

## ON DYNAMIC FEEDBACK COMPENSATION AND COMPACTIFICATION OF SYSTEMS\*

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**Abstract.** This paper introduces a compactification of the space of proper  $p \times m$  transfer functions with a fixed McMillan degree  $n$ . Algebraically, this compactification has the structure of a projective variety and each point of this variety can be given an interpretation as a certain autoregressive system in the sense of Willems. It is shown that the pole placement map with dynamic compensators turns out to be a central projection from this compactification to the space of closed-loop polynomials. Using this geometric point of view, necessary and sufficient conditions are given when a strictly proper or proper system can be generically pole assigned by a complex dynamic compensator of McMillan degree  $q$ .

**Key words.** multivariable systems, dynamic feedback compensation, compactification, central projection, autoregressive systems

**AMS subject classifications.** 93B55, 93C35, 93B25, 54D35, 14M15

**1. Introduction.** In this paper we investigate the pole placement problem with dynamic compensators from a geometric point of view. For this consider a multivariable, time invariant linear system  $\Sigma_n$  of order  $n$  with  $m$ -inputs and  $p$ -outputs. Such a system can be represented with its state space representation

$$(1.1) \quad \Sigma_n : \dot{x} = Ax + Bu, \quad y = Cx.$$

From an engineering point of view, an input–output description is natural. Mathematically, this can be achieved by taking the Laplace transform. The system  $\Sigma_n$  is then described in the frequency domain by the following equation:

$$(1.2) \quad \hat{y} = C(sI - A)^{-1}B \hat{u}.$$

The strictly proper rational matrix  $G(s) := C(sI - A)^{-1}B$  is called the transfer function associated to the system  $\Sigma_n$ . It is well known that the dynamics of the system  $\Sigma_n$  depends in an essential way on the location of the poles of the transfer function  $G(s)$ , which are exactly the eigenvalues of the matrix  $A$ . A fundamental open problem in multivariable linear system theory is the following question: Under which conditions can a  $p$ -input,  $m$ -output system  $F(s)$  of McMillan degree  $q$  be constructed that stabilizes the closed-loop system  $G_F(s) := (I - G(s)F(s))^{-1}G(s)$ ? More generally, we can ask the following question: Given an arbitrary polynomial  $\phi(s) = s^{n+q} + \lambda_{n+q-1}s^{n+q-1} + \dots + \lambda_0$ , under which conditions is it always possible to find a compensator of order  $q$  such that the poles of the closed-loop system  $G_F(s)$  are exactly the roots of the polynomial  $\phi(s)$ ? Willems and Hesselink [37] called a system  $G(s)$  with this property pole assignable in the class of feedback controllers of order  $q$ . Using a dimension argument they showed that

$$(1.3) \quad q(m + p) + mp \geq n + q$$

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is a necessary condition for any system  $G(s)$  to have the pole assignability property in the class of feedback controllers of order  $q$ . In this paper we will show the new result that this numerical condition is not only necessary but also sufficient for a generic system  $G(s)$  if the base field is algebraically closed. To establish this result, we will study for a generic system  $G(s)$  the associated *pole placement map*  $\rho_G$ . The domain of  $\rho_G$  is the space of proper transfer functions of McMillan degree  $q$  and the range of  $\rho_G$  is the space of monic polynomials of degree  $n + q$ . In this language the system  $G(s)$  has the pole assignability property in the class of feedback controllers of order  $q$  if and only if  $\rho_G$  is onto.

The question of pole assignability is fairly well understood if we restrict ourselves to the class of static compensators, in other words, compensators with McMillan degree  $q = 0$ , and if we assume that the base field is algebraically closed. In this case we know that  $mp \geq n$  is a necessary and sufficient condition for the pole placement map  $\rho_G$  to be onto generically. Indeed, Hermann and Martin [13] first showed that  $\rho_G$  is almost onto using the dominant morphism theorem. Brockett and Byrnes [2] later showed that  $\rho_G$  is even onto and the mapping degree of  $\rho_G$  in the case  $mp = n$  is equal to the degree of the Grassmann variety  $\text{Grass}(p, p + m)$ .

The pole placement problem with dynamic compensators ( $q > 0$ ) is much less understood. The following result of Brash and Pearson [1], published 1970, is still one of the strongest results available. For the generic situation their result can be quoted in the following manner. (See, e.g., [3].)

**THEOREM 1.1** (Brash and Pearson [1]). *The generic degree  $n$  linear system with  $m$ -inputs and  $p$ -outputs can be arbitrarily pole assigned (over any field) using a compensator of order  $q$ , where  $q$  is any natural number satisfying*

$$(1.4) \quad \max(m, p)(q + 1) \geq n.$$

It is interesting to see that the necessary condition  $q(m + p) + mp \geq n + q$  of Willems and Hesselink [37] is also sufficient as soon as  $\min(m, p) = 1$ . In §4 we will explain that this is essentially due to the fact that the space of proper transfer functions with fixed McMillan degree is a Zariski open subset of a projective space if  $\min(m, p) = 1$  and  $\rho_G$  is a linear map from this projective space to the space of closed-loop polynomials identified with a projective space as well. If  $\min(m, p) > 1$ , however, this is not the case and  $\rho_G$  is a rather complicated morphism.

An important contribution to understanding the pole placement problem in general was done by Byrnes [4]. In this paper Byrnes introduced a compactification for the quasi-projective variety of proper transfer functions of degree  $q$ , which he denoted by  $C_{m,p}^q$ . He then explained the pole placement problem as an intersection problem in  $C_{m,p}^q$ . Using this point of view he achieved new results for pole assignment with compensators of degree  $q = 1$  not achieved by any other means. Our approach is guided in part by the philosophy of this paper and that is one of the reasons why we have chosen a similar title.

A great deal of research was devoted to the question of understanding the pole placement problem with static compensators over the reals. In 1975 Kimura [16] proved the result that  $m + p - 1 \geq n$  is a sufficient condition for the pole placement map  $\rho_G$  to be generically onto. Since that time, several authors have improved his results and methods in different directions. Using a geometric approach Wang [35] very recently achieved the strong result that the pole placement map  $\rho_G$  is generically onto over the reals as soon as  $mp - 1 \geq n$ . A crucial part in Wang's proof is the fact that the pole placement map  $\rho_G$  is a central projection when  $q = 0$ . As we will show in this paper, the same is true in general.

The paper is structured as follows. After explaining some mathematical preliminaries in §2, we will introduce a projective variety in §3 that can be viewed as a compactification of the space of proper  $p \times m$  transfer functions of McMillan degree  $n$  that we denote with  $K_{p,m}^n$ . This compactification was originally introduced by Rosenthal in [27] and used in [28] to achieve new results for certain low-dimensional feedback problems. In Theorem 3.6 we will describe the defining equations of the variety  $K_{p,m}^n$  and in Theorem 3.10 we give an interpretation in terms of certain autoregressive systems.

In §4 the pole placement problem is formulated in a geometric language. To deal with compensators that are not admissible, the notion of  $q$ -degeneracy, a generalization of the concept of degeneracy [2], is introduced. We will show that for a  $q$ -nondegenerate plant, the pole placement map can be extended in a continuous manner to the whole compactification.

The main results of the paper are given in §5. It is first shown that the  $q$ -degenerate systems form an algebraic subset in the quasi-projective variety of transfer functions. Then necessary and sufficient conditions are given when the  $q$ -nondegenerate systems are generic. We will show that the pole placement map is a central projection. Using this fact we are able to formulate conditions when the pole placement map for a  $q$ -nondegenerate strictly proper (or proper) system is onto (almost onto). These results constitute a generalization of the results of Hermann and Martin [13] and Brockett and Byrnes [2] from the problem of static to the problem of dynamic feedback compensation.

Finally some words about the base field. Most constructions we do in §§3 and 4 can be done over an arbitrary field  $\mathbb{K}$ . For most applications, of course, the relevant base fields are the real or complex numbers, i.e.,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . The results in §5 will use the projective dimension theorem (see, e.g., [9]) and this theorem is only valid if the field is algebraically closed.

**2. Preliminaries.** Let  $\mathbb{K}$  be an arbitrary field. With  $\bar{\mathbb{K}}$  we will denote the algebraic closure of  $\mathbb{K}$ . If  $V$  is a  $\mathbb{K}$ -vector space, we will denote with  $\mathbb{P}(V)$  the set of one-dimensional subspaces of  $V$ .  $\mathbb{P}(V)$  is called the projective space associated to  $V$ . A topology defined on  $V$  induces a topology on  $\mathbb{P}(V)$ , namely, the quotient topology of the canonical projection  $pr : V - \{0\} \rightarrow \mathbb{P}(V)$ . As it is well known,  $\mathbb{P}(\mathbb{C}^{n+1})$  and  $\mathbb{P}(\mathbb{R}^{n+1})$  are compact manifolds with the induced topology coming from the natural topology on  $\mathbb{C}^{n+1}$  or  $\mathbb{R}^{n+1}$ . If  $V = \mathbb{K}^{n+1}$  we sometimes use the notation  $\mathbb{P}_{\mathbb{K}}^n$  or simply  $\mathbb{P}^n$ .

We will identify  $\mathbb{K}^n$  as a subset of  $\mathbb{P}^n$  using the inclusion:

$$(2.1) \quad i : \mathbb{K}^n \rightarrow \mathbb{P}_{\mathbb{K}}^n, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 1).$$

In particular  $\mathbb{K}$  is identified with  $\{(x, 1) \mid x \in \mathbb{K}\} \subset \mathbb{P}_{\mathbb{K}}^1$ . We will call the point  $(1, 0)$ , which is the only point in the difference set  $\mathbb{P}_{\mathbb{K}}^1 - i(\mathbb{K})$ , the point at infinity and  $\mathbb{P}_{\mathbb{K}}^1$  the projective line over  $\mathbb{K}$ .

Consider now the polynomial ring  $\mathbb{K}[s]$  in one indeterminate. Assume the set of polynomials  $\{f_1(s), \dots, f_{n+1}(s)\} \subset \mathbb{K}[s]$  has no common zeroes. Then the map

$$(2.2) \quad f : \mathbb{K} \rightarrow \mathbb{P}_{\mathbb{K}}^n, \quad s \mapsto (f_1(s), \dots, f_{n+1}(s))$$

is well defined and called a rational map. The degree of  $f$  is defined by the highest degree of the polynomials  $f_i(s)$ . Assume  $f$  has degree  $d$ . The homogenization of  $f(s)$

is defined by

$$(2.3) \quad \hat{f}(s, t) := t^d f\left(\frac{s}{t}\right).$$

Note that  $\hat{f}$  extends the rational map  $f$  to the whole projective line  $\mathbb{P}^1_{\mathbb{K}}$ . Moreover, if  $\mathbb{K}$  is algebraically closed, the image  $\text{Im}(\hat{f})$  defines a rational curve in  $\mathbb{P}^n_{\mathbb{K}}$  in the sense of algebraic geometry. Note that over the complex numbers the holomorphic maps from the Riemann sphere  $\mathbb{P}^1_{\mathbb{C}}$  to the complex projective space  $\mathbb{P}^n_{\mathbb{C}}$  are exactly the rational maps corresponding to our definition.

The degree  $d$  of the rational map  $f$  has the following geometric interpretation: Intersect the curve  $\text{Im}(\hat{f})$  with a generic linear hyperplane  $H$  in  $\mathbb{P}^n_{\mathbb{K}}$ , which can be described by a homogeneous linear equation of the form  $\sum c_i x_i = 0$ . By the fundamental theorem of algebra,  $H$  intersects  $\text{Im}(\hat{f})$  over the algebraic closure  $\bar{\mathbb{K}}$  in exactly  $d$  points when counted with multiplicities. In short, the variety  $\text{Im}(\hat{f})$  has degree  $d$ .

Denote with  $\text{Rat}_d(\mathbb{P}^1, \mathbb{P}^n)$  the set of all rational maps of degree  $d$ .  $\text{Rat}_d(\mathbb{P}^1, \mathbb{P}^n)$  can be exhibited as a Zariski open set in  $\mathbb{P}(\mathbb{K}^{d+1} \otimes \mathbb{K}^{n+1})$ . For this consider a particular embedding

$$(2.4) \quad \begin{aligned} \tau : \text{Rat}_d(\mathbb{P}^1, \mathbb{P}^n) &\longrightarrow \mathbb{P}(\mathbb{K}^{d+1} \otimes \mathbb{K}^{n+1}) \\ \left( \sum_{j=0}^d a_{1j} s^j, \dots, \sum_{j=0}^d a_{(n+1)j} s^j \right) &\longmapsto (a_{10}, \dots, a_{1d}, a_{20}, \dots, \dots, a_{(n+1)d}). \end{aligned}$$

The complement of the image of  $\text{Rat}_d(\mathbb{P}^1, \mathbb{P}^n)$  under  $\tau$  in  $\mathbb{P}(\mathbb{K}^{d+1} \otimes \mathbb{K}^{n+1})$  is an algebraic set already described around the turn of the century by Macaulay [21]. If  $n = 1$  this algebraic set is a hypersurface described by the well-known resultant locus of two polynomials:

$$(2.5) \quad \det \text{Res}(f_1, f_2) = 0.$$

A natural generalization of the projective space is the Grassmann variety. Consider again a  $\mathbb{K}$ -vector space  $V$ . The set of  $p$ -dimensional subspaces in  $V$  is called the Grassmann variety which we will denote by  $\text{Grass}(p, V)$ . If  $V = \mathbb{K}^n$  we will just write  $\text{Grass}(p, n)$ . In particular, we have  $\text{Grass}(1, n) = \mathbb{P}^{n-1}$ .

The set  $\text{Grass}(p, n)$  indeed has the structure of a projective variety. For this consider the Plücker embedding  $\varphi$  of the Grassmann variety  $\text{Grass}(p, n)$ , which is defined in the following way:

$$(2.6) \quad \begin{aligned} \varphi : \text{Grass}(p, n) &\longrightarrow \mathbb{P}(\wedge^p \mathbb{K}^n), \\ \text{span}(v_1, \dots, v_p) &\longmapsto v_1 \wedge \dots \wedge v_p. \end{aligned}$$

It is easy to verify that  $\varphi$  is an embedding. Moreover,  $\text{Im}(\varphi)$  is irreducible and described by a famous set of quadratic relations sometimes called “shuffle relations.” (See, e.g., the survey article [17] or [25] for a characteristic free approach.) Finally we say a map  $h : \mathbb{K} \rightarrow \text{Grass}(p, n)$  is a rational map if  $f := \varphi \circ h$  is rational according to the definition above.

The set of all rational maps of degree  $d$  from the projective line  $\mathbb{P}^1_{\mathbb{K}}$  to the Grassmann variety  $\text{Grass}(p, n)$  will be denoted by  $\text{Rat}_d(\mathbb{P}^1_{\mathbb{K}}, \text{Grass}(p, n))$ .

**3. A compactification of the space of proper transfer functions.** In the following denote with  $S_{p,m}^n$  the space of proper  $p \times m$  transfer functions of McMillan degree  $n$ . Algebraically, the set  $S_{p,m}^n$  has the structure of a quasi-projective variety of dimension  $n(m + p) + mp$ . This follows directly from the fact that the space  $\text{Rat}_{n,m,p}$  of strictly proper transfer functions is quasi-projective (even quasi-affine) [10], has dimension  $n(m + p)$ , and  $S_{p,m}^n \cong \text{Rat}_{n,m,p} \times \mathbb{K}^{mp}$ . Analytically, i.e., over the complex numbers, it is well known that  $S_{p,m}^n$  and  $\text{Rat}_{n,m,p}$  are both connected complex manifolds. Many authors already studied topological properties of the spaces  $S_{p,m}^n$ ,  $\text{Rat}_{n,m,p}$ , and very recently Mann and Milgram [22] introduced a new stratification of  $\text{Rat}_{n,m,p}$  enabling them to calculate the additive structure of the homology ring  $H_*(\text{Rat}_{n,m,p})$ .

If we consider feedback problems with high gain compensators or if we want to understand partial system failures, it is of ample importance to understand the boundary structure of the space  $S_{p,m}^n$ . Motivated by those problems, several authors (e.g. [4], [8], [11], [12], [20], [26], [29]) considered the problem of compactifying the space  $S_{p,m}^n$ . In this section we will describe a compactification of the space  $S_{p,m}^n$ , which turns out to be suitable for the study of dynamic feedback compensation. The basic idea is to embed  $S_{p,m}^n$  into a projective space. The closure of the image with respect to the Zariski topology serves as a compactification. Our approach is geometric, indeed, we will view each transfer function  $G(s) \in S_{p,m}^n$  as a rational curve of degree  $n$  into a Grassmann variety. In other words we will identify each  $G(s)$  with its Hermann–Martin curve [23]. Because this curve is of crucial importance for all that follows and because we want to develop our theory over an arbitrary field  $\mathbb{K}$ , we explain this concept in more detail.

Consider a left coprime factorization  $D_L^{-1}(s)N_L(s) = G(s)$ , where  $D_L(s)$  and  $N_L(s)$  are polynomial matrices. The following results are well known and proofs can be found, for example, in [5]. From coprimeness it follows that the  $p \times (m + p)$  polynomial matrix  $(N_L(s) D_L(s))$  is of full rank for all  $s \in \mathbb{K}$ . If  $\tilde{D}_L^{-1}(s)\tilde{N}_L(s) = G(s)$  is a second coprime factorization, then there is a  $p \times p$  unimodular matrix  $U(s)$ , i.e.,  $U(s) \in \text{Gl}_p(\mathbb{K}[s])$ , with  $(\tilde{N}_L(s) \tilde{D}_L(s)) = U(s)(N_L(s) D_L(s))$ ; in other words,  $(\tilde{N}_L(s) \tilde{D}_L(s))$  is row equivalent to  $(N_L(s) D_L(s))$ . From these remarks it now follows that every element  $s \in \mathbb{K}$  is assigned a  $p$ -dimensional subspace in  $\mathbb{K}^{m+p}$ , namely, the rowspace of  $(N_L(s) D_L(s))$ . Identifying each subspace with a point of the Grassmann variety  $\text{Grass}(p, m + p)$  we get a well-defined map  $h$  that is independent of the selected coprime factorization and just depends on the transfer function  $G(s)$ :

$$(3.1) \quad h : \mathbb{K} \longrightarrow \text{Grass}(p, m + p), \quad s \longmapsto \text{rowsp}(N_L(s) D_L(s)).$$

**DEFINITION 3.1.** The map  $h$  is called the Hermann–Martin map associated to the transfer function  $G(s)$ .

We will show that  $h$  is a rational map and  $\text{Im}(h)$  describes a rational curve in the sense of algebraic geometry. It is not hard to see that two different transfer functions  $G(s)$  and  $\tilde{G}(s)$  give rise to two different maps. In this way, the space  $S_{p,m}^n$  is embedded into the space of rational maps into the Grassmannian  $\text{Grass}(p, m + p)$ . As pointed out by Martin and Hermann [23], it is possible to extend  $h$  to “infinity” if we consider a strictly proper transfer function  $G(s)$  and if we work over the complex numbers. In the case of an arbitrary field  $\mathbb{K}$ , we can do something similar. Moreover, we do not have to restrict our considerations to strictly proper transfer functions. We will contemplate the following general setting.

Denote with  $P_{p,m}$  the space of all  $p \times (m + p)$  full rank polynomial matrices

$P(s)$ . We say two elements  $P(s), \tilde{P}(s)$  in  $P_{p,m}$  are (row) equivalent if there is a  $p \times p$  unimodular matrix  $U(s) \in Gl_p(\mathbb{K}[s])$  with the property that  $\tilde{P}(s) = U(s)P(s)$ . Every  $p \times (m+p)$  polynomial matrix defines a system of autoregressive equations of the form

$$(3.2) \quad (P(s)) \cdot \begin{pmatrix} u \\ y \end{pmatrix} (s) = 0.$$

If  $u(s)$  and  $y(s)$  are solutions from the space of rational functions, it is clear that equivalent systems have the same solution set. Using the language of Willems [38], [39] (compare also with [18], [30]) we call an equivalence class in  $P_{p,m}$  an *autoregressive system* and the solution set the behavior of the system. Not all autoregressive systems actually describe a left factorization of a transfer function because the last minor of  $P(s)$  is not necessarily invertible. However, if the polynomial matrix  $P(s)$  can be partitioned into  $(P_1(s) P_2(s))$  with  $P_2(s) \in Gl_p(\mathbb{K}(s))$  (this is the generic situation),  $P(s)$  defines a proper or improper transfer function  $G(s) := P_2^{-1}(s)P_1(s)$  and equivalent systems define the same transfer function.

As shown by Kuijper and Schumacher [18], [19] it is always possible to realize an autoregressive system by a not necessarily regular descriptor system of the form

$$(3.3) \quad E\dot{x} = Ax + Bu, y = Cx + Du.$$

An autoregressive system  $P(s)$  is called irreducible or controllable if  $P(s)$  has full rank for all  $s \in \bar{\mathbb{K}}$ . (Compare with [7], [14], [30], [39].) Every irreducible autoregressive system  $P(s) \in P_{p,m}$  gives rise to a rational map

$$(3.4) \quad h : \mathbb{K} \longrightarrow \text{Grass}(p, m + p); \quad s \longmapsto \text{rowsp}P(s)$$

and this map depends only on the equivalence class in  $P_{p,m}$ . (Compare with [7].) In the following we extend  $h$  to the whole projective line  $\mathbb{P}_{\mathbb{K}}^1$ ; in other words, we extend  $h$  to “infinity.”

Without loss of generality we assume  $P(s)$  is row reduced (see, e.g., [15]). Denote with  $h_i(s)$  the  $i$ th row of  $P(s)$  and with  $\nu_i$  the degree of the polynomial vector  $h_i(s)$ , i.e., the highest degree of all polynomial entries. Consider the homogenization

$$(3.5) \quad \hat{h}_i(s, t) := t^{\nu_i} h_i\left(\frac{s}{t}\right)$$

Denote with  $\hat{P}(s, t)$  the matrix constructed from the rows  $\hat{h}_i(s, t)$ . In this way we receive an extended Hermann–Martin map  $\hat{h}$ :

$$(3.6) \quad \hat{h} : \mathbb{P}_{\mathbb{K}}^1 \longrightarrow \text{Grass}(p, m + p), \quad (s, t) \longmapsto \text{rowsp}\hat{P}(s, t).$$

The map  $\hat{h}$  is in fact rational. For this consider the Plücker embedding  $\varphi$  of the Grassmann variety  $\text{Grass}(p, m + p)$  as defined in (2.6). The combined map  $\hat{f} = \varphi \circ \hat{h}$  is given by  $\hat{f}(s, t) := \hat{h}_1(s, t) \wedge \cdots \wedge \hat{h}_p(s, t)$ . Because the entries of  $\hat{f}(s, t)$  are the principal minors of  $\hat{P}(s, t)$ , it is immediate that  $\hat{f}(s, t)$  is homogeneous in  $(s, t)$  of degree  $n = \sum \nu_i$ . In other words,

$$(3.7) \quad \hat{f} : \mathbb{P}_{\mathbb{K}}^1 \longrightarrow \mathbb{P}(\wedge^p \mathbb{K}^{m+p})$$

defines a rational map. Finally note that  $\hat{f}(s, t)$  is exactly the homogenization of  $f(s) := h_1(s) \wedge \cdots \wedge h_p(s)$ .

Note that the McMillan degree of a proper or even improper transfer function  $G(s)$  represented by a coprime factorization  $D_L^{-1}(s)N_L(s) = G(s)$  is equal to the highest degree of the principal minors of the matrix  $(N_L(s) D_L(s))$  (see, e.g., [7], [14]). Based on this fact we define the McMillan degree of an autoregressive system in the following way.

DEFINITION 3.2. The McMillan degree of an autoregressive system  $P(s)$  is given by the maximal degree of the full size minors of  $P(s)$ .

We are now in a position to describe a compactification of  $S_{p,m}^n$ , the quasi-projective variety of proper  $p \times m$  transfer functions of McMillan degree  $n$ . The Hermann–Martin identification gives rise to an embedding of  $S_{p,m}^n$  into the space of rational maps  $\text{Rat}_n(\mathbb{P}^1_{\mathbb{K}}, \text{Grass}(p, m+p))$ . Using the Plücker embedding (2.6), this set can be identified with a set of rational maps into a projective space, and, as outlined earlier, this set is contained in a Zariski open set of a projective space. All those maps can be summarized by the following diagram of maps [27]:

$$(3.8) \quad \begin{array}{ccc} S_{p,m}^n & \xrightarrow{\text{Her.-Mar.}} & \text{Rat}_n(\mathbb{P}^1, \text{Grass}(p, m+p)) \\ & \xrightarrow{\text{Plücker}} & \text{Rat}_n(\mathbb{P}^1, \mathbb{P}(\wedge^p \mathbb{K}^{m+p})) \\ & \xrightarrow{\tau} & \mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p}). \end{array}$$

DEFINITION 3.3.  $K_{p,m}^n$  is defined as the Zariski closure of  $S_{p,m}^n$  in  $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$ .

$K_{p,m}^n$  is an algebraic set of a projective space by definition. Over the reals ( $\mathbb{K} = \mathbb{R}$ ) or over the complex numbers ( $\mathbb{K} = \mathbb{C}$ ) we have already mentioned in §2 that  $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$  is compact with the induced topology. In this way we can view  $K_{p,m}^n$  as a compactification of  $S_{p,m}^n$ . Note also that  $K_{p,m}^0 = \text{Grass}(p, m+p)$ . In other words, our compactification reduces to the Grassmannian model already widely used to study static output feedback problems (see, e.g., [3], [34]). The following theorem states that  $K_{p,m}^n$  is a projective variety for all natural numbers  $m, p, n$ .

THEOREM 3.4.  $K_{p,m}^n$  is a projective variety of dimension  $n(m+p) + mp$ . If  $S_{p,m}^n$  is irreducible then  $K_{p,m}^n$  is irreducible as well.

Proof. Because  $S_{p,m}^n$  is quasi-projective the dimension of  $S_{p,m}^n$  and its Zariski closure  $K_{p,m}^n$  are the same. The irreducibility of  $K_{p,m}^n$  follows directly from the irreducibility of  $S_{p,m}^n$ . Indeed, consider a decomposition  $K_{p,m}^n = Y^1 \cup Y^2$  into Zariski closed subsets. Then  $S_{p,m}^n = (S_{p,m}^n \cap Y^1) \cup (S_{p,m}^n \cap Y^2)$ . By irreducibility of  $S_{p,m}^n$  it follows that  $S_{p,m}^n \subset Y^1$  or  $S_{p,m}^n \subset Y^2$ . But then  $K_{p,m}^n$  is also contained in one of the sets  $Y^1, Y^2$ .  $\square$

Remark 3.5. 1. Over an algebraically closed field  $S_{p,m}^n$  is always irreducible [10].

2. By a dimension argument it is clear that  $K_{p,m}^n$  is a proper subset of  $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$  as soon as  $\min(m, p) \geq 2$ . On the other hand, we have:

$$(3.9) \quad K_{1,m}^n \cong K_{m,1}^n \cong \mathbb{P}^{mn+m+n}.$$

In the following we want to describe a specific set of equations that generate the homogeneous ideal  $I(K_{p,m}^n)$ . For this consider a polynomial vector  $f(s) \in \text{Rat}_n(\mathbb{P}^1, \mathbb{P}(\wedge^p \mathbb{K}^{m+p}))$  and expand it in terms of its Plücker coordinates with respect to the standard basis:

$$(3.10) \quad f(s) = \sum_{\mathbf{i} \in \binom{m+p}{p}} f_{\mathbf{i}}(s) \cdot e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

To say the map  $f(s)$  factors over the Grassmannian it is necessary that the Plücker coordinates satisfy the “shuffle relations” ( $QR$ ) (see, e.g., [17] or [25] ), when considered as equations of the polynomial ring  $\mathbb{K}[s]$ :

$$(3.11) \quad (QR) \quad \sum_{\lambda=1}^{p+1} (-1)^\lambda \cdot f_{i_1, \dots, i_{p-1}, j_\lambda}(s) \cdot f_{j_1, \dots, \hat{j}_\lambda, \dots, j_{p+1}}(s) = 0.$$

In these equations,  $i_1, \dots, i_{p-1}$  and  $j_1, \dots, j_{p+1}$  are any sequence of integers with  $1 \leq i_\alpha, j_\beta \leq m + p$  and the symbol  $\hat{\phantom{x}}$  means that  $j_\lambda$  must be removed (compare [17]). As shown in [25] the quadratic equations ( $QR$ ) generate the homogeneous ideal if the base field is arbitrary but infinite. Equating polynomial coefficients we receive a set of necessary quadratic equations in  $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$ . The following theorem states that those equations are also sufficient; in other words, they really “cut out”  $K_{p,m}^n$ .

**THEOREM 3.6.** *Let  $\mathbb{K}$  be an infinite field. Then the variety  $K_{p,m}^n \subset \mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$  is the zero set of the ideal generated by the set of quadratic relations obtained from equating the coefficients in the shuffle relations ( $QR$ ).*

*Proof.* Denote with  $\wp$  the homogeneous ideal generated by the equations obtained when equating ( $QR$ ). Because the polynomials of  $\wp$  vanish on  $S_{p,m}^n$ , i.e.,  $\wp \subseteq I(S_{p,m}^n)$ , it follows for the sets of zeros that  $Z(I(S_{p,m}^n)) = K_{p,m}^n \subseteq Z(\wp)$ . It therefore remains to show that  $Z(\wp) \subseteq K_{p,m}^n$ . For this consider in  $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$  the Zariski open subset  $Y$  corresponding to all polynomial vectors  $f(s) = (\dots, f_i(s), \dots)$  that have the property that  $f(s) \neq 0$  for all  $s \in \bar{\mathbb{K}}$  and that have the property that the last Plücker coordinate  $f_{m+1, \dots, m+p}(s)$  has degree  $n$ . Assume now that a point  $f(s) \in Y \subset \mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$  satisfies the equations coming from ( $QR$ ) for all  $s \in \bar{\mathbb{K}}$ . Viewing the entries of  $f(s)$  as elements in the field  $\mathbb{K}(s)$ , it is immediate that there is a rational  $p \times (m+p)$  matrix  $R(s)$  that is mapped under the Plücker embedding on the vector  $f(s)$ . Using the row reduction process introduced by Forney [6], we find a  $p \times (m+p)$  polynomial matrix  $P(s)$  with minimal row indices and a rational matrix  $Q(s) \in Gl_p(\mathbb{K}(s))$  with  $P(s) = Q(s)R(s)$ . The Plücker coordinates  $p(s)$  of the polynomial matrix  $P(s)$  are clearly given by  $p(s) = \det Q(s) f(s)$ . However, it then follows from the assumptions we made that  $\det Q(s) \in \mathbb{K}$ , the last entry of  $p(s)$  is a polynomial of degree  $n$ , and  $P(s)$  is mapped onto  $f(s)$  viewed as a point of projective space. In other words,  $f(s)$  describes a point of  $S_{p,m}^n$ . In short,  $Z(\wp) \cap Y \subseteq S_{p,m}^n$ , but it is then clear that  $Z(\wp) \subseteq K_{p,m}^n$ .  $\square$

The following example illustrates how it is possible to find a describing set of equations in a concrete case.

*Example 3.7* (see [27]).  $K_{2,2}^1 \hookrightarrow \mathbb{P}^{11}$  is the complete intersection of three quadrics.

Indeed,  $K_{2,2}^1$  is defined by  $\{(f_{1,2}(s), \dots, f_{3,4}(s)) \mid f_{1,2}(s)f_{3,4}(s) - f_{1,3}(s)f_{2,4}(s) + f_{1,4}(s)f_{2,3}(s) \equiv 0 \text{ and } f_{i,j}(s) = a_{i,j} + b_{i,j}s \ 1 \leq i < j \leq 4\}$ . In  $\mathbb{P}^{11}$ , we therefore have the following equations:

$$(3.12) \quad a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3} = 0,$$

$$(3.13) \quad b_{1,2}b_{3,4} - b_{1,3}b_{2,4} + b_{1,4}b_{2,3} = 0,$$

$$(3.14) \quad a_{1,2}b_{3,4} + a_{3,4}b_{1,2} - a_{1,3}b_{2,4} - a_{2,4}b_{1,3} + a_{1,4}b_{2,3} + a_{2,3}b_{1,4} = 0.$$

Because  $\dim K_{2,2}^1 = 8$  the intersection must be complete. In particular the degree of  $K_{2,2}^1$  is equal to 8 by the classical Bézout theorem.

In the remaining part of this section we want to give a system theoretic interpretation of the boundary points that were added in the compactification  $K_{p,m}^n$ . For this consider a polynomial vector  $f(s) \in K_{p,m}^n$ . We now distinguish two cases.



*Case 1.* Assume  $f(s) \neq 0$  for all  $s \in \bar{\mathbb{K}}$ . From the proof of Theorem 3.6 it immediately follows that we find a  $p \times (m+p)$  polynomial matrix  $P(s)$  that is mapped onto  $f(s)$  under the Plücker embedding. Because  $P(s)$  has full rank for all  $s \in \bar{\mathbb{K}}$ , it follows that the Kronecker row indices are equal to the minimal row indices in the sense of Forney [6]. (Compare [15].) In other words, if  $\tilde{P}(s)$  is another polynomial matrix that is mapped onto  $f(s)$ , then  $P(s)$  and  $\tilde{P}(s)$  are row equivalent, i.e., there is a unimodular matrix  $U(s)$  with  $\tilde{P}(s) = U(s) \cdot P(s)$ .

*Case 2.* There is an  $s_o \in \bar{\mathbb{K}}$  with  $f(s_o) = 0$ . Because the minimal polynomial of  $s_o$  over  $\mathbb{K}$  divides each coordinate, we find a polynomial  $g(s) \in \mathbb{K}[s]$  with  $f(s) = g(s)\tilde{f}(s)$  and  $\tilde{f}(s) \neq 0$  for all  $s \in \bar{\mathbb{K}}$ . It is obvious that we again find a polynomial  $p \times (m+p)$  matrix  $P(s)$  that is mapped onto  $f(s)$ . Note that  $P(s_o)$  does not have full rank. To describe all other polynomial matrices that are mapped onto  $f(s)$  we introduce the following group:

$$(3.15) \quad H := \{A \in Gl_p(\mathbb{K}(s)) \mid \det A \in \mathbb{K} \setminus \{0\}\}.$$

Clearly the unimodular group is a subgroup of  $H$  consisting of all elements in  $H$  that have polynomial entries. This group enables us to introduce the following equivalence relation.

**DEFINITION 3.8.** Two polynomial matrices  $P(s)$  and  $\tilde{P}(s)$  are called  $H$ -equivalent if there is an element  $U \in H$  with  $\tilde{P}(s) = U \cdot P(s)$ .

Note that row equivalent matrices are always  $H$ -equivalent. Moreover, if  $P(s)$  has full row rank for all  $s \in \bar{\mathbb{K}}$ , it then follows from the proof of Theorem 3.6 that  $P(s)$  and  $\tilde{P}(s)$  are row equivalent if and only if they are  $H$ -equivalent. In other words, the concept of row equivalence and  $H$ -equivalence are the same for the generic set. The following example illustrates the difference of the two concepts.

*Example 3.9.* The following two matrices have the same Plücker coordinates and are therefore  $H$ -equivalent:

$$(3.16) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2s & 3s \end{pmatrix}, \quad B = \begin{pmatrix} s & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix}.$$

On the other hand, it is immediate that the matrices  $A, B$  are not row equivalent; that is, there is no unimodular matrix  $U(s)$  with  $B = UA$ .

From the above it is now clear that every point of  $K_{p,m}^n$  can be viewed as an  $H$ -equivalence class of  $p \times (m+p)$  polynomial matrices and every  $H$ -equivalence class consists of one (generally) or several autoregressive systems. At this point we want to mention that  $K_{p,m}^n$  has singularities and those singularities occur at points where several autoregressive systems form one  $H$ -equivalence class. As shown by Ravi and Rosenthal [26] the set of all “homogeneous autoregressive systems” of degree  $n$  constitutes a desingularisation of  $K_{p,m}^n$  and we refer to [26] for details.

We summarize this section with the following theorem.

**THEOREM 3.10.**  $K_{p,m}^n$  consists of all  $H$ -equivalence classes of autoregressive systems of size  $p \times (m+p)$  and degree less than or equal to  $n$ .

**4. Dynamic feedback and  $q$ -nondegeneracy.** In the last section we introduced a compactification (denoted by  $K_{p,m}^n$ ) of the space of proper transfer functions  $S_{p,m}^n$ . In this section we will show that the pole placement problem with dynamic compensators can be studied as an intersection problem in the variety  $K_{p,m}^n$ .

For this consider a proper transfer function  $G(s) \in S_{p,m}^n$  describing the behavior between an input  $\hat{u}$  and an output  $\hat{y}$  in the frequency domain:

$$(4.1) \quad \hat{y} = G(s)\hat{u}.$$

The feedback compensators that we will consider are proper transfer functions  $F(s) \in S_{m,p}^q$ . The plant and the compensator are combined through the feedback law:

$$(4.2) \quad \hat{u} = F(s)\hat{y} + \hat{v}.$$

If the characteristic matrix  $(I - G(s)F(s))$  is invertible (this is always the case if  $G(s)$  is strictly proper) it is well known that the transfer function between the new input  $\hat{v}$  and the output  $\hat{y}$  is well defined and given by

$$(4.3) \quad G_F(s) := (I - G(s)F(s))^{-1}G(s).$$

The stability of equilibria or periodic motions of the closed-loop system depends on the position of the poles of  $G_F(s)$ . To describe the poles of the closed-loop transfer function, we introduce a left coprime factorization of  $G(s)$  and a right coprime factorization of  $F(s)$ :

$$(4.4) \quad G(s) = D_{LG}^{-1}(s)N_{LG}(s), \quad F(s) = N_{RF}(s)D_{RF}^{-1}(s).$$

A straightforward calculation results in the following form for the closed-loop transfer function:

$$(4.5) \quad G_F(s) = D_{RF}(s)(D_{LG}(s)D_{RF}(s) - N_{LG}(s)N_{RF}(s))^{-1}N_{LG}(s).$$

Note that every pole of  $G_F(s)$  is a zero of the polynomial

$$(4.6) \quad \phi(s) = \det(D_{LG}(s)D_{RF}(s) - N_{LG}(s)N_{RF}(s))$$

and every zero of  $\phi(s)$  is a pole of  $G_F(s)$  if no pole-zero cancellation occurred. Moreover, if  $G(s)$  is a strictly proper system of McMillan degree  $n$  and  $F(s)$  is a proper compensator of McMillan degree  $q$ , then  $\phi(s)$  is a polynomial of degree  $n + q$ . Identifying the vector space  $\mathbb{K}^{n+q}$  with all monic polynomials of degree  $n + q$  we define the pole placement map for a strictly proper system  $G(s)$  by:

$$(4.7) \quad \rho_G : S_{m,p}^q \longrightarrow \mathbb{K}^{n+q}, \quad F(s) \longmapsto \phi(s)_{\text{monic}}.$$

This definition is in many ways unsatisfactory if  $G(s)$  is proper. Indeed, if  $G(s)$  is proper it is possible that  $\phi(s)$  is not of degree  $n + q$  anymore, in particular if  $(I - G(s)F(s))$  is not invertible  $\phi(s) \equiv 0$ .

To extend the definition of the pole placement map to proper systems we first introduce the following set, which Ghosh [8] called the *base locus*:

$$(4.8) \quad B_G := \{F(s) \in S_{m,p}^q \mid \det(I - G(s)F(s)) \equiv 0\}.$$

To avoid difficulties with low-degree polynomials, we identify the space of polynomials with the projective space  $\mathbb{P}^{n+q}$  and use the following definition.

DEFINITION 4.1. The pole placement map for a proper transfer function  $G(s)$  is given by

$$(4.9) \quad \rho_G : S_{m,p}^q - B_G \longrightarrow \mathbb{P}^{n+q}, \quad F(s) \longmapsto \phi(s).$$

It is, of course, an important problem in multivariable linear control theory: under which condition is  $\rho_G$  onto or at least almost onto? In particular, it would be of great interest to know the minimum order  $q$  of a compensator that pole assigns

or stabilizes a given generic system of order  $n$ . Using a dimension argument, we immediately obtain the following necessary condition for  $\rho_G$  to be onto:

$$(4.10) \quad q(m + p) + mp \geq n + q.$$

One of our main goals in this paper is to show that this condition is also sufficient when the field is algebraically closed and the plant  $G(s)$  is generic. To achieve this result, we first give a new description of the polynomial  $\phi(s)$  and this will enable us to reformulate the problem geometrically.

If  $F(s) = D_{LF}^{-1}(s)N_{LF}(s) = N_{RF}(s)D_{RF}^{-1}(s)$  are a left and a right coprime factorization of  $F(s)$  it is obvious that

$$(4.11) \quad (N_{LF}(s) \ D_{LF}(s)) \begin{pmatrix} D_{RF}(s) \\ -N_{RF}(s) \end{pmatrix} \equiv 0_{m \times p}.$$

In some sense we can view the matrix

$$\begin{pmatrix} D_{RF}(s) \\ -N_{RF}(s) \end{pmatrix}$$

as the dual curve of the Hermann–Martin curve  $(N_{LF}(s)D_{LF}(s))$  of  $F(s)$ . The following lemma, which is well known if the compensator is static [2], is now easy to verify and the proof will be omitted.

LEMMA 4.2. *For a particular point  $s_i \in \bar{\mathbb{K}}$  the following conditions are equivalent:*

$$(4.12) \quad \det \left( (D_{LG}(s_i) \ N_{LG}(s_i)) \begin{pmatrix} D_{RF}(s_i) \\ -N_{RF}(s_i) \end{pmatrix} \right) = 0,$$

$$(4.13) \quad \det \begin{pmatrix} D_{LG}(s_i) & N_{LG}(s_i) \\ N_{LF}(s_i) & D_{LF}(s_i) \end{pmatrix} = 0,$$

$$(4.14) \quad \text{rowsp}(D_{LG}(s_i) \ N_{LG}(s_i)) \cap \text{rowsp}(N_{LF}(s_i)D_{LF}(s_i)) \neq \{0\}.$$

Note that two polynomials with the same roots are multiples of each other. In other words the following corollary holds.

COROLLARY 4.3.

$$(4.15) \quad \phi(s) = \det(D_{LG}(s)D_{RF}(s) - N_{LG}(s)N_{RF}(s)) = c \cdot \det \begin{pmatrix} D_{LG}(s) & N_{LG}(s) \\ N_{LF}(s) & D_{LF}(s) \end{pmatrix}$$

The  $(m+p) \times (m+p)$  matrix appearing in this equation has many nice properties. On one side the equation

$$(4.16) \quad \begin{pmatrix} D_{LG}(s) & N_{LG}(s) \\ N_{LF}(s) & D_{LF}(s) \end{pmatrix} \cdot \begin{pmatrix} y \\ -u \end{pmatrix} (s) = 0$$

gives a combined description of the plant and the compensator equations by means of autoregressive equations. This point of view can be found, e.g., in [31], [39].

Geometrically,  $(D_{LG}(s)N_{LG}(s))$  defines a rational curve  $\zeta \in \text{Rat}_n(\mathbb{P}^1, \text{Grass}(p, m + p))$  and  $(N_{LF}(s)D_{LF}(s))$  defines a rational curve  $\psi \in \text{Rat}_q(\mathbb{P}^1, \text{Grass}(m, m + p))$ . Using the Plücker embedding (2.6) we can represent  $\zeta$  by

$$(4.17) \quad g(s) := g_1(s) \wedge \cdots \wedge g_p(s),$$

where again  $g_i(s)$  denotes the  $i$ th row of  $(D_{LG}(s)N_{LG}(s))$ . Similarly,  $\psi$  has a representation

$$(4.18) \quad f(s) := f_1(s) \wedge \cdots \wedge f_m(s).$$

Finally the poles of the closed-loop system are the zeros of the polynomial

$$(4.19) \quad \tilde{\phi}(s) := g_1(s) \wedge \cdots \wedge g_p(s) \wedge f_1(s) \wedge \cdots \wedge f_m(s).$$

Note that  $\tilde{\phi}(s)$  is, of course, a multiple of the polynomial  $\phi(s)$ . In addition the wedge product  $g(s) \wedge f(s)$  defines a bilinear pairing  $\langle \cdot, \cdot \rangle$  that extends linearly to the product space  $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p}) \times \mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p})$ .

We are now in a position to formulate the pole placement problem with dynamic compensators in a geometric language.

*Geometric problem.* Given a rational curve  $\zeta \in \text{Rat}_n(\mathbb{P}^1, \text{Grass}(p, m+p))$  and a divisor  $P = \{s_1, \dots, s_{n+q}\}$ . Is there a curve  $\psi \in \text{Rat}_q(\mathbb{P}^1, \text{Grass}(m, m+p))$  such that  $\psi(s_i) \cap \zeta(s_i) \neq \{0\}$  for all  $s_i \in P$ ? What is the minimal degree  $q$  needed?

*Remark 4.4.* Not all geometric solutions enable us to construct a proper compensator although it is always possible to represent such a solution by an autoregressive system. In addition, we want to find solutions that are admissible (compare with [31]). In geometric terms, we want to exclude a Hermann–Martin curve  $\psi(s)$  with the property that  $\psi(s) \cap \zeta(s) \neq \{0\}$  for all  $s \in \mathbb{K}$ .

To handle these difficulties we make the following definition.

**DEFINITION 4.5.** A rational curve  $\zeta \in \text{Rat}_n(\mathbb{P}^1, \text{Grass}(p, m+p))$  is called  $q$ -degenerate if there is a rational curve  $\psi \in \text{Rat}_i(\mathbb{P}^1, \text{Grass}(m, m+p))$  with  $i \leq q$  and  $\psi(s) \cap \zeta(s) \neq \{0\}$  for all  $s \in \mathbb{K}$ . A curve that is not  $q$ -degenerate is called  $q$ -nondegenerate. A system  $G(s)$  is called  $q$ -(non)degenerate if the corresponding Hermann–Martin curve is  $q$ -(non)degenerate.

Note that our definition is a natural generalization of the concept of the degenerate system as introduced in [2], and this concept itself generalizes the concept of a degenerate curve in projective space. In a concrete example we can use the equivalent formulations in Lemma 4.2 to decide if a particular plant  $G(s)$  is  $q$ -degenerate.

From the definition it now follows immediately that the pole placement map  $\rho_G$  introduced in (4.7) and (4.9) can be extended in a continuous manner to a morphism  $\bar{\rho}_G$  defined on the whole compactification  $K_{m,p}^q$  if the system  $G(s)$  is  $q$ -nondegenerate. In other words, all autoregressive systems  $P(s) \in K_{m,p}^q$  are admissible and the base locus set  $B_G$  introduced in (4.8) is empty:

$$(4.20) \quad \begin{array}{ccc} \bar{\rho}_G : & K_{m,p}^q & \longrightarrow \mathbb{P}^{n+q} \\ & \uparrow & \uparrow \\ \rho_G : & S_{m,p}^q & \longrightarrow \mathbb{K}^{n+q} \end{array}$$

The concept of  $q$ -degeneracy will be of crucial importance in the next section. The following example will illustrate the concept of  $q$ -degeneracy on a 3-input, 1-output system.

*Example 4.6.*

1.  $G(s) = (1/s^5, 1/s^3, 1/s)$  defines a system of order 5, which is 1-degenerate. Indeed, use the first condition in Lemma 4.2 to construct a covector which will make the inner product  $\langle \cdot, \cdot \rangle$  identically zero:

$$(4.21) \quad \langle (1, s^2, s^4, s^5), (0, 0, s, -1) \rangle \equiv 0$$

2.  $\tilde{G}(s) = (1/s^6, 1/s^4, 1/s^2)$  defines a system of order 6 which is 1-nondegenerate because

$$(4.22) \quad \langle (1, s^2, s^4, s^6), (a_1 + b_1s, a_2 + b_2s, a_3 + b_3s, a_4 + b_4s) \rangle \equiv 0$$

implies  $a_i = b_i = 0, \quad i = 1, \dots, 4$ . Actually, we will show in the next section that the generic 1-input, 3-output system of order 6 is 1-nondegenerate.

**5. On the minimal order dynamic compensator.** In this section we will assume that the ground field  $\mathbb{K}$  is algebraically closed. The following theorem, called the projective dimension theorem, will be used several times in this section. Our formulation can be found in Hartshorne [9], where a proof is also given.

**THEOREM 5.1.** *Let  $Y, Z$  be varieties of dimension  $r, s$  in  $\mathbb{P}^N$ . Then every irreducible component of  $Y \cap Z$  has dimension  $\geq r + s - N$ . Furthermore, if  $r + s - N \geq 0$ , then  $Y \cap Z$  is nonempty.*

The next theorem that we present is a strong version of the classical Bézout theorem, which we will need to prove Theorem 5.7. The theorem was originally formulated and proven by Weil [36]. The crucial part for the formulation of the theorem was the “right” definition of the intersection multiplicity  $i$ . For a broader discussion of this theorem and its generalizations we refer the reader to Vogel [33]. The following theorem is a reformulation of [33, Prop. 3.26].

**THEOREM 5.2.** *Let  $Y, Z$  be varieties of dimension  $r, s$  in  $\mathbb{P}^N$ . Assume the intersection  $Y \cap Z$  is proper, i.e.,  $\dim(Y \cap Z) = r + s - N$ . Denote with  $\Omega$  the set of irreducible components of  $Y \cap Z$  and with  $i(Y, Z; C)$  the intersection multiplicity of  $Y$  and  $Z$  along  $C$ . Then we have*

$$(5.1) \quad \deg Y \cdot \deg Z = \sum_{C \in \Omega} i(Y, Z; C) \cdot \deg C.$$

Another important concept in all that follows is the notion of a central projection. Assume  $E, H$  are linear subspaces of dimension  $r, N - r - 1$  and  $E \cap H = \emptyset$ . In this case we can define the following map, which is well defined by basic facts of linear algebra:

$$(5.2) \quad \pi : \mathbb{P}^N - E \longrightarrow H, \quad x \longmapsto \text{span}(x, E) \cap H.$$

$\pi$  is called a central projection onto  $H$  with center  $E$ . As shown by Wang [34], the pole placement map with static compensators is a central projection. As we will show, the same is true in the dynamic case.

Our first goal is a characterization of the  $q$ -nondegenerate systems.

**LEMMA 5.3.** *The set of  $q$ -degenerate systems is algebraic in the quasi-projective variety  $S_{p,m}^n$  of proper systems with McMillan degree  $n$ .*

*Proof.* Consider in  $S_{p,m}^n \times K_{m,p}^q$  the coincidence set

$$(5.3) \quad S := \{ (N_{LG}(s) D_{LG}(s), (N_{RF}(s) D_{RF}(s)) \mid \det(D_{LG}(s) D_{RF}(s) - N_{LG}(s) N_{RF}(s)) \equiv 0 \},$$

which defines an algebraic set in the product. Because  $K_{m,p}^q$  is projective, the projection on the first factor is still an algebraic set by the main theorem of elimination theory (see, e.g., [24]).  $\square$

The next lemma shows that every system is  $q$ -degenerate for some large natural number  $q \in \mathbb{N}$ .

LEMMA 5.4. *If  $q(m + p) + mp > n + q$ , every  $p \times m$  system of order  $n$  is  $q$ -degenerate.*

*Proof.* Assume  $g(s)$  are the Plücker coordinates of a plant  $G(s)$  with McMillan degree  $n$ . Consider in  $\mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p})$  the set

$$(5.4) \quad E_G := \{f(s) \mid \langle g(s), f(s) \rangle \equiv 0\}.$$

$E_G$  defines a plane of codimension at most  $q(m + p) + mp$ , the dimension of the variety  $K_{m,p}^q$ . The plane  $E_G$  intersects  $K_{m,p}^q$  by the projective dimension theorem.  $\square$

So far it has only been shown (Lemma 5.3) that the set of  $q$ -nondegenerate systems form a Zariski-open (possibly empty) set in  $S_{p,m}^n$ . Using the following theorem we will be able to show that this Zariski-open set is nonempty in  $S_{p,m}^n$  if  $q$  is small enough.

THEOREM 5.5. *The dimension of the coincidence set  $S \subset S_{p,m}^n \times K_{m,p}^q$  introduced in (5.3) is given by*

$$(5.5) \quad \dim S = \dim S_{p,m}^n + \dim K_{m,p}^q - n - q - 1.$$

*Proof.* Consider an element of  $S$  given by

$$(5.6) \quad \det \begin{pmatrix} D_{LG}(s) & N_{LG}(s) \\ N_{LF}(s) & D_{LF}(s) \end{pmatrix} \equiv 0.$$

Without loss of generality we assume that the system  $(D_{LG}(s)N_{LG}(s))$  and the compensator  $(N_{LF}(s)D_{LF}(s))$  are both row reduced with minimal indices  $\nu_1 \geq \dots \geq \nu_p$ , respectively,  $\mu_1 \geq \dots \geq \mu_m$  and  $\nu_1 \geq \mu_1$ . In particular we have  $n = \sum_{i=1}^p \nu_i$  and  $q = \sum_{j=1}^m \mu_j$ . As explained in [30] we have a free action on  $(N_{LF}(s)D_{LF}(s))$  with an algebraic group of dimension at least  $m^2$ . This group is characterized as the subgroup of the unimodular group  $Gl_m(\mathbb{K}[s])$  which leaves the row indices  $\mu_1, \dots, \mu_m$  invariant. Similarly there is a free action on  $(D_{LG}(s)N_{LG}(s))$  with an algebraic group of dimension at least  $p^2$ .

Denote with  $S_1$  the parameter space of all  $(m + p - 1) \times (m + p)$  polynomial matrices having row indices  $\nu_2, \dots, \nu_p, \mu_1, \dots, \mu_m$ .  $S_1$  is a vector space of dimension

$$(5.7) \quad \dim S_1 = \left( \sum_{i=2}^p \nu_i + \sum_{j=1}^m \mu_j + m + p - 1 \right) (m + p).$$

Assume now that the last  $m + p - 1$  rows form a minimal basis in the sense of Forney [6] of the  $\mathbb{K}(s)$ -vector space, which these rows generate. Equivalently, the greatest common divisor of the full size  $(m + p - 1) \times (m + p - 1)$  minors is 1. In the following we restrict the dimension calculation to this Zariski-open subset of  $S_1$  because it is not difficult to show that the other cases lead to lower-dimensional subsets. From (5.6) it then follows that the first row is a linear combination of the last  $m + p - 1$  rows

$$(5.8) \quad g_1(s) = \sum_{i=2}^p x_i(s)g_i(s) + \sum_{j=1}^m y_j(s)h_j(s), \quad x_i(s), y_j(s) \in \mathbb{K}(s).$$

From the main theorem in Forney [6] it follows that  $x_i(s), y_j(s)$  are even elements of  $\mathbb{K}[s]$ . Moreover  $\deg x_i(s) \leq \nu_1 - \nu_i$  and  $\deg y_j(s) \leq \nu_1 - \mu_j$ .

Denote with  $S_2$  all polynomial vectors  $g_1(s)$  of degree  $\nu_1$  that are in the row-space of a given set of vectors  $\{g_2(s), \dots, g_p(s), \dots, h_m(s)\}$ . From the above follows that

$$(5.9) \quad \dim S_2 = \sum_{i=2}^p (\nu_1 - \nu_i + 1) + \sum_{j=1}^m (\nu_1 - \mu_j + 1) = (m+p)\nu_1 - n - q + m + p - 1.$$

Finally, taking into consideration the free action of the above-mentioned groups, we obtain

$$(5.10) \quad \dim S \leq \dim S_1 + \dim S_2 - m^2 - p^2$$

$$(5.11) \quad = n(m+p) + mp + q(m+p) + mp - n - q - 1$$

$$(5.12) \quad = \dim S_{p,m}^n + \dim K_{m,p}^q - n - q - 1.$$

Finally, (5.6) imposes at most  $n+q+1$  algebraic conditions because the characteristic equation is a polynomial of degree at most  $n+q$ . The inequality in (5.10) is therefore an equality.  $\square$

**COROLLARY 5.6.** *If  $q(m+p) + mp \leq n+q$ , the generic  $p \times m$  proper system of order  $n$  is  $q$ -nondegenerate.*

*Proof.* Because  $\dim K_{m,p}^q = q(m+p) + mp$  it follows from (5.5) that  $\dim S \leq \dim S_{p,m}^n - 1$ . In particular the projection of  $S$  onto  $S_{p,m}^n$  is a proper algebraic subset in  $S_{p,m}^n$ .  $\square$

The previous corollary was proven for  $q = 0$  (static feedback) by Brockett and Byrnes [2], from which it then followed that the pole placement map with static compensators is generically onto if  $mp = n$ . In the following we will extend this result to the dynamic case. The proof that we present combines ideas from a proof given by Rosenthal in [27] and a proof given by Wang in [34] for the case of static feedback.

**THEOREM 5.7.** *If a system  $G(s)$  is  $q$ -nondegenerate and  $q(m+p) + mp = n+q$ , then the pole placement map*

$$(5.13) \quad \bar{\rho}_G : K_{m,p}^q \longrightarrow \mathbb{P}^{n+q}$$

*is onto of degree  $d_{m,p,q}$ , where  $d_{m,p,q}$  is the degree of the variety  $K_{m,p}^q$ .*

*Proof.* Consider in  $\mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p})$  again the linear subspace

$$(5.14) \quad E_G = \{f(s) \mid \langle g(s), f(s) \rangle \equiv 0\}.$$

Because  $G(s)$  is  $q$ -nondegenerate it follows that  $E_G \cap K_{m,p}^q = \emptyset$  and the codimension of  $E_G$  is equal to  $q(m+p) + mp + 1$ . The linear pairing  $\langle \cdot, \cdot \rangle$  induces a linear map

$$(5.15) \quad L : \mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p}) - E_G \longrightarrow \mathbb{P}^{n+q}$$

$$f(s) \longmapsto \langle g(s), f(s) \rangle,$$

which has to be onto by a linear argument. Note that  $\bar{\rho}_G = L|_{K_{m,p}^q}$ . Denote with  $H$  any linear subspace of  $\mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p})$  for which  $\dim H = n+p$  and  $L(H) = \mathbb{P}^{n+q}$ . We have a central projection

$$(5.16) \quad \pi : \mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p}) - E_G \longrightarrow H.$$

If  $y \in H$  is a particular point, it follows by linear equation theory that the whole fiber  $\pi^{-1}(y)$  (which is a linear plane in  $\mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p})$ ) is mapped under  $L$  onto  $L(y)$ . In other words, we have  $L = L \circ \pi$  and  $\bar{\rho}_G$  is onto if and only if  $\pi|_{K_{m,p}^q}$  is onto.

By the projective dimension theorem  $\pi^{-1}(y) \cap K_{m,p}^q \neq \emptyset$ . Finally every fiber  $\pi^{-1}(y)$  intersects  $K_{m,p}^q$  properly [32, p. 48]. By Theorem 5.2,  $\pi^{-1}(y) \cap K_{m,p}^q$  consists in this case of exactly  $d_{m,p,q}$  points when counted with multiplicities.  $\square$

If the system  $G(s)$  is strictly proper and the compensator  $F(s)$  is admissible and proper, it follows from Corollary 4.3 that the closed-loop characteristic polynomial  $\phi(s)$  has degree exactly equal to  $n + q$ , the sum of the McMillan degrees of  $G(s)$  and  $F(s)$ . In other words the “infinite points,” that is, the points in the set  $K_{m,p}^q - S_{m,p}^q$ , are mapped onto the closed-loop characteristic polynomials of degree strictly less than  $n + q$ . We therefore obtain the following corollary.

**COROLLARY 5.8.** *If  $G(s)$  is  $q$ -nondegenerate and strictly proper and  $q(m + p) + mp = n + q$ , then the pole placement map  $\rho_G : S_{m,p}^q \rightarrow \mathbb{K}^{n+q}$  introduced in (4.7) is onto. Moreover, if counted with multiplicities there are exactly  $d_{m,p,q}$  different compensators  $F(s)$  assigning a specific closed-loop characteristic polynomial.*

The degree of the variety  $K_{m,p}^q$  is therefore equal to the number of compensators that will place the poles of the closed-loop system at a desired location. In particular, if  $G(s)$  is a real plant and the number  $d_{m,p,q}$  would turn out to be odd for certain  $m, p, q$ , we would be able to predict the existence of a real compensator because the solution set must be invariant under complex conjugation. In the case of static feedback, i.e.,  $q = 0$ , we have  $K_{m,p}^0 = \text{Grass}(m, m + p)$  and it is well known when the degree of the Grassmann variety is odd. (Compare, e.g., [3].) As shown in [28] the degree of  $K_{2,3}^1$  is equal to 55 and so such (nontrivial) cases also exist if  $q > 0$ . The following corollary explains the proper situation.

**COROLLARY 5.9.** *If  $G(s)$  is  $q$ -nondegenerate and proper and  $q(m + p) + mp = n + q$ , then the pole placement map  $\rho_G : S_{m,p}^q \rightarrow \mathbb{P}^{n+q}$  introduced in (4.9) is almost onto.*

*Proof.* Because  $G(s)$  is  $q$ -nondegenerate, the lifted map  $\bar{\rho}_G : K_{m,p}^q \rightarrow \mathbb{P}^{n+q}$  exists and is onto by Theorem 5.7. The difference set  $K_{m,p}^q - S_{m,p}^q$  has dimension strictly less than  $n + q$ . Because  $\bar{\rho}_G(K_{m,p}^q) = \mathbb{P}^{n+q}$  and  $\bar{\rho}_G|_{S_{m,p}^q} = \rho_G$  the statement follows.  $\square$

**Remark 5.10.** From the proof it follows in particular that those closed-loop characteristic polynomials that cannot be achieved with a proper compensator can always be achieved with a general autoregressive compensator. (Compare with Remark 4.4.)

So far we have provided only positive results, that is, results when the dimension of the domain and the range are equal. The following theorem explains the situation when the dimension of the domain is larger than the dimension of the range.

**THEOREM 5.11.** *If  $G(s) \in S_{p,m}^n$  is a generic plant and if*

$$(5.17) \quad q(m + p) + mp \geq n + q,$$

*then the pole placement map*

$$(5.18) \quad \rho_G : S_{m,p}^q - B_G \rightarrow \mathbb{P}^{n+q}$$

*introduced in Definition 4.1 is almost onto. Moreover, the extended map*

$$(5.19) \quad \bar{\rho}_G : K_{m,p}^q - E_G \rightarrow \mathbb{P}^{n+q}$$

*is onto.*

*Proof.* Consider again the coincidence set  $S \subset S_{p,m}^n \times K_{m,p}^q$  introduced in the proof of Lemma 5.3. Denote with  $pr : S \rightarrow S_{p,m}^n$  the projection onto the first factor. From



Theorem 5.5 it then follows that for a generic element  $G(s) \in S_{p,m}^n$  the dimension of the fiber  $pr^{-1}(G(s))$  is bounded by

$$(5.20) \quad \dim(pr^{-1}(G(s))) \leq \dim K_{m,p}^q - n - q - 1.$$

In particular, using earlier notation we have

$$(5.21) \quad \dim(E_G \cap K_{m,p}^q) \leq q(m+p) + mp - n - q - 1.$$

Following the proof of Theorem 5.7 and using again the projective dimension theorem it follows that

$$(5.22) \quad \dim(\pi^{-1}(y) \cap K_{m,p}^q) > q(m+p) + mp - n - q - 1.$$

From above two inequalities it now follows in particular that for every closed-loop polynomial  $p(s)$  there is an admissible autoregressive system  $F(s) \in K_{m,p}^q - E_G$  with  $\bar{\rho}_G(F(s)) = p(s)$ . The map  $\bar{\rho}_G$  is therefore onto. Finally because the fibers of  $\bar{\rho}_G$  in  $K_{m,p}^q - E_G$  have dimension at least  $q(m+p) + mp - n - q$  and the dimension of the range of  $\bar{\rho}_G$  is  $n+q$ , the map  $\rho_G : S_{m,p}^q - B_G \rightarrow \mathbb{P}^{n+q}$  is almost onto by a dimension argument.  $\square$

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