

NEW RESULTS IN POLE ASSIGNMENT BY REAL OUTPUT FEEDBACK*

JOACHIM ROSENTHAL†

Abstract. This paper considers the problem of tuning natural frequencies of a linear system by a memoryless controller. Using algebro-geometric methods it is shown how it is possible to improve current sufficiency conditions.

The main result is an exact combinatorial characterization of the nilpotency index of the mod 2 cohomology ring of the real Grassmannian. Using this characterization, new sufficiency results for generic pole assignment for the linear system with m -inputs, p -outputs, and McMillan degree n are given. Among other results it is shown that

$$2.25 \cdot \max(m, p) + \min(m, p) - 3 \geq n$$

is a sufficient condition for generic real pole placement, provided $\min(m, p) \geq 4$.

Key words. output feedback, pole placement, intersection theory, symmetric functions.

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1. Introduction. Consider a linear time invariant system Σ with m -inputs, p -outputs and McMillan degree n . In the time domain Σ can be modelled by the following system of differential equations:

$$(1.1) \quad \Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad x \in \mathbf{R}^n, y \in \mathbf{R}^p, u \in \mathbf{R}^m.$$

The problem of output pole assignment with a static compensator is the problem of finding a feedback law $u = Fy$ in such a way that the closed loop system

$$(1.2) \quad \Sigma_F : \begin{cases} \dot{x} = (A + BFC)x \\ y = Cx \end{cases}$$

is assigned a desired set of eigenvalues. The stability of equilibria or periodic motions of the closed loop system depends on the eigenvalues of the matrix $A + BFC$. In particular the closed loop system is asymptotically stable, if the eigenvalues of $A + BFC$ have negative real parts. In this paper we are interested in under which conditions it is possible to assign a set of real eigenvalues, in particular when it is possible to stabilize a generic system. Because the eigenvalues correspond to the poles of the transfer function under Laplace transform one often refers to this type of problem as the pole placement problem. This question has already been considered by many authors (e.g. [1], [2], [10], [16], [20], [21]), and interesting links to topological questions and Schubert calculus were made. An excellent survey article can be found in [3], where a larger bibliography is also given.

Kimura [10], motivated by the problem of stabilizing and controlling a mechanical system, studied this inverse eigenvalue problem in a systematic way. Typically such systems have m -inputs m -outputs and the dimension of the state is $2m$. More generally, one would hope that $m + p \geq n$ would imply pole assignability, and hence

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† Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556. This research, which constitutes a part of the author's Ph.D. dissertation at Arizona State University, was done during the author's stay at the Department of Systems Science and Mathematics, Washington University, St. Louis, Missouri 63130.

stabilizability for the generic $p \times m$ n -dimensional system. In 1975, Kimura proved a result, which came “within one degree of freedom” of the desired result.

THEOREM 1.1 (KIMURA [10]). *If (A, B) is controllable and (A, C) is observable and if $m + p - 1 \geq n$, an almost arbitrary set of distinct real or complex conjugate poles is assignable by real gain output feedback.*

In 1978, Willems and Hesselink [21] showed that in the case of $m = p = 2$, at most 3 real poles can be assigned arbitrarily for the generic system, so that Theorem 1.1 also gives a necessary condition for this case.

Quite surprisingly, as shown in this paper, the case studied by Willems and Hesselink is the only nontrivial case ($\min(m, p) \geq 2$) where $m + p \geq n$ is not a sufficient condition. This result will follow from a new combinatorial criterion, which will be formulated in the next section. In fact more will be shown. Using an identification of the mod 2 cohomology ring of the real Grassmannian with a quotient of the space of symmetric functions it will be possible to characterize the maximum number of nontrivial terms in a nonzero product of $H^*(Grass(p, m + p), \mathbb{Z}_2)$, sometimes called the cup length of this ring, in a combinatorial way. In the next section the combinatorial criterion is formulated and the main results stated.

2. A new combinatorial criterion. Consider a $m \times p$ array A , where m can be seen as the number of inputs and p as the number of outputs. Let $\mu = (\mu_1, \dots, \mu_s)$ be a partition of mp . This means $\mu_1 \geq \mu_2 \geq \dots \geq \mu_s > 0$ and $\sum \mu_i = mp$. Denote with K_μ the number of possibilities to insert μ_i integers i into the array A under the condition that the rows are increasing and the columns are strictly increasing.

DEFINITION 2.1. $c(m, p) := \max\{s \mid K_{(\mu_1, \dots, \mu_s)} \text{ is odd}\}$.

THEOREM 2.2. *The cup length of the mod 2 cohomology ring of the real Grassmannian $Grass(p, m + p)$ is $c(m, p)$.*

This cup length has an important topological meaning. As was shown by Eilenberg [4], this number gives a lower estimate for the Ljusternik Snirelmann category of a topological space.

In the innovative paper [1], Brockett and Byrnes explained the pole placement problem with static compensators as an intersection problem in some Grassmann variety. Moreover Byrnes [2] showed that the Ljusternik Snirelmann category of the real Grassmannian gives a lower bound for the number of real poles which can be generically assigned. Using Theorem 2.2 therefore, one has immediately the following result.

THEOREM 2.3. *$c(m, p) \geq n$ is a sufficient condition for generic pole placement of a generic, strictly proper linear system Σ_n with m inputs and p outputs and McMillan degree n .*

Clearly not every $m \times p$ system Σ of order n can be pole assigned by output feedback; in particular one needs controllability and observability of the system. The results we present in this paper are stated for a generic system (see, e.g., [21]). Recall that a subset of a variety is called generic if it contains a nonempty Zariski open subset. Before giving the proof of Theorem 2.3, it will be illustrated how it is possible to obtain new sufficiency conditions for generic real pole placement. The following examples were given in [15].

3. Examples and corollaries.

Example 3.1. Two inputs, two outputs or $m = p = 2$. To apply Theorem 2.3, compute K_μ for different partitions of 4:

$K_{(1,1,1,1)} = 2$ (\rightarrow even) given by the two possibilities:

1	2
3	4

1	3
2	4

$K_{(2,1,1)} = 1$ (\rightarrow odd) given by the only possibility:

1	1
2	3

Because $K_{(2,1,1)}$ is odd, $c(2,2) = 3$, consistent with the result of Kimura [10] and Willems and Hesselink [21].

Example 3.2. Two inputs, three outputs, or $m = 2$ and $p = 3$. In this case, one immediately computes $K_{(1^6)} = 5$ (\rightarrow odd) given by the possibilities:

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

In other words $c(2,3)=6$ and up to 6 poles can be placed generically. This result is somewhat surprising, although it was already established in the paper of Brockett and Byrnes [1].

Example 3.3. The following table shows $c(m,p)$ for $\max(m,p) \leq 5$.

(3.1)

$m \backslash p$	1	2	3	4	5
1	1	2	3	4	5
2	2	3	6	7	8
3	3	6	8	9	11
4	4	7	9	10	17
5	5	8	11	17	19

LEMMA 3.4. $m + p - 1 \geq n$ is a sufficient condition for generic real pole assignment.

Proof: Consider the partition $\mu = (p^{m-1}, 1^p)$. As one immediately verifies, $K_\mu = 1$ corresponding to the only possibility:

$$\left(\begin{array}{cccccc} 1 & 1 & \dots & \dots & 1 & \\ 2 & 2 & \dots & \dots & 2 & \\ \vdots & & & & \vdots & \\ m-1 & m-1 & \dots & \dots & m-1 & \\ m & m+1 & \dots & \dots & m+p-1 & \end{array} \right).$$

THEOREM 3.5. The following conditions imply generic real pole assignability. (By duality assume $m \leq p$):

(3.2) $m = 2$ and $1.5p \geq n$

(3.3) $m = 3$ and $2p + 1 \geq n$

(3.4) $m \geq 4$ and $2.25p + m - 3 \geq n$.

The proof of this theorem and Theorem 2.3 is based on an interesting identification of the mod 2 cohomology ring of the real Grassmannian and the space of symmetric functions $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$. A good description of the topology of the real Grassmann manifold can be found for example in [14]. The important properties about the ring structure of $H^*(Grass(p, m + p), \mathcal{Z}_2)$ are given in the next section. Several properties about the ring of symmetric functions $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$ are summarized in an Appendix, where further references are given.

4. The cohomology ring of the real Grassmannian. The collection of m -planes in \mathbf{R}^{m+p} is called the Grassmann manifold and will be denoted by $Grass(p, m + p)$. The Grassmannian $Grass(p, m + p)$ is a smooth, compact manifold of dimension mp . Additively, the cohomology ring $H^*(Grass(p, m + p), \mathcal{Z}_2)$ can be described as a free \mathcal{Z}_2 -module over the set of Schubert cocycles $[a_1, \dots, a_p]$ where $m \geq a_1 \geq \dots \geq a_p \geq 0$. This notation coincides with the notation adopted in Griffith and Harris [6] and is “reverse” to the notation used by Hiller [7], [8].

Denote with $\xi_{p,m+p}$ the canonical p -bundle over $Grass(p, m + p)$. The total space of $\xi_{p,m+p}$ is defined by

$$(4.1) \quad E(\xi_{p,m+p}) = \{(V, x) \in Grass(p, m + p) \times \mathbf{R}^{m+p} \mid x \in V\}$$

and the corresponding bundle map is a projection on the first factor. The orthogonal bundle of $\xi_{p,m+p}$ is an m -plane bundle and will be denoted with $\bar{\xi}_{p,p+m}$. Finally denote with w_k the k th Stiefel Whitney class of $\xi_{p,p+m}$ and with σ_j the j th Stiefel Whitney class of $\bar{\xi}_{p,p+m}$. In terms of Schubert cocycles those Stiefel Whitney classes are described by

$$(4.2) \quad w_k = \underbrace{[1, 1, \dots, 1, 0, \dots, 0]}_k, \quad k = 1, \dots, p$$

$$(4.3) \quad \sigma_j = [j, 0, \dots, \dots, 0], \quad j = 1, \dots, m.$$

The multiplicative structure of $H^*(Grass(p, n), \mathcal{Z}_2)$ is described by the classical formulas of Pieri and Giambelli. Giambelli’s formula expresses a general Schubert cocycle as a polynomial in the special Schubert cocycle σ_j and Pieri’s formula explains how a Schubert cocycle is multiplied with a special Schubert cocycle.

Pieri’s formula:

$$(4.4) \quad [a_1, \dots, a_p] \cdot \sigma_j = \sum_{\substack{a_{i-1} \geq b_i \geq a_i \\ \sum_{i=1}^p b_i = (\sum_{i=1}^p a_i) + j}} [b_1, \dots, b_p]$$

Giambelli’s formula:

$$(4.5) \quad [a_1, \dots, a_p] = \det(\sigma_{a_i + j - i}) = \det \begin{pmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \dots & \sigma_{a_1+p-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{a_p-p+1} & & \dots & \sigma_{a_p} \end{pmatrix}.$$

From Giambelli’s formula it follows in particular that the Stiefel Whitney classes of the orthogonal bundle $\bar{\xi}_{p,p+m}$ generate $H^*(Grass(p, m + p), \mathcal{Z}_2)$. As shown by Hiller [8, p. 530] the same is true for the Stiefel Whitney classes of the canonical bundle $\xi_{p,p+m}$. In fact we will show that a general Schubert cocycle can be expressed in terms of the Stiefel-Whitney classes $\{w_1, \dots, w_p\}$ using a well-known classical formula.

In order to achieve our results the relation between the cohomology ring of the real Grassmannian and the space of symmetric functions will be studied. It will be shown that the cohomology ring $H^*(Grass(p, m + p), \mathcal{Z}_2)$ is isomorphic to a quotient of the space of symmetric functions $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$. Using this identification it is possible to characterize the cup-length of $H^*(Grass(p, m + p), \mathcal{Z}_2)$ in a combinatorial way.

In the case of a complex Grassmannian a connection between the space of symmetric functions $\mathcal{Z}[x_1, \dots, x_p]^{S_p}$ and the cohomology ring $H^*(Grass(p, m + p), \mathcal{Z})$ is well known. According to Stanley [17], Lesieur [12] was the first who recognized a formal similarity between (4.5) and the classical identity of Jacobi and Trudi (see the Appendix). Horrocks [9] showed that this relationship is more than formal and can be explained geometrically.

In this section we work out a similar relationship for the real Grassmannian. From a geometric point of view, this relation can be understood in the following way.

Consider the space $Flag(\mathbf{R}^{m+p})$ of mutually orthogonal and ordered $(m+p)$ -tuples of lines (l_1, \dots, l_{m+p}) . Over $Flag(\mathbf{R}^{m+p})$ are line bundles ξ_i with total space $E(\xi_i)$, where

$$(4.6) \quad E(\xi_i) := \{((l_1, \dots, l_{m+p}); y) \in Flag(\mathbf{R}^{m+p}) \times \mathbf{R}^{m+p} \mid y \in l_i\}.$$

One has a projection

$$(4.7) \quad \begin{aligned} \pi : Flag(\mathbf{R}^{m+p}) &\longrightarrow Grass(p, m + p) \\ (l_1, \dots, l_{m+p}) &\longmapsto \text{span}(l_1, \dots, l_p) \end{aligned}$$

This projection induces an embedding (compare Hiller [8] or Stong [19])

$$(4.8) \quad \begin{aligned} \pi^* : H^*(Grass(p, m + p), \mathcal{Z}_2) &\longrightarrow H^*(Flag(\mathbf{R}^{m+p})) \\ &\cong \mathcal{Z}_2[x_1, \dots, x_{m+p}] / I_{mp}, \end{aligned}$$

where I_{mp} is the ideal generated by the relations

$$(4.9) \quad \prod_{i=1}^{m+p} (1 + x_i) = 1,$$

expressing the triviality of the bundle

$$\xi_1 \oplus \dots \oplus \xi_{m+p}.$$

The projection π can be covered by a bundle map. Indeed, consider the p -bundle $\xi_1 \times \dots \times \xi_p$ over $Flag(\mathbf{R}^{m+p})$. It is immediate that $\pi^*(w(\xi_{p, m+p})) = w(\xi_1 \times \dots \times \xi_p) = \prod_{i=1}^p (1 + x_i)$. Under π^* , the k th Stiefel Whitney class w_k of the canonical p -bundle $\xi_{p, p+m}$ of $Grass(p, p + m)$ is therefore mapped onto the k th elementary symmetric function $e_k = \sum x_{i_1} \dots x_{i_k}$ of $\mathcal{Z}_2[x_1, \dots, x_p]$.

Because the Schubert cocycles $\{w_1, \dots, w_p\}$ generate $H^*(Grass(p, m + p), \mathcal{Z}_2)$ as a ring, $H^*(Grass(p, m + p), \mathcal{Z}_2)$ can be embedded into $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p} / \tilde{I}_{mp}$, where $\tilde{I}_{mp} = I_{mp} \cap \mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$. Because the elementary symmetric functions generate $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$, this last embedding is even an isomorphism.

In the following we will represent the ideal \tilde{I}_{mp} as the kernel of a ring homomorphism. For this denote with h_k the k th complete homogenous symmetric function in

p variables (see the Appendix for details). The set $B := \{h_1, \dots, h_p\}$ is algebraically independent and forms a multiplicative basis of $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$ (compare [13]). Any map defined on B extends therefore in a unique way to a ring homomorphism. Consider now the following ring homomorphism:

$$(4.10) \quad \begin{aligned} \psi : \mathcal{Z}_2[x_1, \dots, x_p]^{S_p} &\longrightarrow H^*(Grass(p, m + p), \mathcal{Z}_2) \\ h_j &\longmapsto \sigma_j. \end{aligned}$$

Here we assume that the j th Stiefel Whitney class σ_j of the orthogonal bundle $\bar{\xi}_{p,p+m}$ is zero for $j > m$. Using the equivalence of (4.5) and the Jacobi–Trudi identity (6.10), it is immediate that a general Schur function s_λ is mapped onto the Schubert cocycle $[\lambda_1, \dots, \lambda_p]$. From Theorem 6.3 it follows that the k th elementary symmetric function e_k is equal to the Schur function $s_{(1^k, 0, \dots, 0)}$, and this element is mapped onto the Stiefel Whitney class w_k . Again from (4.5) it follows that the kernel of the map ψ has an additive basis of Schur functions s_λ with $\lambda_1 > m$. Finally the Nagelbasch–Kostka identity (6.11) gives a formula expressing a general Schubert cocycle as a polynomial in the Stiefel Whitney classes $\{w_1, \dots, w_p\}$.

5. Proof of the theorems. In the following denote by c the cup length of $H^*(Grass(p, m + p), \mathcal{Z}_2)$ and assume that $g = g_1 \cdots g_c$ is a maximal nonzero product. Our first goal will be to show that $g \in H^{mp}$, in other words, $g = [m^p]$. If not, expand g in terms of Schubert cocycles $g = \sum_{i \in I} [\lambda]^i$ and define $d := \max\{b \mid b = m - \lambda_p\}_{i \in I}$. From (4.4) it follows that $g \cdot \sigma_d \neq 0$ contradicting the maximality of the length of the product. It is therefore immediate that $d = 0$ and $g = [m^p]$.

Using the Nagelbasch–Kostka identity (6.11) one can express each factor g_i as a polynomial in the classes $\{w_1, \dots, w_p\}$. In this way, g becomes a polynomial $g = v(w_1, \dots, w_p)$. Because H^{mp} is one-dimensional, v is just a monom, in other words $g = w_\mu = w_{\mu_1} \cdots w_{\mu_k}$. During the substitution process, the number of factors can only increase, in other words $k \geq c$. On the other hand c is equal to the cup length, which shows $k = c$.

In $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$, this product corresponds to a product of elementary symmetric functions $e_\mu = e_{\mu_1} \cdots e_{\mu_c}$. To say w_μ is nonzero is therefore equivalent to the condition that $e_\mu \notin \ker(\psi)$. Using Theorem 6.3, one can expand e_μ in terms of Schur functions:

$$(5.1) \quad e_\mu = \sum_{|\lambda|=mp} K_{\bar{\lambda}\mu} s_\lambda.$$

Because $|\lambda| = |\mu| = mp$, there is exactly one Schur function s_λ not lying in the ideal $\tilde{I}_{mp} = \ker(\psi)$, namely, $s_\lambda = s_{(m^p)}$.

In summary, w_μ is nonzero if and only if the Kostka number $K_{(p^m)\mu}$ is odd. But this number is equal to the number K_μ introduced in §2. This proves Theorem 2.2 and therefore also Theorem 2.3. \square

In fact, one can show a little more. Using (4.5) and the same argument as above one finds a description of g in terms of the special Schubert cocycle σ_j , i.e., $g = \sigma_\nu = \sigma_{\nu_1} \cdots \sigma_{\nu_c}$. In $\mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$. This product can be written as:

$$(5.2) \quad h_\nu = h_{\nu_1} \cdots h_{\nu_c} = \sum_{|\lambda|=mp} K_{\lambda\nu} s_\lambda.$$

To say that σ_ν is nonzero is therefore equivalent to the condition that the Kostka number $K_{(m^p)\nu}$ is odd. In this way we have proved Lemma 5.1.

LEMMA 5.1. $c(m, p) = c(p, m)$.

The proof of Theorem 3.5 is partially based on results obtained by Stong [19]. In this paper Stong calculates explicitly maximal nonzero cup products $w_\mu = w_{\mu_1} \cdots w_{\mu_c}$. In this way he calculates the numbers $c(m, p)$ for $m = 2, 3, 4$.

Putting his results in a little more convenient form one obtains

$$(5.3) \quad c(2, p) = k_2(p) \cdot p \quad \text{where} \quad 1.5 \leq k_2(p) \leq 2,$$

$$(5.4) \quad c(3, p) = k_3(p) \cdot p + 1 \quad \text{where} \quad 2 \leq k_3(p) \leq 2.5,$$

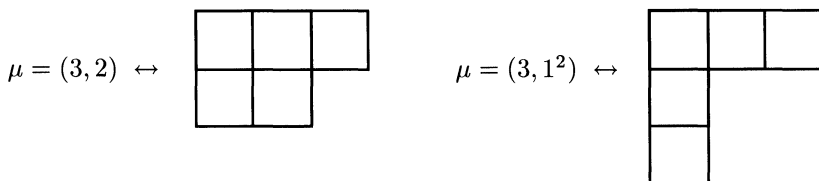
$$(5.5) \quad c(4, p) = k_4(p) \cdot p + 1 \quad \text{where} \quad 2.25 \leq k_4(p) \leq 3.$$

To get a lower bound for $c(m, p)$ in general ($m > 4$), Theorem 2.2 will be used. Consider a partition μ of the number $4p$ of length $c(4, p)$ in such a way that the Kostka number $K_{(p^4)(\mu)}$ is odd. But then it is immediate that the Kostka number $K_{(p^m)(p^{m-4}, \mu)}$ is odd as well. In this way one sees that $c(m, p) \geq c(4, p) + m - 4$, which completes the proof of Theorem 3.5. \square

6. Appendix: Symmetric functions. Let $\mu = (\mu_1, \dots, \mu_s)$ be a partition of n of length s . This means $\mu_1 \geq \mu_2 \geq \dots \geq \mu_s > 0$ and $\sum \mu_i = n$. If the integer μ_i is repeated r_i times in the partition μ , the abbreviated notation $\mu = (\mu_1^{r_1}, \dots, \mu_t^{r_t})$ will be used.

A partition μ defines a diagram D_μ , which can be considered as a left-justified array of boxes with μ_i boxes in the i th row.

Example 6.1. Two partitions with corresponding diagrams are illustrated:



The number $|\mu| = \sum \mu_i$ is sometimes called the *weight* of the partition μ and the numbers μ_i are called the *parts* of the partition. The dual partition $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_t)$ of a partition μ of n is obtained by taking the “transpose” of D_μ . In other words $\bar{\mu}_i$ is defined as the number of boxes in the column i of D_μ . Assume in addition that there is given a set $S \subseteq \mathcal{N}$.

DEFINITION 6.2. A standard Young tableau of shape μ is a diagram D_μ , where each box in D_μ contains a number from S under the constraint that the rows are increasing and the columns are strictly increasing.

Consider now $R = \mathcal{Z}_2[x_1, \dots, x_p]^{S_p}$, the ring of symmetric functions in p variables. R is in a natural way a graded ring:

$$(6.1) \quad R = A_0 + A_1 + \dots + A_n + \dots, \quad A_i A_j \subseteq A_{i+j}.$$

The homogenous component A_n can be described by different classical bases, where each basis is usually parametrized by the set of all partitions μ with weight n . In particular the dimension of A_n is equal to $p(n)$, the partition number of n .

Products of elementary symmetric functions:

$$(6.2) \quad e_\mu := e_{\mu_1} \cdots e_{\mu_s},$$

where $e_k = \sum x_{i_1} \cdots x_{i_k}$ is the k th elementary symmetric function.

Monomial symmetric functions:

$$(6.3) \quad m_\mu := \sum x_1^{\mu_1} \cdots x_p^{\mu_p},$$

where the summation has to be taken over all distinct monomials with exponents μ_1, \dots, μ_p .

Complete homogenous symmetric functions:

$$(6.4) \quad h_\mu := h_{\mu_1} \cdots h_{\mu_s},$$

where $h_k = \sum_{|\lambda|=k} m_\lambda$ is the k th complete symmetric function.

Schur functions: Classically, Schur functions were introduced by Jacobi (~1835) as the quotient of two alternating functions giving a symmetric function:

$$(6.5) \quad s_\mu = \frac{\det[x_i^{\mu_j + p - j}]}{\det[x_i^{p-j}]}, \quad i, j = 1, \dots, p.$$

The denominator of this expression is nothing else than the Vandermonde determinant and the numerator is a generalization of this type of determinant. The importance of those functions became apparent when Schur, a student of Frobenius, developed the character theory of the symmetric group (~1900).

The change of basis between the different bases of A_n is described by a linear transformation. In 1907, Kostka [11] published matrices describing the change of basis and showed that the different transformations are closely related. The following theorem, which is due to Kostka, is proven in [18].

THEOREM 6.3.

$$(6.6) \quad h_\mu = \sum_{|\lambda|=n} K_{\lambda\mu} s_\lambda,$$

$$(6.7) \quad s_\mu = \sum_{|\lambda|=n} K_{\mu\lambda} m_\lambda,$$

$$(6.8) \quad e_\mu = \sum_{|\lambda|=n} K_{\bar{\lambda}\mu} s_\lambda.$$

The coefficients $\{K_{\lambda\mu}\}$ are called the *Kostka coefficients*. The number $K_{\lambda\mu}$ can be described in a combinatorial way as the number of standard Young tableaux with shape λ and content μ . This means the number of ways of filling in μ_i integers i into the diagram D_λ under the condition that the rows are increasing and the columns are strictly increasing (see [18]).

Only in very special cases are formulas for the numbers $K_{\lambda\mu}$ known. For example if $\mu = (1, 1, \dots, 1)$, the number $K_{\lambda\mu}$ can be described by the famous hook formula of Frame, Robinson, and Thrall [5]. If in addition $\lambda = (p^m)$, the number $K_{\lambda\mu}$ is described by the following formula:

$$(6.9) \quad K_{(p^m)(1^m)} = \frac{1! \cdots (p-1)! \cdot (mp)!}{m! \cdots (m+p-1)!}.$$

Finally, the following two classical formulas give a polynomial expression of a Schur function in terms of complete symmetric respective elementary symmetric functions.

Jacobi–Trudi identity:

$$(6.10) \quad s_\lambda = \det(h_{\lambda_i+j-i}), \quad i, j = 1, \dots, p.$$

Nagelbasch–Kostka identity:

$$(6.11) \quad s_\lambda = \det(e_{\lambda_i+j-i}), \quad i, j = 1, \dots, \lambda_1.$$

More details about these identities are given in [13, p. 25] and in [18].

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