition $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd, then ρ_P is also onto over $\mathbb{R}$.

The following example illustrates the concepts introduced and the results derived in this section.

Example 3.16: Consider the two-channel system of McMillan degree 3

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2,$$

$$y_1 = x_1,$$
$$y_2 = x_2.$$

The transfer function of the system is

$$\frac{1}{s^3 - 1} \begin{bmatrix} s^2 + s + 1 & s \\ s^2 + s + 1 & s^2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -1 & s^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ s + 1 & s \end{bmatrix}.$$

It is our goal to tune the natural frequencies of this system using two decentralized output feedback controllers, one static and one of McMillan degree 1

$$u_1 = \alpha_1 y_1,$$
$$\dot{w} = -\beta_1 w + a y_2, \quad u_2 = b w + \beta_2 y_2.$$

We will show that all assumptions of Theorem 3.13 are satisfied, and we therefore will conclude that the associated pole-placement map is onto. The closed-loop characteristic polynomial becomes

$$\begin{aligned} \phi(s) &= \det \begin{bmatrix} s & -1 & 1 & 0 \\ -1 & s^2 & s+1 & s \\ \alpha_1 & 0 & 1 & 0 \\ 0 & \beta_2 s + \beta_3 & 0 & s + \beta_1 \end{bmatrix} \\ &= -\det \begin{bmatrix} s & 1 & -1 & 0 \\ -1 & s+1 & s^2 & s \\ \alpha_1 & 1 & 0 & 0 \\ 0 & 0 & \beta_2 s + \beta_3 & s + \beta_1 \end{bmatrix} \end{aligned}$$

where $\beta_3 = ab + \beta_1 \beta_2$.

By including all possible AR compensators of McMillan degree zero, respectively one, of the form

$$\alpha_0 u_1 = \alpha_1 y_1$$
$$\beta_0 \dot{u}_2 + \beta_1 u_2 = \beta_2 \dot{y}_2 + \beta_3 y_2$$

the closed-loop system in AR representation is given through

$$\begin{bmatrix} \left(\frac{d}{dt}\right) & 1 & -1 & 0 \\ -1 & \left(\frac{d}{dt}\right)+1 & \left(\frac{d}{dt}\right)^2 & \left(\frac{d}{dt}\right) \\ \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & 0 & \beta_2\left(\frac{d}{dt}\right)+\beta_3 & \beta_0\left(\frac{d}{dt}\right)+\beta_1 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ -u_1(t) \\ y_2(t) \\ -u_2(t) \end{bmatrix} = 0. \quad (3.8)$$

One readily computes the closed-loop characteristic polynomial as

$$\begin{aligned} \phi(s) = {} & \alpha_0\beta_0 s^4 + (\alpha_0\beta_1 - \alpha_0\beta_2 - \alpha_1\beta_0)s^3 \\ & + (\alpha_1\beta_2 - \alpha_1\beta_1 - \alpha_1\beta_0 - \alpha_0\beta_3)s^2 \\ & + (\alpha_1\beta_3 - \alpha_1\beta_1 - \alpha_1\beta_0 - \alpha_0\beta_0)s \\ & - \alpha_1\beta_1 - \alpha_0\beta_1. \end{aligned}$$

Next we parameterize the space of all compensators. For this we will identify every static autoregressive system of the form (α_1, α_0) with a point on the projective line $\mathbb{P}^1$ and every dynamic compensator of the form $(\beta_2 s + \beta_3, \beta_0 s + \beta_1)$ with a point in $\mathbb{P}^3$. The total compensator space is then the product variety

$$K^0_{1,1} \times K^1_{1,1} = \mathbb{P}^1 \times \mathbb{P}^3.$$

Thus the dimension of $K^0_{1,1} \times K^1_{1,1}$ is equal to four, which is $n+q$ in this case.

Under the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^3$ can be identified with a subset of $\mathbb{P}^7$

$$\begin{bmatrix} z_0 & z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 & z_7 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} [\beta_0, \beta_1, \beta_2, \beta_3]$$

and the Segre coordinates $z = (z_0, \cdots, z_7) \in \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ must satisfy

$$z_0 z_5 = z_1 z_4, \quad z_0 z_6 = z_2 z_4, \quad z_0 z_7 = z_3 z_4,$$
$$z_1 z_6 = z_2 z_5, \quad z_1 z_7 = z_3 z_5, \quad z_2 z_7 = z_3 z_6.$$

In terms of those Segre coordinates the pole-placement map for the system becomes

$$\rho_P(z) = z_0 s^4 + (z_1 - z_2 - z_4)s^3 + (z_6 - z_5 - z_4 - z_3)s^2 + (z_7 - z_5 - z_4 - z_0)s - z_5 - z_1$$

which is a linear map from $\mathbb{P}^7$ to $\mathbb{P}^4$, i.e., a central projection with center B_P defined through

$$z_0 = 0, \quad z_1 = z_2 + z_4, \quad z_6 = z_5 + z_4 + z_3,$$
$$z_7 = z_5 + z_4 + z_0, \quad z_5 = -z_1. \quad (3.9)$$

Next we verify that the system is $(0, 1)$-nondegenerate in the class of $\vec{m} = (1, 1)$, $\vec{p} = (1, 1)$, $\vec{q} = (0, 1)$ controllers (see Definition 3.12), i.e.,

$$B_P \cap \mathbb{P}^1 \times \mathbb{P}^3 = \emptyset.$$

For this note that if $\alpha_0 = 0$ it follows from (3.9) that $z_4 = 0$, $z_5 = 0$, $z_6 = 0$ and $z_7 = 0$ which is only possible if $\alpha_1 = 0$ or $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$.

If $\alpha_0 \neq 0$ it follows that

$$[z_4, z_5, z_6, z_7] = \alpha_1 \alpha_0^{-1} [z_0, z_1, z_2, z_3]$$

and because of (3.9) this is only possible if $z_i = 0$, $i = 0, \cdots, 7$. But this just means that the system is nondegenerate.

The pole-placement map

$$\rho_P : \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^4$$

is therefore well defined for all two-channel compensators having McMillan degree zero on the first channel and McMillan degree one on the second channel. Moreover ρ_P is onto over $\mathbb{C}$ by Theorem 3.13 and for each closed-loop characteristic polynomial there are $4 = \deg \mathbb{P}^1 \times \mathbb{P}^3$ dynamic compensators which solve the pole-placement problem.

We would like to conclude this example with the remark that the compensators which achieve a particular closed-loop characteristic polynomial of degree four can even be represented by a proper transfer function, because $z_0 \neq 0$ which implies that both α_0 and β_0 are nonzero. This is true for any regular strictly proper system.

We conclude this section with a closed formula of the degree of the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$. By Theorem 3.13 this number is also equal to the number of complex compensators which assigns a given closed-loop characteristic polynomial, i.e., this number is equal to the topological degree of the pole-placement map in the critical dimension $(n + q = \dim K^{\vec{q}}_{\vec{m},\vec{p}})$.

The formula can be readily established by combining [18, Proposition 2.1]

$$\deg(X_1 \times \cdots \times X_r) = \frac{(n_1 + \cdots + n_r)!}{n_1! \cdots n_r!} \prod_{i=1}^{r} m_i$$
$$\text{where } \deg X_i = m_i, \quad \dim X_i = n_i \quad (3.10)$$

and [17, Theorem 5]

$$\deg K^q_{m,p} = (q(m+p)+mp)! \cdot \left| \sum_{n_1+\cdots+n_m=q} \frac{\prod_{1\le k<l\le m}((n_l - n_k)(m+p) + l - k)}{\prod_{j=1}^{m}(p + j + n_j(m+p) - 1)!} \right|. \quad (3.11)$$

By doing so we have the following.

Proposition 3.17: The degree of $K^{\vec{q}}_{\vec{m},\vec{p}}$ is

$$\deg K^{\vec{q}}_{\vec{m},\vec{p}} = (\dim K^{\vec{q}}_{\vec{m},\vec{p}})! \prod_{i=1}^{r} \left| \sum_{n_{i1}+\cdots+n_{im_i}=q_i} \frac{\prod_{1\le k<l\le m_i}((n_{il} - n_{ik})(m_i + p_i) + l - k)}{\prod_{j=1}^{m_i}(p_i + j + n_{ij}(m_i + p_i) - 1)!} \right|. \quad (3.12)$$

$\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd if and only if the degrees of all $\{K^{q_i}_{m_i,p_i}\}$ are odd, and the sets of exponents appearing in the binary representations of $\{\dim K^{q_i}_{m_i,p_i} | i = 1, 2, \cdots, r\}$ are disjoint. (Given a positive integer A, one has $A = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_b}$, $0 \le a_1 < a_2 < \cdots < a_b$ and the set $\{a_1, \cdots, a_b\}$ is the set of exponents of A in its binary representation.)

Example 3.18: $\deg K^1_{2,3} \times K^0_{1,4}$ is odd because $\deg K^1_{2,3}$ and $\deg K^0_{1,4}$ are both odd (see [11] for the method to determine odd or even of $\deg K^q_{m,p}$ without actually computing the degree), and $\dim K^1_{2,3} = 11 = 2^0 + 2^1 + 2^3$ and $\dim K^0_{1,4} = 4 = 2^2$ have disjoint exponents in their binary representations. One could also directly compute the degree using formula 3.12 to get

$$\deg K^1_{2,3} \times K^0_{1,4} = 75075.$$

IV. Generic Pole-Placement Results

Our main goal in this section is to show that "almost all" plants of McMillan degree $n = \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$ satisfy the hypothesis of Theorem 3.13 and "almost all" plants of McMillan degree $n < \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$ satisfy the hypothesis of Theorem 3.15. To make this statement rigorous we will use the concept of genericity.

Firstly, note that each plant $P(s)$ is an element of the parameterizing variety $K^n_{p,m}$. We call a set of systems generic if it contains a nonempty Zariski open set of $K^n_{p,m}$. We say a property holds for the generic system if the set of systems satisfying the property is a generic set.

The following lemma as well as most other results which follow are proved in the Appendix.

Lemma 4.1: The generic system is $\vec{q}$ nondegenerate if

$$n + q = \dim K^{\vec{q}}_{\vec{m},\vec{p}}.$$

Combining this lemma with Theorem 3.13 we then have the following.

Corollary 4.2: If

$$n + q = \dim K^{\vec{q}}_{\vec{m},\vec{p}} \tag{4.1}$$

then the pole-placement map ρ_P is onto over $\mathbb{C}$ (over $\mathbb{R}$ if $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is also odd) for the generic system.

The following corollary considers the case when only proper transfer functions of form (2.2) are allowed in the feedback loop.

Corollary 4.3: Let $\dot{x} = Ax + Bu$, $y = Cx + Du$ be a generic r-channel system as introduced in Section II. If $n + q = \dim K^{\vec{q}}_{\vec{m},\vec{p}}$ and if $\phi(s)$ is a generic polynomial of degree $n+q$ then there exist r proper complex compensators of form (2.2) such that the closed-loop characteristic polynomial is $\phi(s)$. If, in addition, the degree of the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd one can choose a set of r real compensators of the form (2.2). In particular under these assumptions a generic r-channel system can be stabilized by a proper compensator.

Proof: The set of r-channel proper transfer functions is a Zariski dense subset of the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$. By a dimension argument it therefore follows that the pole-placement map ρ_P restricted to the set of proper transfer functions is almost onto, i.e., the generic closed-loop characteristic polynomial $\phi(s)$ is in the image of ρ_P. □

Remark 4.4: If $\dot{x} = Ax + Bu$, $y = Cx$ is a generic strictly proper system, then for every polynomial $\phi(s)$ of degree $n + q = \dim K^{\vec{q}}_{\vec{m},\vec{p}}$ there exist r proper complex compensators of form (2.2) such that the closed-loop characteristic polynomial

$$\det \begin{bmatrix} sI - A - BKC & -BH \\ -EC & sI - F \end{bmatrix} = \phi(s).$$

Indeed in (2.7) the $\det D(s)$ is the only minor of $P(s)$ which has degree n. Therefore if $Q_1(s), \cdots, Q_r(s)$ are solutions for $\phi(s)$, $\det T_{di}$ must have degree q_i, $i = 1, \cdots, r$, which means that there are K_i, H_i, F_i, and E_i such that $T_{di}^{-1}(s)T_{ni}(s) = K_i + H_i(sI - F_i)^{-1}E_i$.

The following corollary is proved in the Appendix.

Corollary 4.5: The pole-placement map ρ_P is onto over $\mathbb{C}$ for the generic system if and only if

$$n \leq \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q. \tag{4.2}$$

Moreover condition (4.2) is also sufficient over $\mathbb{R}$ when $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd.

A little stronger result can also be proved. Recall [18] that $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd if and only if $\{\deg K^{q_i}_{m_i,p_i}\}$ are all odd and the sets of exponents appearing in the binary representations of $\{\dim K^{q_i}_{m_i,p_i}\}$ are disjoint.

Theorem 4.6: The pole-placement map ρ_P is onto over $\mathbb{R}$ for the generic system of degree $n \leq \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$ if $\deg K^{q_i}_{m_i,p_i}$ is odd for $i = 1, \cdots, r$ and if there are positive integers $k_i \leq \dim K^{q_i}_{m_i,p_i}$, $i = 1, \cdots, r$, such that the sets of exponents appearing in the binary representations of $\{k_i\}$ are disjoint and

$$\sum k_i \geq n + q. \tag{4.3}$$

The following example illustrates what type of results are included in the statements of this section.

Example 4.7: Let $P(s)$ be a generic 7×10 autoregressive system of McMillan degree at most 14. Since $n \leq 14 = \dim (K^1_{2,3} \times K^0_{1,4}) - 1$ and since $\deg (K^1_{2,3} \times K^0_{1,4})$ is odd (see Example 3.18), $P(s)$ can be arbitrarily pole assigned with a real 2×5 compensator $Q_1(s)$ of order 1 which acts only on the first five components of the behavior $v(t)$ and a real static 1×5 compensator $Q_2(s)$ which acts only on the last five components of $v(t)$.

Example 4.8: Let $P(s)$ be a generic 5×10 autoregressive system of McMillan degree at most 15. Although $\deg (K^1_{2,3} \times K^1_{3,2})$ is even, but because $8 < \dim K^1_{2,3} = 11$, $7 < \dim K^1_{3,2} = 11$, 8 and 7 have disjoint exponents in their binary representations and $n \leq 8 + 7 = 15$, $P(s)$ can be arbitrarily pole assigned with a real 2×5 compensator $Q_1(s)$ of order 1 which acts only on the first five components of the behavior $v(t)$ and a real 3×5 compensator $Q_2(s)$ of order 1 which acts only on the last five components of $v(t)$.

V. Generic Stabilization of Systems with Fixed McMillan Degree

From the dimension argument one knows that the condition

$$\dim K^{\vec{q}}_{\vec{m},\vec{p}} \geq n + q$$

is a necessary condition for arbitrary pole-placement by decentralized dynamic compensators. The question is: Can generic systems be stabilized by decentralized dynamic compensators if

$$\dim K^{\vec{q}}_{\vec{m},\vec{p}} < n + q?$$

As we will show in this section, the answer is no. For the one-channel problem such a result has been proven in [10].

Theorem 5.1: If $n+q > \dim K^{\vec{q}}_{\vec{m},\vec{p}}$ there exists a nonempty classical open neighborhood U of $K^n_{p,m}(\mathbb{R})$ whose elements cannot be stabilized by any real (or even complex) decentralized dynamic compensator in $K^{\vec{q}}_{\vec{m},\vec{p}}$.

Let $S^n_{p,m}(\mathbb{R})$ be the real manifold of all $p \times m$ proper transfer functions of McMillan degree n. Since $S^n_{p,m}(\mathbb{R})$ is Zariski dense in $K^n_{p,m}(\mathbb{R})$ we immediately have the following corollary.

Corollary 5.2: If $n+q > \sum_{i=1}^r (q_i(m_i+p_i)+m_ip_i))$ there exists a nonempty classical open neighborhood U of $S^n_{p,m}(\mathbb{R})$ whose elements cannot be stabilized by any real (or even complex) decentralized dynamic compensator in $K^{\vec{q}}_{\vec{m},\vec{p}}$.

Combining Theorem 5.1 with Corollary 4.5 we have the following.

Corollary 5.3: The set of m-input, p-output linear systems of McMillan degree n is generically pole assignable (over $\mathbb{C}$) using r-channel decentralized controllers of type $\vec{m}$, $\vec{p}$, $\vec{q}$ if, and only if this set is generically stabilizable.

We conclude this section with an illustrative example explaining the ingredients of Theorem 5.1 and Corollaries 5.2 and 5.3.

Example 5.4: Consider the two-channel system of McMillan degree 3

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x. \tag{5.1}$$

This observable and controllable system was provided by Davison and Chang [4] to illustrate the concept of decentralized fixed mode (DFM).

Note that it is impossible to stabilize this system with two decentralized static controllers of the form

$$u_1 = \alpha_1 y_1, \quad u_2 = \beta_1 y_2$$

since the closed-loop system

$$\dot{x} = \begin{bmatrix} -1+\alpha_1 & \alpha_1 & 0 \\ 0 & 1 & \beta_1 \\ 0 & 0 & -2+\beta_1 \end{bmatrix} x$$

has always an eigenvalue (DFM) at one independent of the applied feedback. It is our goal to show that even after a small perturbation in the system parameters, the resulting system can still not be stabilized even with general decentralized static AR compensators.

For this we rewrite the dynamics in autoregressive form

$$\begin{bmatrix} \left(\frac{d}{dt}\right)^2+1 & \left(\frac{d}{dt}\right)-1 & 0 & \left(\frac{d}{dt}\right)+1 \\ 0 & 0 & \left(\frac{d}{dt}\right)+2 & 1 \\ \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_0 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ -u_1(t) \\ y_2(t) \\ -u_2(t) \end{bmatrix} = 0. \tag{5.2}$$

One readily verifies that for any compensators $((\alpha_1, \alpha_0), (\beta_1, \beta_0)) \in \mathbb{P}^1 \times \mathbb{P}^1$ system (5.2) is autonomous, i.e., every static decentralized compensator is admissible or equivalently $B_P \cap \mathbb{P}^1 \times \mathbb{P}^1 = \emptyset$. The pole-placement map

$$\rho_P : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

is therefore well defined. The fact that (5.1) has a fixed mode at one is expressed through the fact that every polynomial in the image of ρ_P contains a root at one. From the proof of Theorem 5.1 it therefore follows that (5.1) cannot be stabilized even after a small perturbation. In other words

$$\dot{x} = \begin{bmatrix} \epsilon_1 - 1 & \epsilon_2 & \epsilon_3 \\ \epsilon_4 & \epsilon_5+1 & \epsilon_6 \\ \epsilon_7 & \epsilon_8 & \epsilon_9 - 2 \end{bmatrix} x + \begin{bmatrix} \epsilon_{10}+1 & \epsilon_{11} \\ \epsilon_{12} & \epsilon_{13}+1 \\ \epsilon_{14} & \epsilon_{15}+1 \end{bmatrix} u$$
$$y = \begin{bmatrix} \epsilon_{16}+1 & \epsilon_{17}+1 & \epsilon_{18} \\ \epsilon_{19} & \epsilon_{20} & \epsilon_{21}+1 \end{bmatrix} x$$

cannot be stabilized, as long as the numbers ϵ_i are sufficiently small. ($|\epsilon_i| \leq 0.1$ seems sufficient, based on numerical calculations). Notice that the perturbed systems do not have in general a fixed mode. Also, these systems cannot be stabilized even with high gain feedback since there are no static decentralized AR compensators available.

It is a matter of future research to determine exact perturbation bounds, and we believe that the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$ will be of importance in this task.

VI. Conclusion

In this paper we studied the pole-placement problem and the stabilizability problem for a generic linear system of McMillan degree n using a decentralized dynamic control scheme acting on r decentralized channels. We establish new necessary and sufficient conditions which guarantee arbitrary pole placement with dynamic compensators of a bounded McMillan degree.

We develop the theory for general autoregressive systems since we believe that a complete theory should incorporate improper transfer functions as well as more general descriptor systems.

In the last section we prove that if a generic system P cannot be arbitrarily pole assigned by compensators of fixed degrees $\vec{q}$, then the set of systems which can be stabilized by these compensators is not a generic set. Further, there is a small ϵ neighborhood of P inside the manifold of proper transfer functions with the property that any element in this neighborhood cannot be stabilized by compensators of degree $\vec{q}$.

Appendix

Proof of Theorem 3.13: The linear subspace B_P defined by a nondegenerate plant P has dimension at least $N-n-q-1$ since it is defined by $n+q+1$ equations arising out of setting the $n+q+1$ coefficients of the characteristic polynomial of the closed-loop system (3.5) to be zero. On the other hand, since P is nondegenerate $B_P \cap K^{\vec{q}}_{\vec{m},\vec{p}} = \emptyset$. By the projective dimension theorem (see, e.g., [6, p. 48]), if $\dim B_P > N-n-q-1$ then B_P must intersect $K^{\vec{q}}_{\vec{m},\vec{p}}$. Thus $\dim B_P = N-n-q-1$, and we can apply Proposition 3.9.

The only thing remaining to prove is the last sentence in the statement of the theorem. For this, we first observe that the variety $K^{\vec{q}}_{\vec{m},\vec{p}}$ and B_P are defined over the real numbers. Also under the assumptions on the closed-loop polynomial, the point $b \in \mathbb{P}^{n+q}$ in the image has real coordinates. Thus the linear subspace L_b spanned by b and B_P is also defined over the reals. Now, as in the proof of Proposition 3.9, the compensators that give the closed-loop polynomial b are found by intersecting L_b and $K^{\vec{q}}_{\vec{m},\vec{p}}$. Since both these varieties are defined over the reals, all intersections occur in complex conjugate pairs. Thus if the number of solutions, which is $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$, is odd, at least one of the solutions is real. □

Proof of Theorem 3.15: Under the given condition, there exists a plane $L \subset \mathbb{P}^N$ having codimension equal to $\dim K^{\vec{q}}_{\vec{m},\vec{p}} - q - n$ and having the property that $K^{\vec{q}}_{\vec{m},\vec{p}} \cap B_P \cap L = \emptyset$ [9, Corollary (2.29)].

Let π_1 be the central projection with center $B_P \cap L$. Then $\pi_1 : K^{\vec{q}}_{\vec{m},\vec{p}} \to \mathbb{P}^{\dim K^{\vec{q}}_{\vec{m},\vec{p}}}$ is onto over $\mathbb{C}$ and over $\mathbb{R}$ if $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd. So

$$\rho_P = \pi_2 \circ \pi_1 : K^{\vec{q}}_{\vec{m},\vec{p}} \to \mathbb{P}^{n+q}$$

is onto over $\mathbb{C}$ and over $\mathbb{R}$ if $\deg K^{\vec{q}}_{\vec{m},\vec{p}}$ is odd, where $\pi_2 : \mathbb{P}^{\dim K^{\vec{q}}_{\vec{m},\vec{p}}} \to \mathbb{P}^{n+q}$ is the central projection with center $\pi_1(B_P)$. □

Proof of Lemma 4.1: Since the set of $\vec{q}$ nondegenerate systems is Zariski open, we only need to prove that this set is nonempty. For this consider an r-tuple of systems $P_i(s)$ which are q_i nondegenerate and have size $p_i \times (m_i + p_i)$ and McMillan degree $n_i = q_i(m_i + p_i - 1) + m_i p_i$. By [12, Corollary 5.6] such systems exist. One readily verifies that

$$P(s) := \begin{bmatrix} P_1(s) & 0 & \cdots & 0 \\ 0 & P_2(s) & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & P_r(s) \end{bmatrix} \tag{A.1}$$

is a $\vec{q}$ nondegenerate system. □

Proof of Corollary 4.5: The only case that remains to be considered is when $n < \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$. The proof of this case turns out to be quite technical. To structure the proof we first show a lemma (Lemma A.1) and a proposition (Proposition A.2). The proof of Corollary 4.5 will then be a direct consequence of Theorem 3.15 and Proposition A.2.

We first recall the following.

Let E_P be the center of the centralized pole-placement map defined by formula (5.4) of [12]. Then E_P is a plane in

$$\mathbb{P}^{(q+1)\binom{m+p}{m}-1} \supset K^q_{m,p} \tag{A.2}$$

and it follows from [12, Theorem 5.5] that

$$\dim E_P \cap K^q_{m,p} = q(m+p) + mp - n - q - 1 \tag{A.3}$$

for the generic system P of McMillan degree $n < q(m+p-1) + mp$.

Lemma A.1: For any positive integer l, there exist l distinct hyperplanes $H_1, \cdots, H_l$ in the projective space $\mathbb{P}^{(q+1)\binom{m+p}{m}-1}$ such that for any integer $k \leq l$ and any choice of k distinct integers $\{i_1, \cdots, i_k\} \subset \{1, \cdots, l\}$

$$\dim K^q_{m,p} \cap H_{i_1} \cap \cdots \cap H_{i_k} = q(m+p) + mp - k \tag{A.4}$$

and

$$\dim E_P \cap K^q_{m,p} \cap H_{i_1} \cap \cdots \cap H_{i_b} = q(m+p) + mp - n - q - k - 1 \tag{A.5}$$

for the generic system $P \in K^n_{p,m}$, with the convention that the empty set has negative dimension.

Proof: We first consider the case when $n \geq q(m+p-1) + mp$. In this case, for a generic system P, $E_P \cap K^q_{m,p} = \emptyset$, so the second condition (A.5) is vacuously true for any choice of hyperplanes. So we only need to choose l hyperplanes, that satisfy the first condition (A.5). For this, it suffices to choose l hyperplanes $H_1, \cdots, H_l$ such that for any $k \leq l$, the codimension of the intersection of any k of these hyperplanes is k. Now, a hyperplane in a projective space is given by a homogeneous linear equation. So, to choose our hyperplanes we need a system of l homogeneous linear equations, such that the row rank of any submatrix of k rows is maximal. This is clearly satisfied by a generic system of l homogeneous equations.

Now, we consider the case when $n \leq q(m+p-1) + mp$. We first choose a generic plant P such that $\dim E_P \cap K^q_{m,p} = q(m+p) + mp - n - q - 1$. We choose the hyperplanes inductively. Now, let us assume that we have chosen l hyperplanes that satisfy the hypothesis. We will use the fact that given any projective variety $V \in \mathbb{P}$, the set of hyperplanes that intersect with V to give a variety of dimension one less that of V, form a Zariski open set in the variety of hyperplanes in $\mathbb{P}$. To choose H_{l+1}, consider all possible subsets I of $\{1, \cdots, l\}$. For each subset the intersection of H_{l+1} with $V_I = E_P \cap K^q_{m,p} \cap (\bigcap_{i \in I} H_i)$ and with $W_I = K^q_{m,p} \cap (\bigcap_{i \in I} H_i)$ must have dimension one less than that of V_I and W_I. So for each I, the hyperplanes H_{l+1} that satisfy this condition form a nonempty open set. Thus we can choose H_{l+1} to be any hyperplane that is in the intersection of these 2^l Zariski open sets. Having chosen the l hyperplanes, the set of plants P for which E_P satisfies A.4 is again a Zariski open subset of $K^n_{p,m}$. □

Proposition A.2: If $n < \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q$, then

$$\dim K^{\vec{q}}_{\vec{m},\vec{p}} \cap B_P = \dim K^{\vec{q}}_{\vec{m},\vec{p}} - q - n - 1 \tag{A.6}$$

for the generic system $P \in K^n_{p,m}$.

Proof: Let $l = \dim K^{\vec{q}}_{\vec{m},\vec{p}} - n - q$. Choose l hyperplanes $H^j_1, \cdots, H^j_l$ as in Lemma A.1 for each of the r factors $K^{q_j}_{m_j,p_j}$ in $K^{\vec{q}}_{\vec{m},\vec{p}}$. Assume that H^j_i is defined by $f^j_i = 0$ for some linear homogeneous polynomial f^j_i. The product f_i of f^j_i, $j = 1, \cdots, r$, is a linear equation in the coordinates of $\mathbb{P}^N$ and thus defines a hyperplane H_i in $\mathbb{P}^N$. Now, $K^{\vec{q}}_{\vec{m},\vec{p}} \cap (\bigcap_{i=1}^l H_i)$ is a union of algebraic sets of the form $Z_{I_1,k_1} \times \cdots \times Z_{I_r,k_r}$ where $\sum k_j = l$, I_j is a subset of k_j distinct integers between one and l, and $Z_{I_j,k_j} = K^{q_j}_{m_j,p_j} \cap (\bigcap_{i \in I_j} H^j_i)$. By

Lemma A.1, $E_{P_j} \cap Z_{I_j, k_j}$ is empty for the generic system P_j of degree $n_j = q_j(m_j + p_j) + m_j p_j - q_j - k_j$ in $K^{n_j}_{p_j, m_j}$. Then $P := \text{block diag}(P_1, \cdots, P_r)$ has McMillan degree $\dim K^{\vec{q}}_{\vec{m}, \vec{p}} - q - \sum k_i = n$ and has the property that $B_P \cap (Z_{I_1, k_1} \times \cdots \times Z_{I_r, k_r}) = \emptyset$. Hence there is a nonempty Zariski open set $U_I \in K^n_{p, m}$ such that for all $P \in U_I$, $B_P \cap (Z_{I_1, k_1} \times \cdots \times Z_{I_r, k_r}) = \emptyset$. Each of the finite components of $K^{\vec{q}}_{\vec{m}, \vec{p}} \cap (\bigcap_{i=1}^{l} H_i)$ gives rise to such a nonempty Zariski open set. Let U be the intersection of all these open sets. Then for $P \in U$

$$B_P \cap K^{\vec{q}}_{\vec{m}, \vec{p}} \cap \left(\bigcap_{i=1}^{l} H_i\right) = \emptyset.$$

So by the projective dimension theorem [9, Corollary (3.30)]

$$\dim B_P \cap K^{\vec{q}}_{\vec{m}, \vec{p}} \leq l - 1 = \dim K^{\vec{q}}_{\vec{m}, \vec{p}} - n - q - 1.$$

On the other hand, the equations defining B_P impose at most $n + q + 1$ algebraic conditions on $K^{\vec{q}}_{\vec{m}, \vec{p}}$ because the characteristic polynomial has degree at most $n + q$. So

$$\dim B_P \cap K^{\vec{q}}_{\vec{m}, \vec{p}} \geq \dim K^{\vec{q}}_{\vec{m}, \vec{p}} - n - q - 1$$

for any $P \in K^n_{p, m}$. Therefore

$$\dim B_P \cap K^{\vec{q}}_{\vec{m}, \vec{p}} = \dim K^{\vec{q}}_{\vec{m}, \vec{p}} - n - q - 1$$

for all systems P in the nonempty Zariski open set U of $K^n_{p, m}$. □

Combining Theorem 3.15 and Proposition A.2 results in the proof of Corollary 4.5. □

Proof of Theorem 4.6: Let $l_i = \dim K^{q_i}_{m_i, p_i} - k_i$ and $H^i_1, \cdots, H^i_{l_i}$ be the hyperplanes defined in Lemma A.1 for $K^{q_i}_{m_i, p_i}$. Since $\deg K^{q_i}_{m_i, p_i}$ is odd, by Bézout's Theorem (see, e.g., [5, p. 227]), $Z_i := K^{q_i}_{m_i, p_i} \cap (\bigcap_{j=1}^{l_i} H^i_j)$ contains at least one irreducible component X_i of odd degree. So $\deg X_1 \times \cdots \times X_r$ is odd and

$$\dim X_1 \times \cdots \times X_r = \sum k_i \geq n + q.$$

Replacing $K^{\vec{q}}_{\vec{m}, \vec{p}}$ by $X_1 \times \cdots \times X_r$ and using the same argument one shows that

$$\rho_P : X_1 \times \cdots \times X_r \to \mathbb{P}^{n+q}$$

is onto over $\mathbb{R}$. □

Proof of Theorem 5.1: We first construct a $\vec{q}$ nondegenerate system $P_0(s)$ in $K^n_{p, m}(\mathbb{R})$ which cannot be stabilized by any compensator in $K^{\vec{q}}_{\vec{m}, \vec{p}}$. Take a real $\vec{q}$ nondegenerate system $P_1(s)$ of degree $\dim K^{\vec{q}}_{\vec{m}, \vec{p}} - q$. Such a system exists because when $k = \dim K^{\vec{q}}_{\vec{m}, \vec{p}} - q$ the set of $\vec{q}$ degenerate systems is a proper algebraic subset of $K^k_{p, m}(\mathbb{C})$, and $K^k_{p, m}(\mathbb{R})$, which has a Zariski open subset isomorphic to $\mathbb{R}^{q(m+p)+mp}$, cannot be contained in any proper algebraic subset of $K^k_{p, m}(\mathbb{C})$. Let

$$P_0(s) = P_1(s) \begin{bmatrix} s-1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{A.7}$$

Then $P_0(s) \in K^n_{p, m}(\mathbb{R})$, $P_0(s)$ is $\vec{q}$ nondegenerate and is not stabilizable by any compensator $K^{\vec{q}}_{\vec{m}, \vec{p}}$. Let $O \subset K^n_{p, m}$ be the open set of all $\vec{q}$ nondegenerate systems and define

$$\phi : K^{\vec{q}}_{\vec{m}, \vec{p}} \times O \to \mathbb{P}^{n+q}, \quad (z, P) \mapsto \rho_P(z). \tag{A.8}$$

Then ϕ is continuous in both the classical and Zariski topologies. Let $S \subset \mathbb{P}^{n+q}$ be the classical open set of all the polynomials which have at least one root whose real part is greater than one-half. Since $\phi(z, P_0) \in S$ there exists for any $z \in K^{\vec{q}}_{\vec{m}, \vec{p}}$ classical open neighborhoods $U_z \subset O$ of P_0 and $W_z \subset K^{\vec{q}}_{\vec{m}, \vec{p}}$ of z such that

$$\phi(W_z \times U_z) \subset S.$$

By construction $\{W_z | z \in K^{\vec{q}}_{\vec{m}, \vec{p}}\}$ is an open cover and by compactness of $K^{\vec{q}}_{\vec{m}, \vec{p}}$ there exists a finite subcover

$$\{W_{z_i} | i = 1, 2, \cdots, l\}.$$

Let

$$U = \bigcap_{i=1}^{l} U_{z_i}.$$

Then $\phi(K^{\vec{q}}_{\vec{m}, \vec{p}} \times U) \subset S$ which implies that all the systems in U are not stabilizable by $K^{\vec{q}}_{\vec{m}, \vec{p}}$. □

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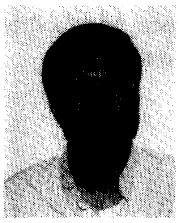

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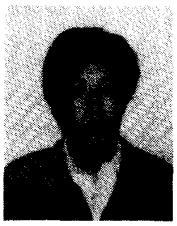

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