Output Feedback Pole Placement with Dynamic Compensators

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***Abstract*—In this paper we derive a new rank condition which guarantees the arbitrary pole assignability of a given system by dynamic compensators of degree at most q. By using this rank condition we establish several new sufficiency conditions which ensure the arbitrary pole assignability of a generic system. Our proofs also come with a concrete numerical procedure to construct a particular compensator which assigns a given set of closed-loop poles.**

I. Introduction

ONE of the main open problems in linear system theory is to determine the minimum order q of a dynamic compensator which can arbitrarily assign the closed-loop poles of a generic m-input, p-output system of McMillan degree n. This problem has the pole placement problem by memoryless feedback as a special case, a subject which was studied by many researchers.

A classical result of Brash and Pearson [1] states that arbitrary pole assignment for a controllable and observable system can be achieved using dynamic compensators of order

$$q = \min(\kappa_{\max}, \nu_{\max}) \tag{1}$$

where $\kappa_{\max}$ and $\nu_{\max}$ are the largest controllability and observability index of the plant, respectively. A corollary of this result is that arbitrary pole assignment is possible for the generic system if (see, e.g., [2])

$$\max(m, p)(q + 1) \geq n. \tag{2}$$

Despite many attempts and despite many small improvements, one has to say that the result of Brash and Pearson remained one of the major results until a couple of years ago. Of course the question arises "how much better" can one possibly do, and this question was answered by Willems and Hesselink [3] who showed through a "parameter count" that

$$q(m + p - 1) + mp \geq n \tag{3}$$

i.e., $q \geq \frac{n-mp}{m+p-1}$ is a necessary condition for generic pole assignment.

We would like to note at this point that the problem at hand is clearly a nonlinear problem. Since the constraints involved can all be expressed through polynomial equations, it is not surprising that one way of studying this problem is by means of algebraic geometry. Unfortunately some of the most powerful theorems available in algebraic geometry require that the field is algebraically closed, a property which the complex numbers have but the reals do not have. Still, it was possible to show through the use of the so-called dominant morphism theorem (Hermann and Martin [4]) and the use of Schubert calculus (Brockett and Byrnes [5], see also [2]) that (3) is also a sufficient condition for the static pole placement problem (i.e., q has to be zero) if complex compensators can be implemented in the feedback loop.

For the general dynamic problem (i.e., q can be larger than zero) Rosenthal [6] proved by applying the so-called projective dimension theorem that (3) is a necessary and sufficient condition if complex compensators can be implemented.

Of course from an engineering point of view the "real problem," i.e., the problem which requires that the compensators have real parameters, remained a long way away from its complexified counterpart. Only recently has significant progress been made.

A first breakthrough was established in 1992 by Wang [7] for the static ($q = 0$) pole placement problem. Using algebraic geometric techniques, Wang established the result that the generic system has arbitrary pole assignability if

$$mp > n \tag{4}$$

missing the necessary condition (3) by only one degree of freedom.

Again, using geometric techniques, Wang and Rosenthal [8] were able to derive the result that

$$q \max(m, p) + mp > n \tag{5}$$

implies generic pole assignability over the reals. Note that (5) comes close to (3) as long as q and $\min(m, p)$ are small.

Seemingly independently Ariki [9], Leventides and Karcanias [10], [11], and Wang [12] all realized that the geometric techniques employed in [7] and [8] are essentially based on a linearization procedure around a so-called *dependent compensator*. While Leventides and Karcanias [11] and Wang [12] were using this idea mainly to derive numerical schemes and formulas capable of constructing static ($q = 0$) feedback compensators, it was used by Ariki [9] to derive a simplified proof of the sufficiency result (5). Finally independent of [9] and [11], Rosenthal *et al.* [13] derived a proof of Wang's result which also describes through explicit polynomial equation an

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open set of matrices A, B, C which have the pole assignability property.

In this paper we elaborate on the linearization techniques developed in [9]–[14], and we derive a new rank condition which ensures arbitrary pole assignability. Using this condition we prove that the generic system has arbitrary pole assignability if

$$q(m+p-1)+mp-\min(r_m(p-1), r_p(m-1)) > n \quad (6)$$

where $r_m = q - m[q/m]$ and $r_p = q - p[q/p]$ are the remainders of q divided by m and p, respectively. In particular, when q is a multiple of either m or p, one has the sufficient condition

$$q(m+p-1)+mp > n \quad (7)$$

which misses the necessary condition of (3) by only one degree of freedom. Note also that Willems and Hesselink [3] already showed that when $q(m+p-1)+mp = n$, then a real solution may not exist, i.e., (6) is the best possible bound in many situations.

The paper is structured as follows. In the next section we will develop the theory using system descriptions as they have been recently considered in the "behavioral literature." There are many advantages for this approach, and we will say more about this in the course of the paper. Using this quite general setup, we will extend and summarize in this section several of the major theorems which exists in the area of dynamic pole placement.

The main results will be given in Section III. Using generalized first-order representations, we will formulate first the main theorem of this paper and several of its corollaries. The results are mainly based on a careful study of the so-called *pole placement map*. Of particular importance will be the linearization around a so-called *dependent compensator*. In Section III we present the main ideas of the proofs, and we illustrate the results on several examples. The more technical parts of the proofs are given in the Appendix.

In Section IV we describe an effective computational method to compute the dynamic compensators assigning any self-conjugate set of poles for any given system which satisfies the rank condition described in Section III.

II. Preliminary Results and Problem Formulation

In this section we collect some preliminary results and simultaneously establish our notation. It is our intention to develop the theory in a behavioral framework, since the essence of the problems and their solutions are most transparent in this language. For the connection between the behavioral point of view and the classical state-space formulation, we refer to [15]–[17] and also to the recent preprint [18].

As it was already stated in the introduction, it is the goal of this paper to provide a strong sufficiency condition which guarantees that a "generic system" describing "some generic behavior" is arbitrary pole assignable using real dynamic compensators of a bounded McMillan degree.

Recall from [17] that by definition a *dynamical system* Σ is a triple $\Sigma = (T, W, \mathcal{B})$, where $T \subset \mathbb{R}$ is the time axis, W is the signal space, and $\mathcal{B} \subset W^T$ is called the behavior. In this paper we will only consider dynamical systems Σ whose time axis $T = \mathbb{R}$, whose signal space $W = \mathbb{R}^{m+p}$, and whose behavior $\mathcal{B} \subset C^\infty(\mathbb{R}, \mathbb{R}^{m+p})$ has a so-called "kernel representation," i.e., there exists a polynomial matrix $P(s)$ such that

$$\mathcal{B} = \left\{ w(t) \in C^\infty(\mathbb{R}, \mathbb{R}^{m+p}) \mid P\left(\frac{d}{dt}\right) w(t) = 0 \right\}. \quad (8)$$

Behaviors having particular kernel representation of the form of (8) are sometimes called "AR"-systems (compare with [19] and [17]), and we will abbreviate it through $\mathcal{B}(P)$. Every such AR-system comes with two important invariants called the *rank* $r(\Sigma)$ and the *McMillan degree* $n(\Sigma)$ which are defined as follows: $r(\Sigma)$ is equal to the minimal number of rows needed for a polynomial matrix $P(s)$ which describes the behavior $\mathcal{B}$ in a representation of the form of (8). We will call such a representation a (row) minimal representation.

If the polynomial matrix $P(s)$ is row minimal, then we define the McMillan degree $n(\Sigma)$ as the maximal degree of the full-size minors in one and therefore any minimal representation.

The rank $r(\Sigma)$ and the McMillan degree $r(\Sigma)$ are quite "rough" system invariants. A much finer set of (projective) system invariants are the set of all full-size minors of $P(s)$ in a minimal polynomial representation. This set of system invariants is of particular importance if the system Σ is *autonomous*. Recall that a system $\Sigma = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B})$ is called autonomous if the rank $r(\Sigma) = m+p$. If the $(m+p) \times (m+p)$ matrix $P(s)$ describes an autonomous behavior, we define $\det P(s)$ as the *characteristic polynomial* of Σ which we will abbreviate with χ_Σ. As we just mentioned, χ_Σ is a projective invariant, i.e., if

$$\det P(s) = a_0 + a_1 s + \cdots + a_{n+q} s^{n+q}$$

then $(a_0, \cdots, a_n)$ defines a unique one-dimensional subspace, i.e., a point in the projective space $\mathbb{P}^n$ (see, e.g., [20]). This point then only depends on the autonomous system Σ and does not depend on the particular representation. The roots of $\det P(s)$ are by definition the *poles* of Σ.

Next we would like to introduce feedback. For this, assume that $\Sigma_1 = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B}_1)$ and $\Sigma_2 = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B}_2)$ are two AR-systems. Then the interconnected system $\Sigma_1 \wedge \Sigma_2$ is defined as

$$\Sigma_1 \wedge \Sigma_2 := (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B}_1 \cap \mathcal{B}_2).$$

We say $\Sigma_1 \wedge \Sigma_2$ is a regular interconnection (see [18]) if the ranks "add up," i.e., if

$$r(\Sigma_1 \wedge \Sigma_2) = r(\Sigma_1) + r(\Sigma_2)$$

and we speak of a singular interconnection if this is not the case.

As it is immediate from the definition, the interconnected system $\Sigma_1 \wedge \Sigma_2$ is defined through

$$\begin{pmatrix} P_1\left(\frac{d}{dt}\right) \\ P_2\left(\frac{d}{dt}\right) \end{pmatrix} w(t) = 0$$

where P_1, P_2 are polynomial matrices representing Σ_1, respectively, Σ_2. If P_1, P_2 are in addition row minimal representations, then one verifies that $\Sigma_1 \wedge \Sigma_2$ is a regular interconnection if and only if $\binom{P_1}{P_2}$ is row minimal.

The fundamental question, which is in fact a generalized pole placement problem and which implies many of the "traditional" pole placement questions, now follows.

Problem 2.1: Let m, p, n, q be fixed positive integers. Under what condition is it true that for a generic set of systems system $\Sigma_1 = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B}_1)$ having rank $r(\Sigma_1) = p$ and McMillan degree $n(\Sigma_1) = n$ the following holds: For every polynomial $\phi \in \mathbb{R}[s]$ of degree $n+q$ there exists a system $\Sigma_2 = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B}_2)$ having rank $r(\Sigma_2) = m$ and McMillan degree $n(\Sigma_2) \leq q$ such that $\chi_{\Sigma_1 \wedge \Sigma_2} = \phi$.

If Problem 2.1 has a positive answer, we will say that the generic rank p system of McMillan degree n is arbitrary pole assignable in the class of feedback compensators of McMillan degree at most q. As it will turn out, in this case the generic m-input, p-output system of McMillan degree n is then also arbitrary pole assignable using compensators of McMillan degree q only (compare with Theorem 3.3).

A major difficulty in the formulation of Problem 2.1 is of course the term "generic" which we now want to make precise. For this note Kuijper and Schumacher [21] have shown that every AR-system $\Sigma = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B})$ has an equivalent first-order representation, i.e., a "realization." To be precise they showed [21, Th. 4.1] (compare also with [22], [23], [13]) for every system Σ having rank $r(\Sigma) = p$ and McMillan degree $n(\Sigma) = n$ the existence of $n \times (m+n)$ matrices F, G and a $(m+p) \times (m+n)$ matrix H such that the behavior $\mathcal{B}$ of Σ is equivalently described through the first-order representation

$$G\dot{z}(t) = Fz(t), \quad w(t) = Hz(t). \tag{9}$$

In this representation $z(t) \in Z \simeq \mathbb{R}^{m+n}$ describes the set of "internal variables" and $w(t) \in \mathbb{R}^{m+p}$ describes the behavior $\mathcal{B}$. The matrices F, G are linear maps from the space of internal variables $Z \simeq \mathbb{R}^{m+n}$ to the state-space $X \simeq \mathbb{R}^n$. The triple (F, G, H) is essentially the same as "a linear machine" as defined in [23, p. 178]. Clearly (9) describes a behavior $\mathcal{B} \subset C^\infty(\mathbb{R}, \mathbb{R}^{m+p})$, and we will abbreviate this behavior with $\mathcal{B}(F, G, H)$. Corresponding to a change of coordinates in X and Z, one has a natural equivalence among pencil representations

$$(F, G, H) \sim (SGT^{-1}, SFT^{-1}, HT^{-1}) \tag{10}$$

where $S \in Gl_n$ and $T \in Gl_{m+n}$. Note that the behavior $\mathcal{B}(F, G, H)$ is invariant under those two changes of base.

Recall that a subset S of a vector space V is called a generic set if the complement of S in V is contained in a proper algebraic set. In the sequel we will view triples of matrices F, G, H, where F, G are of size $n \times (m+n)$ and H is of size $(m+p) \times (m+n)$ simply as points in the vector space $\mathbb{R}^{(m+p+2n)(m+n)}$. Based on (9) we will give the following definition.

Definition 2.2: Let $A^n_{p,m}$ denote the set of all AR-systems $\Sigma = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B})$ having rank $r(\Sigma) = p$ and McMillan degree $n(\Sigma) = n$. A subset $S \subset A^n_{p,m}$ is called a generic set of systems if the set

$$\{(F, G, H) \mid \mathcal{B}(F, G, H) \in S\} \tag{11}$$

forms a generic set of the vector space $\mathbb{R}^{(m+p+2n)(m+n)}$.

We would like to motivate this definition a little. First note that the behaviors of the form $\mathcal{B}(F, G, H)$ represent a slightly more general class of behaviors than the set of behaviors representable by a kernel representation of the form of (8), and we refer to [24] and [25] for a detailed treatment. There is an important notion of minimality which extends the notion of observability in a natural way (compare with [25, Th. 2.3] and [26, Def. 2.1]).

Definition 2.3: System (9) is called minimal if the homogeneous pencil $[sG - tF]$ has (generically) full row rank and if the pencil $\left[\begin{smallmatrix} sG-tF \\ H \end{smallmatrix}\right]$ has full column rank for all $(s, t) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

One of the main results derived in [25, Th. 4.4] states that two minimal triples (F_1, G_1, H_1) and (F_2, G_2, H_2) define the same (impulsive) behavior if and only if those triples are equivalent in the sense of (10).

Let $Y \subset \mathbb{R}^{(m+p+2n)(m+n)}$ denote the set of all minimal triples (F, G, H). Theorem 4.4 in [25] essentially says that the orbit space $Y/(Gl_n \times Gl_{m+n})$ parameterizes behaviors having a fixed rank and a fixed McMillan degree. Moreover, the main theorem in [26] states that $Y/(Gl_n \times Gl_{m+n})$ is isomorphic to the set of so-called homogeneous AR-systems having rank p and McMillan degree n, and because of this we will denote this set with $\mathcal{H}^n_{p,m}$. The main result in [27] states that $\mathcal{H}^n_{p,m}$ has the structure of a smooth projective variety, i.e., it also has the structure of a compact manifold. In the algebraic geometry literature $\mathcal{H}^n_{p,m}$ is also referred to as a *Quot scheme*. For further reference we will quickly calculate the dimension of $\mathcal{H}^n_{p,m}$. This one verifies that the group $Gl_n \times Gl_{m+n}$ acts freely on the set Y. We therefore conclude that

$$\begin{aligned} \dim \mathcal{H}^n_{p,m} &= \dim Y/(Gl_n \times Gl_{m+n}) \\ &= \dim Y - \dim Gl_n - \dim Gl_{m+n} \\ &= n(m+p) + mp. \end{aligned} \tag{12}$$

From the above remarks one sees that a subset $S \subset \mathbb{R}^{(m+p+2n)(m+n)}$ is a generic set if and only if $\pi_1(S \cap Y) \subset \mathcal{H}^n_{p,m}$ is a generic set, where $\pi_1 \colon Y \longrightarrow \mathcal{H}^n_{p,m}$ is the canonical projection.

Crucial in our investigation will be the so-called *pole placement map* associated to a particular plant $P(s) := P_1(s)$. Roughly speaking, the pole placement map assigns to a compensator $P_2(s)$ the closed-loop characteristic polynomial $\det\binom{P_1(s)}{P_2(s)}$. To make this precise, we will again use first-order representations. For this we will represent the polynomial matrix $P_2(s)$ which is of size $m \times (m+p)$ through a triple (F, G, H), where F, G are of size $q \times (p+q)$ and H is of size $(m+p) \times (p+q)$ and has the property that $\mathcal{B}(F, G, H) = \mathcal{B}(P_2)$. Let

$$N := (m+p+2q)(p+q).$$

Identify a polynomial

$$\phi(s) \in a_0 + a_1 s + \cdots + a_{n+q} s^{n+q} \in \mathbb{R}[s]$$

of degree at most $n+q$ with the point $(a_0, a_1, \ldots, a_{n+q})$ in the vector space $\mathbb{R}^{n+q+1}$. Then the pole placement map is defined as

$$\begin{aligned}\chi\colon \mathbb{R}^N &\longrightarrow \mathbb{R}^{n+q+1}\\ (F,G,H) &\longmapsto \det\begin{pmatrix} sG-F\\ P(s)H\end{pmatrix}.\end{aligned} \tag{13}$$

The map χ is a polynomial map and it is characterized through the following important properties:

1) For every $\lambda \in \mathbb{R}$ one has
$$\chi(\lambda F, \lambda G, \lambda H) = \lambda^{p+q}\chi(F,G,H)$$
i.e., χ is homogeneous of degree $p+q$.

2) If $S \in Gl_q$ and $T \in Gl_{p+q}$, then
$$\chi(SGT^{-1}, SFT^{-1}, HT^{-1}) = \frac{\det S}{\det T}\chi(F,G,H).$$

The second property just states that the roots of $\chi(F,G,H)$ (i.e., the poles of the closed-loop system!) do not depend on the representation. In this way we like to see χ as a map which associates to a particular behavior $\mathcal{B}(F,G,H)$ a point in the projective space $\mathbb{P}^{n+q}$. To clarify this we give the following definition.

Definition 2.4: A compensator (F,G,H) (i.e., the behavior $\mathcal{B}(F,G,H)$) is called a dependent compensator if $\chi(F,G,H) = 0$. The set of all dependent compensators in $\mathbb{R}^N$ will be denoted with Y_D.

Using those notations, we have a well-defined polynomial map

$$\begin{aligned}\bar{\chi}:\ \mathbb{R}^N\setminus Y_D &\longrightarrow \mathbb{P}^{n+q}\\ (F,G,H) &\longmapsto \det\begin{pmatrix} sG-F\\ P(s)H\end{pmatrix}\end{aligned} \tag{14}$$

which assigns to every behavior $\mathcal{B}(F,G,H)$ a unique point in $\mathbb{P}^{n+q}$. We conclude this section by formulating two main sufficiency conditions.

First note that part 2) in the previous remark also states that $\bar{\chi}$ "factors" over the orbit space

$$Y/(Gl_q \times Gl_{p+q}) = \mathcal{H}^q_{p,m}.$$

Since

$$\dim \mathcal{H}^q_{p,m} = q(m+p) + mp$$

and since $\dim \mathbb{P}^{n+q} = n+q$, we immediately conclude that the McMillan degree of the compensators must satisfy

$$q \geq \frac{n - mp}{m+p-1}. \tag{15}$$

The main theorem in [6] essentially states that as soon as "complex compensators" are allowed in the feedback loop, then (15) is necessary and sufficient. In our context this means that the domain of the pole placement map χ is extended to the whole complex vector space $\mathbb{C}^N$ resulting in an extended pole placement map $\chi^c\colon \mathbb{C}^N \longrightarrow \mathbb{C}^{n+q+1}$. With this preliminary we have an extension of some results reported in [16].

Proposition 2.5: For a generic set $S \subset A^n_{p,m}$ of (real) systems $\Sigma = (\mathbb{R}, \mathbb{R}^{m+p}, \mathcal{B}(P)) \in A^n_{p,m}$ having rank $r(\Sigma) = p$ and McMillan degree $n(\Sigma) = n$, the extended pole placement map $\chi^c\colon \mathbb{C}^N \longrightarrow \mathbb{C}^{n+q+1}$ is onto as soon as (15) is satisfied. Moreover, if the number

$$d(m,p,q) := (mp + q(m+p))! \cdot c \tag{16}$$

where the constant c is defined as

$$c := \left| \sum_{n_1+\cdots+n_m = q} \frac{\prod_{k<j}(j-k+(n_j-n_k)m+p))}{\prod_{j=1}^{m}(p+j+n_j(m+p)-1)!}\right|$$

is odd, then the (real) pole placement map χ is onto as well.

Proof: See the Appendix. □

The techniques developed in [6] and [16] also provide results for the "traditional dynamic pole placement problem." For this assume that the plant Σ_1 is a m-input, p-output linear system of McMillan degree n described through

$$\dot{x} = Ax + Bu, \quad y = Cx$$

and assume that the compensator Σ_2 is a p-input, m-output linear system of McMillan degree q described through

$$\dot{z} = Fz + Gy, \quad u = Hz + Ky. \tag{17}$$

The closed-loop behavior is then described through

$$\begin{bmatrix} \frac{d}{dt}I_n - A - BKC & -BH\\ -GC & \frac{d}{dt}I_q - F\end{bmatrix}\begin{bmatrix} x\\ z\end{bmatrix}. \tag{18}$$

In this situation the following proposition (see [16, Th. 5.1]) is true.

Proposition 2.6: Let $q \geq \frac{n-mp}{m+p-1}$. Then for a generic set of matrices $(A,B,C) \in \mathbb{R}^{n(n+m+p)}$ the following is true: For every monic polynomial $\phi(s) \in \mathbb{R}[s]$ of degree $n+q$ there exists a complex dynamic compensator of the form of (17) resulting in the closed-loop characteristic polynomial $\phi(s)$. If in addition the number $d(m,p,q)$ introduced in (16) is odd, then there exists a real compensator assigning the closed-loop characteristic polynomial $\phi(s)$.

Note that Propositions 2.5 and 2.6 state that for certain triples m,p,q (namely the ones for which $d(m,p,q)$ is odd) (15) is necessary and sufficient. For the static pole placement problem this has been observed earlier by Brockett and Byrnes [5]. If $d(m,p,q)$ is even, then the best known sufficient condition for the static pole placement problem misses the complex bound $n = mp$ only by one degree of freedom [7]. In the next section we show that the same is true when q is a multiple of either m or p.

III. Main Results

In the sequel we will assume that m,n,p,q are fixed positive integers. p,n characterizes the rank and the McMillan degree of the plant, and m,q will characterize the rank and the McMillan degree of the compensator. Let $[x]$ denote the Gauss bracket, i.e., $[x]$ stands for the largest integer smaller than or equal to x. Let $r_m = q - m[q/m]$ and $r_p = q - p[q/p]$ be the remainders of q divided by m and p, respectively. In all following sufficiency criterions we will assume that

$$q(m+p)+mp-\min(r_m(p-1),r_p(m-1))>n+q. \quad (19)$$

Theorem 3.1: Let m,n,p,q be fixed integers satisfying (19). Then for a generic set of (real) systems $\Sigma = (\mathbb{R},\mathbb{R}^{m+p},\mathcal{B}(P)) \in A^n_{p,m}$ having rank $r(\Sigma)=p$ and McMillan degree $n(\Sigma)=n$, the pole placement map χ introduced in (14) is onto.

Proof: See the Appendix. □

We would like to remark that (19) is always at least as good as (5). Moreover, if q is an integer multiple of either m or p, then (19) reduces to

$$q(m+p)+mp>n+q$$

missing in this way the complex bound of (3) by only one degree of freedom. As a direct consequence we obtain a sufficient condition for Problem 2.1.

Corollary 3.2: Let m,n,p,q be fixed integers satisfying (19). Then every element of a generic set $S\subset A^n_{p,m}$ of plants is arbitrary pole assignable in the class of feedback compensators of McMillan degree at most q. In other words, Problem 2.1 has a positive answer in this situation.

Proof: See the Appendix. □

As it will turn out, the two statements above also give a strong sufficiency criterion for the "traditional" pole placement as formulated before Proposition 2.6. For the "real situation" we have the following theorem.

Theorem 3.3: Let (A,B,C) be a generic set of real matrices of size $n\times n$, $n\times m$, and $p\times n$, respectively. Let q be a number satisfying (19). Then for every monic polynomial $\phi\in\mathbb{R}[s]$ of degree $n+q$ there exist real matrices F,G,H, and K of size $q\times q$, $q\times p$, $m\times q$, and $m\times p$, respectively, such that

$$\phi(s)=\det\begin{bmatrix} sI_n-A-BKC & -BH\\ -GC & sI_q-F\end{bmatrix}.$$

Proof: See the Appendix. □

Instead of using an AR-description for the compensator, we can also use an *image* or MA-description.

For this let $P(s)$ be a $p\times(m+p)$ polynomial matrix which describes the behavior of the plant Σ_1 through the system of autoregressive equations

$$P\left(\frac{d}{dt}\right)w(t)=0. \quad (20)$$

Next consider a $(m+p)\times m$ polynomial matrix $Q(s)$ which describes the behavior of the compensator Σ_2 through the image representation

$$w(t)=Q\left(\frac{d}{dt}\right)\ell(t). \quad (21)$$

Here $\ell(t)$ is a so-called latent variable [17]. The behavior of the interconnected system is described through

$$P\left(\frac{d}{dt}\right)Q\left(\frac{d}{dt}\right)\ell(t)=0,\quad w(t)=Q\left(\frac{d}{dt}\right)\ell(t). \quad (22)$$

As we can describe the behavior (20) of Σ_1 through first-order representations of the form (9), it is also possible to describe the behavior (21) of the compensator through a state-space system (compare with [17, p. 269] and [22]) of the form

$$K\dot{x}(t)+Lx(t)+Mw(t)=0,\quad x(t)\in\mathbb{R}^n, w(t)\in\mathbb{R}^{m+p}. \quad (23)$$

Here K,L are matrices of size $(q+m)\times q$, and M is a matrix of size $(q+m)\times(m+p)$. Then we have the following theorem.

Theorem 3.4: Let F,G be generic matrices of size $n\times(m+n)$, and let H be a generic matrix of size $(m+p)\times(m+n)$. Let q be a number satisfying (19). Then for every polynomial $\phi\in\mathbb{R}[s]$ of degree at most $n+q$, there exist real matrices K,L,M of size $(q+m)\times q$, $(q+m)\times q$, and $(q+m)\times(m+p)$, respectively, such that

$$\phi(s)=\det\begin{bmatrix} O_{n\times q} & sG-F\\ sK+L & MH\end{bmatrix}.$$

Proof: See the Appendix. □

The next theorem shows that over the reals "not even" the bound $q(m+p)+mp>n+q$ is in general sufficient.

Theorem 3.5: $q(m+p)+mp>n+q$ does not guarantee even the almost arbitrary pole assignability for the generic system generally if q is not an integer multiple of both m and p.

Proof: See the Appendix. □

After having formulated all those results we will now explain the main ingredients of the proof. The main idea is based on a polynomial map ψ which will be closely related to the pole placement map χ. For this let m,n,p,q be fixed integers. Let $n=dp+l$, where $l=n-p[n/p]$ is the remainder of n divided by p, and let

$$\nu_1=\cdots=\nu_l=d+1, \text{ and } \nu_{l+1}=\cdots=\nu_p=d. \quad (24)$$

In the sequel we will identify the set of all $p\times(m+p)$ polynomial matrices $P(s)$ whose ith row degree is at most ν_i with a point in the vector space $\mathbb{R}^{(n+p)(m+p)}$. In this way it also makes sense to speak about a "generic" $p\times(m+p)$ polynomial matrix $P(s)$ of order n.

Similarly let $q=kp+r$, where $r=q-p[q/p]$ is the remainder of q divided by p, and let

$$\mu_1=\cdots=\mu_r=k+1, \text{ and } \mu_{r+1}=\cdots=\mu_p=k. \quad (25)$$

Using those definitions we can identify a $(m+p)\times p$ polynomial matrices $Q(s)$ whose ith column degree is at most μ_i with a point in the vector space $\mathbb{R}^{(q+p)(m+p)}$.

For every polynomial matrix $P(s)$, we define now a polynomial map ψ which will be closely related to the pole placement map χ

$$\begin{aligned}\psi\colon \mathbb{R}^{(q+p)(m+p)} &\longrightarrow \mathbb{R}^{n+q+1}\\ Q(s) &\longmapsto \det P(s)Q(s).\end{aligned} \quad (26)$$

In analogy to the properties of χ, the map ψ satisfies:

1) For every $\lambda\in\mathbb{R}$ one has $\psi(\lambda Q(s))=\lambda^p\psi(Q(s))$, i.e., ψ is homogeneous of degree p.

2) If $S \in Sl_p$, i.e., if $S \in \mathbb{R}^{p \times p}$ is a matrix with determinant one, then $\psi(Q(s)S) = \psi(Q(s))$.

The main technical result, which we will prove in the appendix and which will be the starting point for many proofs in this paper, is shown.

Proposition 3.6: Let $P(s) \in \mathbb{R}^{(n+p)(m+p)}$ be a generic $p \times (m+p)$ matrix of order n. If (19) is satisfied, then ψ is onto.

In the remainder of this section we study the linearization of the map ψ, and we will show that if this linearization around a so-called dependent matrix $Q(s)$ is an onto linear map, then the map itself is onto. For this we give the following definition in analogy to [13].

Definition 3.7: A matrix $Q(s)$, whose Jacobian $d\psi_Q$ is an onto linear map, will be a *full compensator* for the plant $P(s)$.

A priori full compensators are only defined for the generic set of Kronecker indexes since the map ψ was introduced in this way. By linearizing, e.g., the pole placement map χ introduced in (14), the definition also extends to compensators not having the generic set of indexes.

Theorem 3.8: The map ψ introduced in (26) is onto as soon as there is a dependent and full compensator $Q(s)$.

Proof: By the Inverse Function Theorem, ψ maps a neighborhood Q onto a neighborhood of 0 in $\mathbb{R}^{n+q+1}$. Since ψ is homogeneous, ψ is onto. □

Remark 3.9: If ψ has no full compensators, then according to Sard's theorem (see, e.g., [20]) the image $\mathrm{Im}(\psi)$ has measure zero. The existence of a full compensator is therefore also necessary. The domain and the range of the map ψ can be naturally extended to the complex vector spaces $\mathbb{C}^{(q+p)(m+p)}$ and $\mathbb{C}^{n+q+1}$, respectively. We will denote this resulting map with ψ^c. By using the dominant morphism theorem (see [4]), it follows that ψ^c is almost onto over $\mathbb{C}$ if and only if there is a full compensator.

In the sequel we will derive conditions which guarantee the existence of full dependent compensators. First recall that the *adjoint* of a $p \times p$ matrix A, denoted by $\mathrm{adj}\, A$, is the $p \times p$ matrix defined by

$$(\mathrm{adj}\, A)_{ij} := \left((-1)^{i+j} A_{ji}\right)_{i,j=1}^{p} \tag{27}$$

where A_{ji} denotes the determinant of the $(p-1) \times (p-1)$ matrix obtained from A by removing the jth row and the ith column. Finally let tr denote the trace of a matrix.

Theorem 3.10: The Jacobian

$$d\psi_Q\colon \mathbb{R}^{(q+p)(m+p)} \to \mathbb{R}^{(n+q+1)}$$

is given by

$$X(s) \longmapsto \mathrm{tr}(\mathrm{adj}(P(s)Q(s))P(s)X(s)) \tag{28}$$

where $X(s)$ is an arbitrary $(m+p) \times p$ polynomial matrix whose ith column degree $\leq \mu_i$, i.e., $X(s)$ describes an arbitrary element of the tangent space $\mathbb{R}^{(q+p)(m+p)}$.

Proof: Consider the Taylor series expansion in direction of $X(s)$, i.e.,

$$\psi(Q(s) + \varepsilon X(s)) = \psi(Q(s)) + \varepsilon d\psi_Q X(s) + \text{terms of higher order in } \varepsilon. \tag{29}$$

Let $q_1, \cdots, q_p$ and $x_1, \cdots, x_p$ be the columns of $Q(s)$ and $X(s)$, respectively. Computing the quotient difference

$$\lim_{\varepsilon \mapsto 0} \frac{\psi(Q(s) + \varepsilon X(s)) - \psi(Q(s))}{\varepsilon}$$

immediately gives

$$\begin{aligned} d\psi_Q X(s) = \det P(s)[x_1, q_2, \ldots, q_p] \\ + \det P(s)[q_1, x_2, \ldots, q_p] + \cdots \\ + \det P(s)[q_1, q_2, \ldots, x_p]. \end{aligned} \tag{30}$$

Using the definitions of adjoint and trace, one gets the result. □

Corollary 3.11: Let B be the $(p+q) \times p$ matrix defined through

$$B := \text{block diag}[b_1, \ldots, b_p], \quad b_i := \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\mu_i} \end{bmatrix}.$$

Then a compensator $Q(s)$ is a full compensator if and only if the polynomial entries of the matrix

$$B(\mathrm{adj}(P(s)Q(s)))P(s) \tag{31}$$

generate the vector space $\mathbb{R}^{n+q+1}$ viewed as the space of all polynomials of degree at most $n+q$.

Remark 3.12: It seems that there are $(q+p)(m+p)$ freedoms in $\mathbb{R}^{(q+p)(m+p)}$. Since for any matrix $S \in Sl_p$ one has $\psi(Q(s)S) = \psi(Q(s))$, the rank of the Jacobian $d\psi_Q$ can be at most $(q+p)(m+p) - (p^2 - 1)$, i.e., a necessary condition for the existence of full compensators is

$$q(m+p) + mp \geq n + q.$$

When a compensator $Q(s)$ is in addition dependent, then the tangent line in direction of $Q(s)$ is in the kernel of $d\psi_Q$. So a necessary condition for the existence of dependent full compensators is

$$q(m+p) + mp > n + q.$$

For a dependent compensator $Q(s)$, the expression for the Jacobian becomes much simpler. Indeed if $P(s)Q(s)$ has rank $\leq p-2$, then $\mathrm{adj}\, P(s)Q(s) = 0$ and it follows that $d\psi_Q = 0$ as well. If $P(s)Q(s)$ has rank $p-1$, then one can show that $\mathrm{adj}\, P(s)Q(s)$ has rank one and can be factored into $\mathrm{adj}\, P(s)Q(s) = r(s)l(s)$, where $l(s)$ is a specific vector in the left kernel of $P(s)Q(s)$ and $r(s)$ is a specific vector in the right kernel of $P(s)Q(s)$. As a consequence the Jacobian $d\psi_Q$ has the simple form $X(s) \mapsto l(s)P(s)X(s)r(s)$. Since we will not need this result later, we only derive a weaker form of it.

Corollary 3.13: Let $P(s) = \begin{bmatrix} P_1(s) \\ \alpha(s) \end{bmatrix}$, where $\alpha(s)$ denotes the last row of $P(s)$. Let $Q(s)$ be a dependent compensator satisfying $\alpha(s)Q(s) = 0$, and let $\beta(s)$ be the last column of $\mathrm{adj}\, P(s)Q(s)$. Then the Jacobian $d\psi_Q\colon \mathbb{R}^{(q+p)(m+p)} \to \mathbb{R}^{(n+q+1)}$ is given by

$$X(s) \longmapsto \alpha(s)X(s)\beta(s) \tag{32}$$

and $Q(s)$ is full if and only if

$$\operatorname{span}_{\mathbb{R}}\{s^k\alpha_i(s)\beta_j(s)\} = \mathbb{R}^{n+q+1} \tag{33}$$

where $i = 1,\ldots,m+p,\ j = 1,\ldots,p,\ k = 0,1,\ldots,\mu_j$.

Proof: Only the last column $\beta(s)$ of the matrix $\operatorname{adj} P(s)Q(s)$ is nonzero. So

$$\begin{aligned} d\psi_Q(X(s)) &= \operatorname{tr}((\operatorname{adj} P(s)Q(s))P(s)X(s)) \\ &= \operatorname{tr}(\beta(s)\alpha(s)X(s)) \\ &= \alpha(s)X(s)\beta(s). \end{aligned}$$

□

Though we introduced the map ψ simply as a map between real vector spaces, it should be apparent from the paragraph before Theorem 3.4 that the surjectiveness of ψ has direct consequences for the pole placement problem. For this note that if $Q(s)$ is a $(m+p)\times p$-dependent compensator having column degrees $\mu_1 \geq \cdots \geq \mu_p$ (not necessarily the generic set of indexes), then it follows immediately that the rank condition (33) guarantees the arbitrary pole assignability in the class of feedback compensators of McMillan degree at most $q := \sum_{i=1}^{p}\mu_i$. The following example gives a first illustration.

Example 3.14: Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} x - \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x.$$

We will use a dynamic compensator of McMillan degree $q = 1$ (which is the minimum) to assign the closed-loop poles. Generally when $m = p = 2$, $n = 6$, $q = 1$, there is a nonempty open set of systems which do not have any dependent compensator and whose pole placement maps are not almost onto (see Example 3.19 and the proof of Theorem 3.5). On the other hand, there also exists a nonempty open set of systems which do have dependent compensators and whose pole placement maps are onto (one can easily write down the conditions). This system is such an example.

By eliminating the state variable x, the system becomes

$$\begin{bmatrix} 1 & 3-(d/dt)^2 & (d/dt)^3 & d/dt \\ 0 & 1+d/dt & (d/dt)^2 & (d/dt)^3 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = 0.$$

Let

$$P(s) = \begin{bmatrix} p_1(s) \\ p_2(s) \end{bmatrix} = \begin{bmatrix} 1 & 3-s^2 & s^3 & s \\ 0 & 1+s & s^2 & s^3 \end{bmatrix}$$

and

$$Q(s) = [q_1(s) \quad q_2(s)].$$

Then the (pole placement) map is given by $\psi(Q) = \det P(s)Q(s)$. Let

$$Q(s) := \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ s & 0 \\ -1 & 0 \end{bmatrix}.$$

Then $p_1(s)Q(s) = [s^4 - s, 1]$ and $p_2(s)Q(s) = 0$, hence $Q(s)$ is a dependent compensator. Using the notations of Corollary 3.13, we have $\alpha(s) = p_2(s) = [0, 1+s, s^2, s^3]$, $\beta_1 = -1$, $\beta_2 = s^4 - s$. Since $\{\alpha_i\beta_1,\ s\alpha_i\beta_1,\ \alpha_i\beta_2 \mid i = 1,\cdots,4\}$ span the space of all polynomials of degree ≤ 7, $Q(s)$ is also a full compensator, and according to Theorem 3.8 the map ψ is onto. Using Lemma A.1 one concludes that the map χ introduced in (14) is surjective as well, i.e., the poles of the closed-loop system can be arbitrarily placed by compensators of McMillan degree ≤ 1.

To apply Theorem 3.8 to a system $P(s)$, one needs to find a dependent compensator first. Following are some known results from [6] and [7] about existence (and nonexistence) of dependent compensators.

Proposition 3.15:

1) A complex dependent compensator always exists if

$$q(m+p) + mp > q + n.$$

2) When $q = 0$, a real dependent compensator always exists if

$$mp > n.$$

In the following we give a new condition which ensures the existence of a real dependent compensator.

Lemma 3.16 Assume a plant $P(s)$ has row indexes $\nu_1 \geq \cdots \geq \nu_p$ and McMillan degree $n = \sum \nu_i$. Let $q = kp + r$ where $k = [q/p]$. Then a real dependent compensator of degree at most q exists as soon as

$$\nu_p < k(m+p-1) + m + r_p.$$

Proof: Let $p(s)$ be the last row of a row-reduced [28] representation of the plant, and let the column vectors

$$\{q_1(s),\ldots,q_{m+p-1}(s)\}$$

be a minimal basis of the kernel ker $p(s)$ in the sense of Forney [28], i.e., $p(s)q_j(s) = 0$ for all j.

Let $\mu_j := \deg q_j(s)$, $\mu_1 \leq \cdots \leq \mu_{m+p-1}$. Then

$$\sum_{j=1}^{m+p-1} \mu_j = \nu_p < k(m+p-1) + m + r_p. \tag{34}$$

We claim that $\sum_{j=1}^{p}\mu_j \leq q$. By contradiction, assume that $\sum_{j=1}^{p}\mu_j > q$. Then $\mu_p \geq k+1$ and

$$\begin{aligned} \sum_{j=1}^{m+p-1}\mu_j &= \sum_{j=1}^{p}\mu_j + \sum_{j=p+1}^{m+p-1}\mu_j > q + (m-1)(k+1) \\ &= k(m+p-1) + m + r_p - 1 \end{aligned}$$

which contradicts (34). Let

$$Q(s) := [q_p(s),\ldots,q_1(s)].$$

Then $Q(s)$ is a dependent compensator of degree $\sum_{j=1}^{p}\mu_j \leq q$. □

Theorem 3.17: Let $r_m = q - m[q/m]$ and $r_p = q - p[q/p]$ be the remainders of q divided by m and p, respectively. If

$$q(m+p) + mp - \min(r_m(p-1), r_p(m-1)) > n + q \tag{35}$$

then a real dependent compensator of McMillan degree at most q exists.

Proof: First we will assume that $r_p(m-1) = \min(r_m(p-1), r_p(m-1))$. Let $P(s)$ be row reduced with row degrees $\nu_1 \geq \cdots \geq \nu_p$. Then, since

$$\begin{aligned}\nu_1 + \cdots + \nu_p \\ &= n < q(m+p-1) + mp - r_p(m-1) \\ &= p(k(m+p-1) + m + r_p)\end{aligned}$$

one must have $\nu_p < k(m+p-1) + m + r_p$, and the result follows from the previous Lemma. Now assume that $r_m(m-1) = \min(r_m(p-1), r_p(m-1))$. According to Lemma A.1, there exists a $(m+p)\times m$ polynomial matrix $\tilde{P}(s)$ having the property that $\det(\tilde{P}(s), Q(s)) = \det P(s)Q(s)$. Using the previous argument once more, one readily establishes the existence of a $m \times (m+p)$ compensator $\tilde{Q}(s)$ having the property that $\det \tilde{Q}(s)\tilde{P}(s) = 0$. But this establishes the existence of a dependent compensator of McMillan degree at most q. □

Remark 3.18:

1) The condition

$$q(m+p) + mp - \min(r_m(p-1), r_p(m-1)) > n + q$$

improves the condition

$$q \max(m,p) + mp > n$$

given in [8].

2) When q is an integer multiple of either m or p, the condition reduces to

$$q(m+p) + mp > n + q$$

which is also necessary for the existence of a dependent compensator for a generic system.

3) In general, $q(m+p) + mp > n + q$ does not guarantee the existence of a real dependent compensator of degree at most q (compare it with Proposition 3.15).

The following example proves the third claim of the previous remark.

Example 3.19: Consider the system with $m = p = 2$, $n = 6$

$$P(s) = \begin{bmatrix} -s^3 & s & -2s^2 & 2 \\ s & s^3 & 1 & s^2 \end{bmatrix}. \tag{36}$$

We will show that all the compensators of degree ≤ 1 are not dependent. For this assume $\det P(s)Q(s) = 0$. Then

$$\begin{aligned}&2(s^4+1)c_{12}(s) + s^3 c_{13}(s) + (2s^5 + s)c_{14}(s) \\ &\quad - (s^5 + 2s)c_{23}(s) - s^3 c_{24}(s) + (s^6 + s^2)c_{34}(s) = 0\end{aligned} \tag{37}$$

where $c_{ij}(s) = a_{ij} + b_{ij}s$ is the 2×2 minor of $Q(s)$ consisting of the ith and jth rows. The solutions of (37) are given by

$$\begin{array}{ll} a_{34} = -3b_{14}, & a_{13} = a_{24} \\ a_{14} = -a_{23}, & a_{12} = 0 \\ b_{34} = 0, & b_{13} = b_{24} \\ b_{14} = -b_{23}, & b_{12} = -(3/2)a_{14}. \end{array} \tag{38}$$

The minors of $Q(s)$ must satisfy [6]

$$c_{12}(s)c_{34}(s) - c_{13}(s)c_{24}(s) + c_{14}(s)c_{23}(s) = 0$$

i.e.,

$$\begin{aligned} &a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0 \\ &b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23} = 0 \\ &a_{12}b_{34} - a_{13}b_{24} + a_{14}b_{23} + b_{12}a_{34} \\ &\quad - b_{13}a_{24} + b_{14}a_{23} = 0. \end{aligned} \tag{39}$$

Substituting (38) into the first two equations of (39) results in $a_{24}^2 + a_{23}^2 = 0$, $b_{24}^2 + b_{23}^2 = 0$. Since the equations have only the trivial solution 0, $Q(s)$ does not have full rank and therefore is not a compensator.

Remark 3.20: Here is the use the language of [16]: (39) defines the projective variety $K_{2,2}^1$ in $\mathbb{P}^{11}$, where the homogeneous coordinates are given by $\{a_{ij}, b_{ij} \mid 1 \leq i < j \leq 4\}$. If $\{a_{ij}, b_{ij}\}$ are considered as affine coordinates, then (39) defines the affine cone of $K_{1,1}^1$ in $\mathbb{R}^{12}$. The center of the pole placement map is defined by (38).

Example 3.21: Consider the system

$$\dot{x} = \begin{bmatrix} 0&0&0&0&0&0&0&0&-1 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&-1 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{bmatrix} x - \begin{bmatrix} 0&1 \\ 1&0 \\ 0&0 \\ 0&0 \\ 0&1 \\ 0&0 \\ 0&1 \\ 0&0 \\ 1&0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} x.$$

By eliminating the state variable x, the dynamics of the system is equivalently described through $P(d/dt)[{u \atop y}] = 0$, where

$$P(s) = \begin{bmatrix} s & 1+s^4 & s^5 & 1+s^2 \\ s^3 & s & 1 & s^4 \end{bmatrix}.$$

By Theorem 3.17, any two-input, two-output system of degree nine has a dependent compensator $Q(s)$ of degree ≤ 2 which can be solved from the equation $\alpha(s)Q(s) = 0$, where $\alpha(s)$ is the row of degree ≤ 4 of $P(s)$. For this system $\alpha(s) = [s^3, s, 1, s^4]$ and the solution of $\alpha(s)Q(s) = 0$ (which is unique up to equivalence) are given by

$$Q(s) = \begin{bmatrix} s & 0 \\ 0 & -1 \\ 0 & s \\ -1 & 0 \end{bmatrix}.$$

Let $p_1(s)$ be the first row of $P(s)$. Then $p_1(s)Q(s) = [-1, s^6 - s^4 - 1]$, i.e., $\beta_1 = -s^6 + s^4 + 1$ and $\beta_2 = -1$.

Since $\{s^k\alpha_i(s)\beta_j(s) \mid i = 1, \cdots, 4,\ j = 1, 2,\ k = 0, 1\}$ span the space of all polynomials of degree ≤ 11, $Q(s)$ is full and the closed-loop poles of the system can be arbitrarily placed by compensators of McMillan degree ≤ 2.

IV. Computation of the Compensators

Note that Theorem 3.8 gives not only an existence condition but also provides a concrete numerical procedure on how to construct a particular compensator which assigns a given closed-loop characteristic polynomial. Indeed if $d\psi_Q$ has full rank, then by the Inverse Function Theorem a small neighborhood of $Q(s)$ is mapped to a small neighborhood of the origin of $\mathbb{R}^{n+q+1}$. Since the solution exists locally, Newton's Method is very effective.

To find the compensator, one can choose an $n+q+1$-dimensional linear subspace $\mathcal{S} \subset \mathbb{R}^{(q+p)(m+p)}$ such that $Q \in \mathcal{S}$, the Jacobian of $\psi|_\mathcal{S}$ at Q is onto, and $\mathcal{S} \cong \mathbb{R}^{n+q+1}$ locally around Q. Since we assume that Q is a full compensator, such a subspace $\mathcal{S}$ always exists.

Let $x = [x_0, \ldots, x_{n+q}]^t$ be a point in $\mathcal{S}$ and $J(x)$ be the Jacobian matrix of $\psi|_\mathcal{S}$ at x. Let ϕ be the desired closed-loop characteristic polynomial whose coefficients we will represent as a column vector. Starting with the initial value $x^0 := Q$, Newton's method computes iteratively

$$x^{k+1} = x^k - J^{-1}(x^k)(\psi(x^k) - \delta\phi).$$

As long as δ is chosen sufficiently close to zero, the procedure converges and in the limit one has

$$\lim_{k\mapsto\infty} \psi(x^k) = \delta\phi.$$

If $P(s) = [-N_\ell(s)\ D_\ell(s)]$ is a strictly proper system, then $\det D_\ell(s)$ is the only $p \times p$ minor which has degree n. If $\phi(s)$ has been chosen monic and if

$$Q(s) = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}$$

is a solution of

$$\psi(Q(s)) = \delta\phi(s) \tag{40}$$

then the determinant of the high-order coefficient matrix of $F_2(s)$ must be δ which means that some of the coefficients of the feedback

$$u(s) = F_1(s)F_2^{-1}(s)y(s)$$

may be very large if $|\delta|$ is too small. To find the solution of (40) for not too small $|\delta|$, one can choose a sequence $\delta_1, \delta_2, \ldots$ and use Newton's method to solve

$$\psi(Q(s)) = \delta_i\phi(s)$$

with initial point the solution of

$$\psi(Q(s)) = \delta_{i-1}\phi(s).$$

At each step, as long as the Jacobian has full rank and $|\delta_i - \delta_{i-1}|$ is not too large, Newton's method will always converge to the solution. The following example will illustrate this procedure.

Example 4.1: We have computed the compensators of degree 2 to assign all the closed-loop poles at -1, i.e., to assign the closed-loop characteristic polynomial to

$$\phi(s) = (s+1)^{11} \tag{41}$$

for the system of Example 3.21. Let

$$\mathcal{S} = \left\{ \begin{bmatrix} s + x_1 & x_2 \\ x_3 s + x_4 & x_5 s + x_6 - 1 \\ x_7 & s + x_8 \\ x_9 s + x_{10} - 1 & x_{11}s + x_{12} \end{bmatrix} \right\} \simeq \mathbb{R}^{12} \subset \mathbb{R}^{16}$$

and $\psi(x)$ is equal to the determinant of the matrix

$$\begin{bmatrix} s & 1+s^4 & s^5 & 1+s^2 \\ s^3 & s & 1 & s^4 \end{bmatrix} \times \begin{bmatrix} s + x_1 & x_2 \\ x_3 s + x_4 & x_5 s + x_6 - 1 \\ x_7 & s + x_8 \\ x_9 s + x_{10} - 1 & x_{11}s + x_{12} \end{bmatrix}.$$

Then from Example 3.21 we know that $\psi(0) = 0$, and the Jacobian $d\psi_0$ is invertible.

We have chosen 10 δ's ranging from $\delta = 0.0001$ to $\delta = 100$. The whole computation for the 10 compensators took less than 1 s using the Matlab Program implemented on a Sun Sparc Station 5. The error of the coefficients of the closed-loop characteristic polynomials are less than 10^{-12}. The solution for $\delta = 0.0001$ is given by

$$x = \begin{bmatrix} 0.002\,140\,043\,415\,81 \\ -0.021\,045\,047\,523\,24 \\ -0.008\,843\,102\,883\,05 \\ -0.029\,449\,284\,981\,49 \\ -0.022\,786\,638\,001\,98 \\ -0.039\,125\,421\,731\,45 \\ -0.054\,020\,706\,789\,22 \\ -0.059\,229\,254\,433\,54 \\ -0.000\,100\,000\,000\,00 \\ 0.003\,550\,554\,736\,49 \\ -0.074\,108\,717\,718\,78 \\ -0.083\,836\,874\,067\,81 \end{bmatrix}.$$

A realization of the corresponding compensator having the form $\dot{z} = Fz + Gy$, $u = Hz + Ky$ is given through

$$F = \begin{bmatrix} -10\,004.528\,505\,739\,3 & -8.822\,627\,816\,5 \\ 5.402\,070\,678\,9 & 0.059\,229\,254\,4 \end{bmatrix}$$

$$G = \begin{bmatrix} -741.087\,177\,187\,8 & -100 \\ 100 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} -10\,004.526\,365\,695\,8 & -8.822\,838\,267\,0 \\ 88.440\,394\,637\,3 & 0.067\,614\,654\,9 \end{bmatrix}$$

$$K = \begin{bmatrix} -741.087\,177\,187\,8 & -10\,000 \\ 6.530\,723\,515\,2 & 88.431\,028\,830\,5 \end{bmatrix}.$$

One can see that some of the coefficients are pretty large. A better compensator occurs at $\delta = 0.007$

$$x = \begin{bmatrix} 0.070\,239\,655\,233\,22 \\ -1.642\,632\,887\,325\,98 \\ 0.293\,192\,630\,816\,87 \\ 0.282\,885\,245\,493\,30 \\ -0.524\,743\,904\,250\,59 \\ -6.198\,400\,119\,921\,10 \\ -0.352\,799\,083\,822\,39 \\ -8.072\,575\,136\,594\,06 \\ -0.007\,000\,000\,000\,00 \\ 0.244\,765\,209\,812\,81 \\ -6.407\,803\,582\,329\,09 \\ -3.589\,828\,548\,477\,21 \end{bmatrix}.$$

A realization of the compensator is

$$F = \begin{bmatrix} -430.843\,146\,192\,391 & -55.317\,304\,427\,365 \\ 50.399\,869\,117\,485 & 8.072\,575\,136\,594 \end{bmatrix}$$
$$G = \begin{bmatrix} -640.780\,358\,232\,909 & -100 \\ 100 & 0 \end{bmatrix}$$
$$H = \begin{bmatrix} -615.389\,866\,481\,653 & -79.041\,146\,939\,395 \\ -180.317\,542\,035\,331 & -23.283\,810\,082\,511 \end{bmatrix}$$
$$K = \begin{bmatrix} -915.400\,511\,761\,299 & -142.857\,142\,857\,143 \\ -268.913\,428\,198\,654 & -41.884\,661\,545\,267 \end{bmatrix}.$$

Although it is quite straightforward to compute a particular compensator, we would like to note that practically it might be infeasible to implement those compensators to such accuracy.

We also did some computations for the system of Example 3.14. The following is a compensator of degree one, having the form $\dot{z} = fz + Gy, \quad u = Hz + y$, and assigning the closed-loop characteristic polynomial to $(s+1)^7$ with the error less than 10^{-12}

$$f = [20.629\,539\,551\,519\,00,]$$
$$G = [-20.739\,048\,585\,621\,70 \quad 17.069\,005\,415\,038\,21]$$
$$H = \begin{bmatrix} -1.262\,953\,955\,151\,90 \\ -0.425\,907\,910\,303\,79 \end{bmatrix}.$$

V. Conclusion

In this paper we studied the dynamic pole placement problem using dynamic compensators of bounded McMillan degree. We provided a strong new sufficiency condition which guarantees arbitrary pole placement with real compensators and which comes very close to the best known necessary condition. We showed the implications of this sufficiency condition through several new theorems, and we illustrated the theory through several examples. In a last section we outlined a way of computing compensators which are capable of assigning a desired closed-loop characteristic polynomial.

Appendix

We start with a technical lemma which will be needed at several points in this paper.

Lemma A.1: Let $P(s)$ be an arbitrary polynomial matrix of size $p \times (m+p)$. If $Q(s)$ is a polynomial matrix of size $(m+p) \times p$, then there exists a not necessarily unique polynomial matrix $\tilde{Q}(s)$ having the property that

$$\det(P(s)Q(s)) = \det \begin{pmatrix} P(s) \\ \tilde{Q}(s) \end{pmatrix}.$$

Proof: If $Q(s)$ has not-full column rank, then $\det(P(s)Q(s)) = 0$, and the statement is trivial. So we assume that $Q(s)$ has full column rank. In this case there exists a factorization $Q(s) = Q_1(s)R_1(s)$ having the property that $Q_1(s)$ has full column rank for all $s \in \mathbb{C}$, and $R_1(s)$ is a $p \times p$ polynomial matrix. Define a $m \times (m+p)$ matrix $\tilde{Q}_1(s)$ and a $m \times m$ matrix $\tilde{R}_1(s)$ through the requirements $\mathrm{Ker}\tilde{Q}_1(s) = \mathrm{Im}(Q_1(s)$ and $\det \tilde{R}_1(s) = \det R_1(s)$. Then $\tilde{Q}(s) := \tilde{R}_1(s)\tilde{Q}_1(s)$ has the desired properties. □

Remark A.2: From the proof it also follows that the "McMillan degrees," i.e., the degrees of the full-size minors of $\tilde{Q}(s)$ and $Q(s)$ are the same.

Proof of Proposition 2.5: The proof of this proposition largely follows from [16, Th. 2.15]. Some of the notations rely on those papers. Let $K^q_{p,m}$ be the variety introduced in [6] and studied in [16] and let π_2: $\mathcal{H}^q_{p,m} \to K^q_{p,m}$ be the canonical projection. (π_2 assigns to a homogeneous AR-system of size $p \times (m+p)$ all the $p \times p$ full-size minors.) Let $\bar{\rho}$: $K^q_{p,m} \to \mathbb{P}^{n+q}$ be the associated pole placement map, and let Y_D be the set of dependent compensators introduced in Definition 2.4. Those maps are related through the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^q_{p,m} & \xrightarrow{\pi_2} & K^q_{p,m} \\ \uparrow \pi_1 & & \downarrow \bar{\rho} \\ Y \setminus Y_D & \xrightarrow{\bar{\chi}} & \mathbb{P}^{n-q}. \end{array} \tag{42}$$

By the assumptions of the Proposition and by [16, Th. 2.15], $\bar{\rho}$ is surjective of mapping degree $d(m,p,q)$ (over $\mathbb{C}$), but then the same is true for $\bar{\chi}$. □

The following proofs all will rely on Proposition 3.6 which we will prove at the end of this Appendix.

Proof of Theorem 3.1: We give a short proof which relies on some properties of the commutative diagram (42). In the proof of Theorem 3.3 we outline a different way which can also be used to proof Theorem 3.1.

Let $S \subset \mathbb{R}^{(n+p)(m+p)}$ be the set of polynomial matrices whose associated pole placement map ψ introduced in (26) is onto. According to Proposition 3.6, S is a generic set in the vector space $\mathbb{R}^{(n+p)(m+p)}$ and clearly every $P(s) \in S$ has full rank p. Consider the map π: $S \to K^n_{p,m}$ which assigns to $P(s)$ the full-size minors of $P(s)$. Since S is generic $\pi(S)$ is also generic in the variety $K^n_{p,m}$. Let π_1, π_2 be the morphisms used in (42). Then by "continuity of the Zariski topology" $\pi_1^{-1}(\pi_2^{-1}(\pi(S))) \subset Y \subset \mathbb{R}^N$ is a generic set. This shows that for the generic system Σ_1 (generic this time with respect to Definition 2.2) represented through the polynomial matrix $P(s)$, the map ψ is surjective. Applying Lemma A.1 and "realizing" the resulting compensator $\tilde{Q}(s)$ through a first-order triple (F,G,H) establishes the result. □

Proof of Corollary 3.2: We have to show that for the generic $p \times (m+p)$ polynomial matrix $P_1(s)$ of degree n and for every polynomial $\phi(s) \in \mathbb{R}[s]$ of degree at most $n+q$, there exists a $m \times (m+p)$ polynomial matrix $P_2(s)$ of degree at most q having the property that $\det\left(\begin{smallmatrix} P_1(s) \\ P_2(s) \end{smallmatrix}\right) = \phi(s)$. Due to Theorem 3.1 there exist matrices F, G, H such that

$$\det\left(\begin{bmatrix} 0 & P_1(s) \\ I_q & 0 \end{bmatrix}\begin{bmatrix} sG-F \\ H \end{bmatrix}\right) = \phi(s).$$

By applying Lemma A.1 with $P(s) := \left[\begin{smallmatrix} 0 & P_1(s) \\ I_q & 0 \end{smallmatrix}\right]$ and $Q(s) := \left[\begin{smallmatrix} sG-F \\ H \end{smallmatrix}\right]$, the proof is readily established. □

Proof of Theorem 3.3: We outline here a proof which extends techniques developed in [13]. The same techniques can also be used for an alternative proof of Theorem 3.1. Consider the set of triples F, G, H which represent strictly proper systems of McMillan degree n, i.e., consider all triples of the form

$$\begin{bmatrix} sG-F \\ H \end{bmatrix} := \begin{bmatrix} sI_n - A & B \\ C & 0 \\ 0 & I_m \end{bmatrix}$$

where A, C forms an observable pair. (F, G, H is then observable as well [26, Lemma 2.2].) Let the rows of the $p \times (n+m+p)$ matrix $L(s)$ be a minimal basis of the left kernel of $\left[\begin{smallmatrix} sG-F \\ H \end{smallmatrix}\right]$, and let $P(s)$ represent the last $m+p$ columns of $L(s)$. Realization theory (compare, e.g., with [21, Th. 4.1] and [24, Th. 5.15]) tells us that the behaviors $\mathcal{B}(P)$ and $\mathcal{B}(F, G, H)$ are the same. Without loss of generality we can assume that the row degrees of $P(s)$ are ordered. As in Corollary 3.13 we will denote the last row of $P(s)$ with $\alpha(s)$. The crucial point is now that using the generalized observability matrix defined in [26, Lemma 2.2], it is possible to express $\alpha(s)$ and therefore the dependent compensator $Q(s)$ polynomially in the parameters of A, B, C. (This was done for the static problem in [13, Lemma 4.2]). As for the proof of Proposition 3.6, one shows that the rank condition for the Jacobian becomes purely polynomial in terms of the parameters A, B, C. As in [13], the proof is complete if we can construct one example. Since the examples constructed in the proof of Proposition 3.6 all represent strictly proper systems, the same example can be used. □

Proof of Theorem 3.4: The proof of Theorem 3.1 also showed that ψ is onto for a polynomial matrix $P(s)$ representing a generic system $\Sigma_1 \in A^n_{p,m}$. ($P(s)$ does not have, in this case, the generic row indexes.) Rewriting both $P(s)$ and $Q(s)$ in terms of first-order representations establishes the result. □

Proof of Theorem 3.5: We will show that for $m = p = 2$, $q = 1$, and $n = 6$, there exists a nonempty open set of systems such that the set of unassignable polynomials of each system is a nonempty open set. Since the set of all 2×4 systems of degree ≤ 6 is a compact set (see [19]) and the pole placement map ψ is continuous, one needs only to find one system with at least one unassignable polynomial. Consider the system (36) in Example 3.19. We will show that the polynomial $s^7 - s^5 + s^3 - s$ cannot be achieved by any compensator of degree ≤ 1. As in Example 3.19, one needs to solve

$$\begin{aligned} &2(s^4+1)c_{12}(s) + s^3 c_{13}(s) + (2s^5+s)c_{14}(s) \\ &\quad - (s^5+2s)c_{23}(s) - s^3 c_{24}(s) + (s^6+s^2)c_{34}(s) \\ &\quad = s^7 - s^5 + s^3 - s \end{aligned} \tag{43}$$

subject to the constraints (39) for $c_{ij}(s) = a_{ij} + b_{ij}s$. The solutions of (43) are given by

$$\begin{aligned} a_{34} &= 3b_{23}, & a_{13} &= a_{24} \\ a_{14} &= -a_{23}, & a_{12} &= 0 \\ b_{34} &= 1, & b_{13} &= b_{24} \\ b_{14} &= -b_{23}, & b_{12} &= -(1-3a_{23})/2. \end{aligned} \tag{44}$$

Substitute (44) into the first two equations of (39)

$$a_{24}^2 + a_{23}^2 = 0, \quad \frac{1-3a_{23}}{2} + b_{24}^2 + b_{23}^2 = 0.$$

The equations have no real solution. □

Proof of Proposition 3.6: Without loss of generality we may assume (compare with the proof of Theorem 3.17) that

$$r_p(m-1) = \min(r_m(p-1), r_p(m-1))$$

and q is the smallest integer such that

$$n < q(m+p-1) + mp - r_p(m-1). \tag{45}$$

To simplify the notation we use r for r_p. Let k, r, d, l, ν, μ satisfy the definitions as given before Proposition 3.6. Then by (45), we have that

$$d \leq k(m+p-1) + m + r - 1. \tag{46}$$

Since we assume that q is the smallest integer satisfying (45), it follows that

$$\begin{aligned} &k(m+p-1) + m + r - 2 < d \\ &\quad \leq k(m+p-1) + m + r - 1, && \text{if } r > 0 \\ &\text{and} \\ &(k-1)(m+p-1) + m + p - 2 < d \\ &\quad \leq k(m+p-1) + m - 1, && \text{if } r = 0 \end{aligned} \tag{47}$$

i.e.,

$$\begin{aligned} e &:= k(m+p-1) + m + r - 1 - d \\ &= \begin{cases} = 0 & \text{if } r > 0 \\ \leq m-1 & \text{if } r = 0. \end{cases} \end{aligned} \tag{48}$$

Let $S \subset \mathbb{R}^{(p+n)(m+p)}$ be the set of $p \times (m+p)$ full rank polynomial matrices $P(s)$ of row degrees ν_i such that

1) the first $k(m+p) + m + r$ entries of

$$[\alpha(s), \ldots, s^k\alpha(s)]$$

generate all polynomials of degree $\leq d + k$ [note that $d + k + 1 \leq k(m+p) + m + r$ by (46)];

2) at least one of the first m entries of $\alpha(s)$ is a polynomial of degree d;

where $\alpha(s)$ is the last row of a matrix. S is a Zariski open set of $\mathbb{R}^{(p+n)(m+p)}$, and the generic system can be represented by a matrix in S. For any $P(s) \in S$ let $Q(s)$ be the $(m+p) \times p$ full rank matrix of column degrees μ such that

$$\alpha(s)Q(s) = 0.$$

Then the coefficients of $Q(s)$ are polynomials of the coefficients of $\alpha(s)$. Therefore the rank condition (33) defines a Zariski open set of S. To finish the proof we just need to find one system in S which satisfies (33).

Since $m = 1$ is covered by Proposition 2.5 and since $n < mp$ was done in [7], we may assume that

$$n \geq mp \quad \text{and} \quad m \geq 2. \tag{49}$$

Then

$$d \geq m \geq 2. \tag{50}$$

We first consider the case

$$k > 0. \tag{51}$$

See (52), shown at the bottom of the page, and

$$Q(s) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ s^{\mu_1} & -1 & \cdots & 0 \\ 0 & s^{\mu_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & -1 \\ 0 & 0 & \cdots & s^{\mu_p} \end{bmatrix} \tag{53}$$

where $b_i(s)$ is defined by

$$\begin{array}{lll} b_1(s) = 1 & & \\ b_i(s) = s^k b_{i-1}(s) & \text{for } 2 \leq i \leq p-r+1 \text{ and} & \\ & \quad p+2 \leq i \leq p+e+1 & \\ b_i(s) = s^{k+1} b_{i-1}(s) & \text{for } p-r+2 \leq i \leq p+1 \text{ and} & \\ & \quad p+e+2 \leq i & \end{array} \tag{54}$$

(note that $b_p(s) = s^{k(m+p-1)+m+r-1-e} = s^d$), and choose nonzero $a_1, \cdots, a_{p-1}$ such that the rows of

$$P_1(s)Q(s) = \begin{bmatrix} -a_1 & s^{\nu_1+\mu_2} & -s^{\nu_1} & 0 & \cdots & 0 \\ a_2 s^{\mu_1} & -a_2 & s^{\nu_2+\mu_3} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -s^{\nu_{p-2}} \\ 0 & \cdots & 0 & a_{p-1}s^{\mu_{p-2}} & -a_{p-1} & s^{\nu_{p-1}+\mu_p} \end{bmatrix} \tag{55}$$

form a minimal basis in the sense of Forney [28] (one can prove by induction on p that such a_i's exist). Let $\beta(s)$ be the last column of $\operatorname{adj} P_1(s)Q(s)$. Then according to (33), we have to show that

$$\begin{aligned} \dim \operatorname{sp}_{\mathbb{R}} \{ s^i \beta_j(s) \mid i = 0, \ldots, \tau_j, \\ j = 1, \ldots, p \} = n + q + 1 \end{aligned}$$

where $\tau_1 = \cdots = \tau_r = d + k + 1$, and where $\tau_{r+1} = \cdots = \tau_p = d + k$. For

$$\begin{array}{ll} i = 0, \cdots, \nu_p + \mu_1 & \text{if } j = 1 \\ i = 0, \cdots, \nu_{j-1} + \mu_j - 1 & \text{if } j = 2, \cdots, p \end{array}$$

consider the polynomials $\{s^i \beta_j(s)\}$. It is enough to show that this set of polynomials forms a linearly independent set over $\mathbb{R}$. (Note that there are exactly $n+q+1$ polynomials). If not, then see (56) shown at the bottom of the previous page, for some nonzero polynomials $\{f_i(s)\}$ with

$$\deg f_1(s) \leq \nu_p + \mu_1 \quad \text{and} \quad \deg f_i(s) \leq \nu_{i-1} + \mu_i - 1, \qquad i = 2, \cdots, p. \tag{57}$$

Since the $p-1$ rows of $P_1(s)Q(s)$ form a minimal basis, by the main result of [28] there are polynomials $h_1(s), \ldots, h_{p-1}(s)$ such that

$$[h_1(s), \cdots, h_{p-1}(s)] P_1(s)Q(s) = [f_1(s), \cdots, f_p(s)].$$

$$P(s) = \begin{bmatrix} P_1(s) \\ \alpha(s) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & a_1 & 0 & s^{\nu_1} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & a_2 & 0 & s^{\nu_2} & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & a_{p-1} & 0 & s^{\nu_{p-1}} \\ b_{m+p}(s) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_3(s) & b_2(s) & b_1(s) \end{bmatrix} \tag{52}$$

$$\det \begin{bmatrix} -a_1 & s^{\nu_1+\mu_2} & -s^{nu_1} & 0 & \cdots & 0 \\ a_2 s^{\mu_1} & -a_2 & s^{\nu_2+\mu_3} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -s^{\nu_{p-2}} \\ 0 & \cdots & 0 & a_{p-1}s^{\mu_{p-1}} & -a_{p-1} & s^{\nu_{p-1}+\mu_p} \\ f_1(s) & \cdots & \cdots & \cdots & \cdots & f_p(s) \end{bmatrix} = 0 \tag{56}$$

$$
P(s) = \begin{bmatrix}
0 & \cdots & a_1 & 0 & s^{\nu_1} & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & a_2 & 0 & s^{\nu_2} & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & a_{r-1} & 0 & s^{\nu_{r-1}} & 0 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a_r & 0 & s^{\nu_r} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & a_{r+1} & s^{\nu_{r+1}} & \cdots & 0 \\
\vdots & & & & & & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & a_{p-1} & s^{\nu_{p-1}} \\
s^d & \cdots & \cdots & \cdots & \cdots & \cdots & s^2 & s & 1 & 0 & \cdots & 0 & 0
\end{bmatrix} \tag{58}
$$

$$
P_1(s)Q(s) = \begin{bmatrix}
-a_1 & s^{\nu_1+1} & -s^{\nu_1} & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
a_2 s & -a_2 & s^{\nu_2+1} & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & -s^{\nu_{r-2}} & \vdots & & & & \vdots \\
0 & \cdots & 0 & a_{r-1}s & -a_{r-1} & s^{\nu_{r-1}+1} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & a_r s & -a_r & s^{\nu_r} & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & 0 & a_{r+1} & s^{\nu_{r+1}} & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & 0 & 0 & & \ddots & s^{\nu_{p-2}} & 0 \\
0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & a_{p-1} & s^{\nu_{p-1}}
\end{bmatrix}. \tag{60}
$$

Let i be the smallest such that

$$
\deg h_i(s) \begin{cases} > \deg h_j(s) & \text{for all } j < i \\ \geq \deg h_j(s) & \text{for all } j > i. \end{cases}
$$

Then

$$
\begin{aligned}
f_{i+1}(s) = &-s^{\nu_{i-1}} h_{i-1}(s) + s^{\nu_i+\mu_{i+1}} h_i(s) \\
&- a_{i+1} h_{i+1}(s) + a_{i+2} s^{\mu_{i+1}} h_{i+2}(s).
\end{aligned}
$$

(Let the corresponding term be zero if there is no such subscript.) By (48) and (49), $\nu_{i-1} \leq \nu_i + \mu_{i+1}$ and $\mu_{i+1} < \nu_i + \mu_{i+1}$. So $\deg f_{i+1}(s) = \nu_i + \mu_{i+1} + \deg h_i(s) \geq \nu_i + \mu_{i+1}$ which is a contradiction.

Now we consider the case $k = 0$. For this see (58) shown at the top of the page [note that $d + 1 = m + r$ by (45) and (47)], and

$$
Q(s) = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & & \vdots \\
0 & & & & & \vdots \\
-1 & & & & & \vdots \\
s^{\mu_1} & \ddots & & & & \vdots \\
\vdots & \ddots & -1 & & & \vdots \\
\vdots & & s^{\mu_r} & 0 & & \vdots \\
\vdots & & & s^{\mu_{r+1}} & \ddots & \vdots \\
\vdots & & & & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & s^{\mu_p}
\end{bmatrix}. \tag{59}
$$

Then see (60) shown at the top of the page. The same argument as in the case $k > 0$ can be applied as well. Hence a full dependent compensator exists for the generic polynomial matrix $P(s) \in \mathbb{R}^{(p+n)(m+p)}$ and the proof is complete. □

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