

# A Cell Structure for the Set of Autoregressive Systems

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<sup>1</sup>Supported in part by the Research Enhancement Fund from College of Art and Sciences, Texas Tech University.

<sup>2</sup>Supported in part by NSF grant DMS-9201263.

### **Abstract**

The set of autoregressive systems generalizes the set of transfer functions in a natural way. In this paper we describe a topology for the set of all autoregressive systems of fixed size and bounded McMillan degree. We show that this topological space has the structure of a finite CW-complex.

**Abbreviated Title:** A Cell Structure for Autoregressive Systems

# 1 Introduction

In the last two decades a lot of research was devoted to the study of topological and algebraic properties of the set of transfer functions. This research was motivated by problems arising in adaptive and robust control, system identification, dynamic pole placement and more general interpolation problems.

Let  $S_{p,m}^n(\mathbb{K})$  be the set of all proper  $p \times m$  transfer functions of a fixed McMillan degree  $n$  defined over the field  $\mathbb{K}(s)$ . Over the reals ( $\mathbb{K} = \mathbb{R}$ ) Clark [5] showed that  $S_{p,m}^n$  has the structure of a smooth manifold of dimension  $n(m+p) + mp$ . Over a general algebraically closed field  $\mathbb{K}$  Hazewinkel [10] showed that  $S_{p,m}^n$  has the structure of a quasi affine variety. Hermann and Martin [21] established an isomorphism between the subset of strictly proper transfer functions and the set of base point preserving holomorphic maps from the Riemann sphere  $S^2$  to the Grassmann manifold  $\text{Grass}(p, m+p)$  making in this way the connection to geometry.

Since this time there has been a great effort to understand the topological properties of the class of linear systems. Important contributions include e.g. the articles of Byrnes and Hurt [3], Byrnes and Duncan [4], Delchamps [6] and Helmke [12]. More recently a cellular decomposition of  $S_{p,m}^n$  was constructed independently by Manthey [20] (compare also with [8, 12, 14]) and by Mann and Milgram [19]. Using their decomposition Mann and Milgram were able to calculate topological invariants of  $S_{p,m}^n$  like the integral homology groups. For a good survey and a comparative study of different topologies on  $S_{p,m}^n$  we recommend the dissertation of Glüsing-Lüerßen [9].

A natural generalization of the set  $S_{p,m}^n$  of  $p \times m$  transfer functions of a fixed McMillan degree  $n$  is the set of  $p \times (m+p)$  autoregressive systems with McMillan degree at most  $n$  which we will denote by  $A_{p,m}^{\leq n}$ . This set of systems represents the class of time invariant, continuous-time linear systems in the behavioral framework of Willems [29, 30, 31].

In the next section we will review the notion of an autoregressive system and we will define a topology on  $A_{p,m}^{\leq n}$  which extends the topology on the set  $S_{p,m}^n$  in a natural way. In Section 3 we will show that the Kronecker indices and pivot indices as introduced by Forney [7] can be used to construct a cellular decomposition. In order to understand the boundary structure of each cell we will define for each set of Kronecker and pivot indices a new set of indices. On this set of indices we will define a partial order which corresponds on the topological side to the closure inclusion. In other words we will show (Section 4, Theorem 4.12) that the closure of each cell consists of cells with smaller index.

Finally in the last section we will show that  $A_{p,m}^{\leq n}$  is a compact topological space and the constructed cellular decomposition gives rise to a finite CW-complex. Using this information we will describe the singular homology of  $A_{p,m}^{\leq n}(\mathbb{C})$ .

## 2 The Set of Autoregressive Systems

Let  $\mathbb{K}$  denote either the field of real ( $\mathbb{K} = \mathbb{R}$ ) or complex ( $\mathbb{K} = \mathbb{C}$ ) numbers and consider a  $p \times (m + p)$  matrix with entries in the polynomial ring  $\mathbb{K}[s]$ .  $P(s)$  defines a system of autoregressive equations in the sense of Willems [29, 31] through:

$$P\left(\frac{d}{dt}\right)w(t) = 0 \tag{2.1}$$

Under a solution of 2.1 we understand a vector valued distribution  $\varphi : \mathbb{K} \rightarrow \mathbb{K}^{m+p}$ . (Compare with [31, page 279]). Using again the language of Willems [29, 31] (see also [17, 26]) we call the set of solutions of 2.1 the *behavior* of the system.

Clearly elementary row operations on  $P(s)$  have no affect on the behavior. Moreover an important result formulated in Schumacher [26, Corollary 2.5] (compare also with [29, Section 5] and with [25]) states that two full rank polynomial matrices  $P(s)$  and  $\tilde{P}(s)$  are row equivalent if, and only if the systems  $P\left(\frac{d}{dt}\right)w(t) = 0$  and  $\tilde{P}\left(\frac{d}{dt}\right)w(t) = 0$  have the same behavior. Based on this result we define: (Compare also with [9, 25, 31])

**Definition 2.1** Two  $p \times (m + p)$  polynomial matrices  $P(s)$  and  $\tilde{P}(s)$  are called (row) equivalent if there is a unimodular matrix  $U(s)$  with  $\tilde{P}(s) = U(s)P(s)$ . An equivalence class of full rank polynomial matrices is called an autoregressive system.

In the sequel we often will not distinguish between the matrix  $P(s)$  and the autoregressive system this matrix defines. The context will make the meaning clear.

**Definition 2.2** An autoregressive system  $P(s)$  is called irreducible or controllable if  $P(s)$  has full rank for all  $s \in \mathbb{C}$ .

**Definition 2.3** The McMillan degree of a full rank  $p \times (m + p)$  polynomial matrix  $P(s)$  is given by the maximal degree of the full size minors of  $P(s)$ . The degree of an autoregressive system

$$P\left(\frac{d}{dt}\right)w(t) = 0$$

is defined to be the degree of  $P(s)$ .

Clearly Definitions 2.2 and 2.3 do not depend on the particular representant of the equivalence class.

The set of autoregressive systems generalizes the set of transfer functions in the following sense: Assume  $G(s)$  is a proper or improper  $p \times m$  transfer function of fixed McMillan degree  $n$ . Let  $D^{-1}(s)N(s) = G(s)$  be a left coprime factorization. Then it

is well known that  $\tilde{D}^{-1}(s)\tilde{N}(s) = G(s)$  is a second left coprime factorization if, and only if the  $p \times (m + p)$  polynomial matrices  $(N(s) \ D(s))$  and  $(\tilde{N}(s) \ \tilde{D}(s))$  are row equivalent. Moreover the polynomial matrix  $(N(s) \ D(s))$  is in this case irreducible and the McMillan degree of  $(N(s) \ D(s))$  as defined in Definition 2.3 is equal to the McMillan degree of the transfer function  $D^{-1}(s)N(s) = G(s)$ . For a proof of these results we refer the interested reader to [16]. Finally we want to mention that after a partitioning of the vector  $w(t)$  into an input part  $u(t)$  and an output part  $y(t)$  it is always possible to find a state space representation of Equation 2.1 (see [30, Theorem 4.1]).

Our interest in topological questions of autoregressive systems originates in our work on feedback compensation [24, 28]. If one wants to understand the pole placement problem or if one studies degeneration phenomena one is immediately led to the study of topological properties of the space of all autoregressive systems. Moreover it is possible to study the pole placement problem and more general interpolation problems (compare with [1]) even in the set of autoregressive systems.

In the next section we define a topology on the set of autoregressive systems of fixed size and fixed McMillan degree. Using the ordered Kronecker indices and pivot indices we will then introduce a cellular decomposition of this space. Note that already earlier Hazewinkel and Martin [11] considered closure inclusions involving the set of Kronecker indices of a controllable system of the form  $\dot{x} = Ax + Bu$  and Helmke [12] (see also [15]) used the set of Kronecker indices to produce a cellular decomposition of this class of linear systems.

### 3 A Cellular Decomposition

Let  $A_{p,m}^{\leq n}$  be the set of all  $p \times (m + p)$  autoregressive systems of degree at most  $n$ . The following Lemma is well known and characterizes the Kronecker (or row degree) indices of  $P(s)$ .

**Lemma 3.1** *Given any  $p \times (m + p)$  polynomial matrix  $\tilde{P}(s)$  of McMillan degree  $n$ , there exist unique  $\nu = (\nu_1, \dots, \nu_p)$  with  $\nu_1 \leq \dots \leq \nu_p$  and  $\sum_{i=1}^p \nu_i = n$  and a  $p \times p$  unimodular matrix  $U(s)$  such that the matrix  $P(s) = U(s)\tilde{P}(s)$  has row degrees  $\nu_1 \leq \dots \leq \nu_p$ .*

A proof can be found for example in [2, page 330].

**Definition 3.2** The numbers  $\nu = (\nu_1, \dots, \nu_p)$  are called the ordered Kronecker indices of the autoregressive system  $P(s)$ . A matrix  $P(s)$  is called row reduced if the ordered Kronecker indices are equal to the degrees of the rows of  $P(s)$ .

Note that  $\nu_1, \dots, \nu_p$  are not the ordered (minimal) indices of a ‘minimal basis’ in the sense of Forney [7] unless  $P(s)$  is irreducible.  $P(s)$  is row reduced if and only if the high order coefficient matrix of  $P(s)$  has full rank, where the high order coefficient matrix of a polynomial matrix  $P(s)$  is a matrix whose entries of the  $i$ -th row are the coefficients of  $s^{\nu_i}$  of the  $i$ -th row of  $P(s)$  where  $\nu_i$  is the highest power of  $s$  in the  $i$ -th row of  $P(s)$ .

Our definition of pivot indices is only slightly different from the one given by Forney [7].

**Definition 3.3** Given an autoregressive system with  $P(s)$  row reduced and with  $P_h$  the high order coefficient matrix of  $P(s)$ , the  $i$ -th pivot index  $\mu'_i$  is the largest integer such that the submatrix of  $P_h$  formed from the intersection of columns  $\mu'_1, \dots, \mu'_i$  with the rows corresponding to indices  $\leq \nu_i$  has rank  $i$ . The ordered pivot indices  $\mu = (\mu_1, \mu_2, \dots, \mu_p)$  are the indices obtained from  $(\mu'_1, \mu'_2, \dots, \mu'_p)$  by reordering such that  $\mu_i < \mu_{i+1}$  if  $\nu_i = \nu_{i+1}$ .

The ordered Kronecker and pivot indices are invariant under row equivalence (see [7]).

In the following we would like to combine the indices  $\nu$  and  $\mu$  in a single set  $\alpha$  of indices defined through:

**Definition 3.4**  $\alpha_i := \nu_i(m + p) + \mu_i$ .

From the properties of  $\nu$  and  $\mu$  we can see that:

1.  $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_p \leq (n + 1)(m + p)$ .
2.  $\mu_i \equiv \alpha_i \pmod{m + p}$

From property 2. in particular follows that the assignment  $(\nu, \mu) \mapsto \alpha$  is one-one. Based on this observation we denote with  $C_\alpha$  the subset of all the equivalence classes in  $A_{p,m}^{\leq n}$  with indices  $\alpha = (\alpha_1, \dots, \alpha_p)$ . The following proposition is then a direct consequence of Forney’s echelon form (see [7]).

**Proposition 3.5** *Each equivalence class in  $C_\alpha$  can be represented by a unique polynomial matrix*

$$P(s) = P_0 + P_1 s + P_2 s^2 + \dots + P_n s^n. \quad (3.1)$$

*such that the matrix  $Q := [P_0 | P_1 | \dots | P_n]$  has the following special echelon form:*

1. *The  $(i, \alpha_i)$  entry is 1, all the other entries on the  $\alpha_i$ -th column and all the entries to the right of the  $(i, \alpha_i)$  entry on the  $i$ -th row are zero.*
2. *If  $j > \alpha_i$  and  $j \equiv \alpha_i \pmod{m + p}$  then the  $j$ -th column of  $Q$  is zero.*

Moreover every  $p \times (m + p)(n + 1)$  matrix  $Q$  with the above echelon form defines a unique element of  $C_\alpha$ .

The following example illustrates this correspondence:

**Example 3.6** Consider the set of  $2 \times 4$  matrices with indices  $\nu = (0, 2)$  and  $\mu = (2, 4)$ , i.e. the subset  $C_{(2,12)}$  of  $A_{2,2}^{\prec 2}$ . Then the corresponding echelon forms are:

$$\left[ \begin{array}{cccc|cccc|cccc} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & * & 0 & * & * & * & 0 & * & 1 \end{array} \right].$$

Similar the systems with indices  $\alpha = (3, 10)$  ( $\nu = (0, 2)$  and  $\mu = (3, 2)$ ) have a corresponding echelon form

$$\left[ \begin{array}{cccc|cccc|cccc} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & * & 1 & 0 & 0 \end{array} \right].$$

For each set  $C_\alpha$  let  $d_\alpha$  be the number of “free parameters” appearing in the corresponding echelon form of the matrices  $Q$ . Proposition 3.5 in particular implies that  $C_\alpha$  is in one to one correspondence to the euclidean space  $\mathbb{R}^{d_\alpha}$ . So far we didn’t define any topology on the set  $A_{p,m}^{\prec n}$  and of course we would like that the set theoretic identification coming from Proposition 3.5 is in addition continuous. With this in mind we define now a topology on  $A_{p,m}^{\prec n}$ :

Let  $P_{p,m}^{\prec n}$  be the set of all  $p \times (m + p)$  full rank polynomial matrices of degree at most  $n$  and let  $P_{p,m}^{\prec n,q}$  be the subset of  $P_{p,m}^{\prec n}$  formed by all matrices whose entries are polynomials of degree at most  $q$ . Then

$$P_{p,m}^{\prec n,0} \subset P_{p,m}^{\prec n,1} \subset P_{p,m}^{\prec n,2} \subset \dots$$

with union  $P_{p,m}^{\prec n} = \bigcup_{q=0}^{\infty} P_{p,m}^{\prec n,q}$ . Note that each  $P_{p,m}^{\prec n,q}$  is a subset of  $\mathbb{R}^{p(m+p)(q+1)}$ . Take the topology on  $P_{p,m}^{\prec n,q}$  induced by the natural topology on  $\mathbb{R}^{p(m+p)(q+1)}$ . The direct limit of the topologies on  $P_{p,m}^{\prec n,q}$   $q = 0, 1, \dots$  defines a topology on  $P_{p,m}^{\prec n}$ . In other words, a subset of  $P_{p,m}^{\prec n}$  is open if and only if its intersection with  $P_{p,m}^{\prec n,q}$  is open as a subset of  $P_{p,m}^{\prec n,q}$  for each  $q$ . The topology which we will take on  $A_{p,m}^{\prec n}$  will be the quotient topology under the equivalence induced by the unimodular group, i.e. one has the definition given in [25]:

**Definition 3.7** A subset  $U$  of  $A_{p,m}^{\prec n}$  is open if, and only if the subset  $V$  of  $P_{p,m}^{\prec n}$  formed by all the polynomial matrices in the equivalent classes of  $U$  is open.

With respect to this topology one has now immediately the following Lemma which is easy to verify:

**Lemma 3.8** *The set of equivalence classes  $C_\alpha$  are cells, i.e. homeomorphic to an euclidean space. Moreover one has  $A_{p,m}^{\leq n} = \bigcup_\alpha C_\alpha$  and  $C_\alpha \cap C_\beta = \emptyset$  if  $\alpha \neq \beta$ .*

The dimension of the cell  $C_\alpha$ , earlier denoted by  $d_\alpha$  can be determined by counting the numbers of “free” entries in the echelon form. For this denote for any real number  $x$  with  $[x]$  the largest integer which is smaller or equal to  $x$ . Then the numbers of free entries on the  $i$ -th row is

$$\alpha_i - 1 - \sum_{j=1}^{i-1} \left( \left[ \frac{\alpha_i - \alpha_j}{m+p} \right] + 1 \right) = \alpha_i - i - \sum_{j=1}^{i-1} \left[ \frac{\alpha_i - \alpha_j}{m+p} \right].$$

Therefore

$$d_\alpha = \left( \sum_{i=1}^p \alpha_i \right) - \frac{p(p+1)}{2} - \sum_{i=2}^p \sum_{j=1}^{i-1} \left[ \frac{\alpha_i - \alpha_j}{m+p} \right]. \quad (3.2)$$

The formula (3.2) can be written in terms of the Kronecker indices and pivot indices.

$$\begin{aligned} d_\alpha &= n(m+p) + \sum_{i=1}^p \mu_i - \sum_{i,j=1}^p \max(\nu_i - \nu_j + 1, 0) + \\ &\quad + \#\{(i, j) \mid \nu_i \leq \nu_j, \mu_i > \mu_j\} \end{aligned} \quad (3.3)$$

$$\begin{aligned} &= n(m+p) + \sum_{i=1}^p \mu'_i - \sum_{i,j=1}^p \max(\nu_i - \nu_j + 1, 0) + \\ &\quad + \#\{(i, j) \mid \nu_i \leq \nu_j, \mu'_i > \mu'_j\} \end{aligned} \quad (3.4)$$

Let  $A_{p,m}^\nu$  be the subset of all the systems with Kronecker indices  $\nu$ . The “thickest” open cell in  $A_{p,m}^\nu$  has pivot indices  $(\mu'_1, \mu'_2, \dots, \mu'_p) = (m+p, m+p-1, \dots, m+1)$ . Its dimension by above formula is

$$(n+p)(m+p) - \sum_{i,j=1}^p \max(\nu_i - \nu_j + 1, 0)$$

which is the dimension of  $A_{p,m}^\nu$  obtained in [25]. In particular, the “thickest” open cell of  $A_{p,m}^{\leq n}$  has indices

$$\alpha_i = \begin{cases} a(m+p) + m + b + i, & i = 1, \dots, p-b \\ (a+1)(m+p) + m + b - p + i, & i = p-b+1, \dots, p \end{cases} \quad (3.5)$$

and dimension  $n(m+p) + mp = \dim A_{p,m}^{\leq n}$ , where  $a$  and  $b$  are the integers such that  $n = ap + b$ ,  $0 \leq b < p$ .



## 4 Closure of An Open Cell

In this section we will describe the closure of a cell  $C_\alpha$  in  $A_{p,m}^{\prec n}$ . The following example shows what type of phenomena we can expect:

**Example 4.1** Assume  $a, b, c, d, f$  are real constants and  $a \neq 0$ . Then every element in the cell  $\begin{bmatrix} a & b & 1 & 0 \\ s+f & h & 0 & l \end{bmatrix}$  can be represented by  $\begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & bs+c & s+f & d \end{bmatrix}$ . In the limit as  $a \rightarrow \infty$ , we have the cell  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & bs+c & s+f & d \end{bmatrix}$ .

In order to characterize the closure of a cell in general we define first a partial order on the set of all indices. For each  $\alpha = (\alpha_1, \dots, \alpha_p)$  we associate with a infinite sequence:

$$f(\alpha) = (f_1(\alpha), f_2(\alpha), \dots) \quad (4.1)$$

where

$$\{f_i(\alpha)\} = \{\alpha_j + k(m+p) \mid k = 0, 1, 2, \dots; \quad j = 1, \dots, p\} \quad (4.2)$$

and arrange the order such that

$$f_1(\alpha) < f_2(\alpha) < \dots$$

**Example 4.2** Consider in  $A_{2,2}^{\prec 2}$ .

$$f(2, 12) = (2, 6, 10, 12, 14, 16, \dots),$$

$$f(3, 10) = (3, 7, 10, 11, 14, 15, \dots).$$

The following definition establishes a partial order on the set of indices  $\alpha$  introduced in Definition 3.4:

**Definition 4.3**

$$\alpha \leq \beta \text{ if and only if } f_i(\alpha) \leq f_i(\beta) \text{ for all } i. \quad (4.3)$$

**Example 4.4** Consider  $A_{2,2}^{\prec 2}$ .

$$(3, 12) < (4, 11) < (7, 8)$$

because

$$f(3, 12) = (3, 7, 11, 12, \dots),$$

$$f(4, 11) = (4, 8, 11, 12, \dots),$$

$$f(7, 8) = (7, 8, 11, 12, \dots).$$

The partial order can be characterized in another way.

**Lemma 4.5** *Let*

$$g(\alpha, k) = \#\{f_i(\alpha) \mid f_i(\alpha) \leq k\}. \quad (4.4)$$

*Then  $\alpha \leq \beta$  if and only if  $g(\alpha, k) \geq g(\beta, k)$  for all positive integers  $k$ .*

**Definition 4.6**  $\beta$  is called to cover  $\alpha$  if  $\alpha < \beta$  and there is no  $\gamma$  such that  $\alpha < \gamma < \beta$ .

**Lemma 4.7** *If  $\beta$  covers  $\alpha$  then  $\alpha$  and  $\beta$  must take one of the following forms:*

1. *There exists an index  $j$  such that*

- $\alpha_i = \beta_i$  for all  $i \neq j$  and
- $\alpha_j = \max\{r \mid r < \beta_j, r \not\equiv \beta_i \pmod{m+p}\}$  for all  $i < j$ .

2. *There exist  $j$  and  $l$  with  $j < l$  such that*

- $\alpha_i = \beta_i$  for all  $i \neq j, l$  and
- $\alpha_j = \beta_l - (1 + \lceil \frac{\beta_l - \beta_j}{m+p} \rceil)(m+p)$ ,  $\alpha_l = \beta_j + (1 + \lceil \frac{\beta_l - \beta_j}{m+p} \rceil)(m+p)$  and
- $\lceil \frac{\beta_l - \beta_i}{m+p} \rceil = \lceil \frac{\alpha_l - \alpha_i}{m+p} \rceil$  for all  $i \in (j, l)$

*Proof:* Let  $j$  be the smallest number such that  $\alpha_j \neq \beta_j$ , then  $\alpha_j < \beta_j$ . Assume

$$\alpha_i = a_i(m+p) + b_i, \quad \beta_i = c_i(m+p) + d_i, \quad 0 < b_i, d_i \leq (m+p).$$

Since both 1) and 2) are unchanged under translation (i.e. add a fixed integer to all the indices), without loss of generality, assume  $d_j = m+p$ . Then  $c_i > c_j$  for all  $i > j$ .

We first prove that  $a_j = c_j$ . Define  $\gamma$ :

$$\gamma_i = \begin{cases} c_j(m+p) + d_{j+1} & \text{if } i = j \\ c_{j+1}(m+p) + m+p & \text{if } i = j+1 \\ \beta_i & \text{otherwise.} \end{cases}$$

Then  $\gamma < \beta$ . If  $\alpha \leq \gamma$ , then  $\alpha = \gamma$  which implies that  $a_j = c_j$ . If  $\alpha \not\leq \gamma$ , then  $g(\alpha, k_0) < g(\gamma, k_0)$  for some integer  $k_0 = a_0(m+p) + b_0$ ,  $0 < b_0 \leq m+p$  where  $g$  is defined by (4.4). Notice that for any integer  $k = a(m+p) + b$

$$g(\gamma, k) = \begin{cases} g(\beta, k) + 1 & \text{if } c_j \leq a < c_{j+1}, d_{j+1} \leq b < m+p \\ g(\beta, k) & \text{otherwise.} \end{cases}$$

So we must have

$$g(\beta, k_0) \leq g(\alpha, k_0) < g(\gamma, k_0) = g(\beta, k_0) + 1,$$

and

$$c_j \leq a_0 < c_{j+1}, \quad (4.5)$$

$$d_{j+1} \leq b_0 < m + p, \quad (4.6)$$

$$g(\alpha, k_0) = g(\beta, k_0). \quad (4.7)$$

By (4.5),  $\beta_i > k_0$  for all  $i > j$ . So

$$g(\beta, k_0) = \sum_{i=1}^j \#\{n | \beta_i \leq n \leq k_0, n = \beta_i \bmod m + p\}.$$

By (4.6),

$$\#\{n | \beta_j \leq n \leq k_0, n = \beta_j \bmod m + p\} = a_0 - c_j.$$

So

$$\begin{aligned} g(\beta, k_0) &= \sum_{i=1}^{j-1} \#\{n | \beta_i \leq n \leq k_0, n = \beta_i \bmod m + p\} + a_0 - c_j \\ &= \sum_{i=1}^{j-1} \#\{n | \alpha_i \leq n \leq k_0, n = \alpha_i \bmod m + p\} + a_0 - c_j \\ &\leq g(\alpha, k_0) - \#\{n | \alpha_j \leq n \leq k_0, n = \alpha_j \bmod m + p\} + a_0 - c_j \\ &\leq g(\alpha, k_0) - (a_0 - a_j) + a_0 - c_j \end{aligned}$$

which means that

$$a_j \geq c_j$$

because of (4.7). On the other hand,

$$a_j \leq c_j$$

because  $\alpha_j < \beta_j$ . Therefore

$$a_j = c_j.$$

If there is a  $l > j$ ,  $d_l \in [b_j, m + p)$ , choose such  $l$  so that  $\beta_l$  is the smallest. Define  $\eta$ :

$$\eta_i = \begin{cases} c_j(m + p) + d_l & \text{if } i = j \\ c_l(m + p) + m + p & \text{if } i = l \\ \beta_i & \text{otherwise.} \end{cases}$$

Then  $\eta < \beta$ . If  $\alpha \not\leq \eta$ , the same argument shows that  $g(\alpha, k_0) = g(\beta, k_0)$  for some  $k_0 = a_0(m+p) + b_0$  with  $c_j \leq a_0 < c_l$ ,  $d_l \leq b_0 < m+p$ . Since  $\beta_l$  is the smallest and  $a_0 < c_l$ , one has

$$\begin{aligned} g(\beta, a_0(m+p) + b_j - 1) &= g(\beta, k_0) - \#\{d_i \mid d_i \in [b_j, b_0], i < j\} \\ &= g(\alpha, k_0) - \#\{b_i \mid b_i \in [b_j, b_0], i < j\} \\ &\geq g(\alpha, a(m+p) + b_j - 1) + 1 \\ &\geq g(\beta, a(m+p) + b_j - 1) + 1 \end{aligned}$$

which is a contradiction. So  $\alpha \leq \eta$  which implies  $\alpha = \eta$  and it is case 2.

If there is no  $l > j$ ,  $d_l \in [b_j, m+p)$ , define  $\kappa$ :

$$\kappa_i = \begin{cases} c_j(m+p) + d_0 & \text{if } i = j \\ \beta_i & \text{if } i \neq j. \end{cases}$$

where  $d_0 = \max\{d < d_j \mid d \neq d_i, i < j\}$ . Notice that  $b_j \leq d_0$ . If  $\alpha \not\leq \kappa$ , the same argument shows that  $g(\alpha, k_0) = g(\beta, k_0)$  for some  $k_0 = a_0(m+p) + b_0$  with  $c_j \leq a_0$  and  $d_0 \leq b_0 < m+p$ . Then

$$\begin{aligned} g(\beta, a_0(m+p) + b_j - 1) &= g(\beta, k_0) - \#\{d_i \mid d_i \in [b_j, b_0], i < j\} \\ &= g(\alpha, k_0) - \#\{b_i \mid b_i \in [b_j, b_0], i < j\} \\ &\geq g(\alpha, a(m+p) + b_j - 1) + 1 \\ &\geq g(\beta, a(m+p) + b_j - 1) + 1 \end{aligned}$$

which is a contradiction. So  $\alpha \leq \kappa$  which implies  $\alpha = \kappa$  and it is case 1. ■

**Corollary 4.8** *If  $\beta$  covers  $\alpha$  then*

$$d_\alpha = d_\beta - 1$$

*Proof:* We prove it for the two cases of Lemma 4.7 respectively.

**Case 1:** Let  $i_r < j$  be the integer such that

$$\alpha_{i_r} = \alpha_j + r \pmod{m+p}, \quad r = 1, \dots, k = \beta_j - \alpha_j - 1. \quad (4.8)$$

Then

$$\left[ \frac{\alpha_r - \alpha_i}{m+p} \right] = \begin{cases} \left[ \frac{\beta_r - \beta_i}{m+p} \right] - 1 & \text{if } r = j \text{ and } i = i_1, \dots, i_k \\ \left[ \frac{\beta_r - \beta_i}{m+p} \right] & \text{otherwise} \end{cases}$$

Therefore by the formula (3.2),

$$d_\beta - d_\alpha = (\beta_j - \alpha_j) - k = 1.$$

**Case 2:** Direct computation shows that

$$\sum_i \alpha_i = \sum_i \beta_i, \quad (4.9)$$

$$\left[ \frac{\alpha_l - \alpha_j}{m+p} \right] = \left[ \frac{\beta_l - \beta_j}{m+p} \right] + 1, \quad (4.10)$$

$$\begin{aligned} \left[ \frac{\alpha_j - \alpha_i}{m+p} \right] &= \left[ \frac{\beta_l - \beta_i}{m+p} \right] - (1 + \left[ \frac{\beta_l - \beta_j}{m+p} \right]), \\ \left[ \frac{\alpha_l - \alpha_i}{m+p} \right] &= \left[ \frac{\beta_j - \beta_i}{m+p} \right] + (1 + \left[ \frac{\beta_l - \beta_j}{m+p} \right]), \end{aligned} \quad (4.11)$$

for all  $i < j$ ,

$$\begin{aligned} \left[ \frac{\alpha_i - \alpha_j}{m+p} \right] &= \left[ \frac{\beta_i - \beta_l}{m+p} \right] + (1 + \left[ \frac{\beta_l - \beta_j}{m+p} \right]), \\ \left[ \frac{\alpha_i - \alpha_l}{m+p} \right] &= \left[ \frac{\beta_i - \beta_j}{m+p} \right] - (1 + \left[ \frac{\beta_l - \beta_j}{m+p} \right]) \end{aligned} \quad (4.12)$$

for all  $i > l$

and

$$\begin{aligned} \left[ \frac{\alpha_i - \alpha_j}{m+p} \right] &= \left[ \frac{\beta_l - \beta_j}{m+p} \right] - \left[ \frac{\beta_l - \beta_i}{m+p} \right], \\ \left[ \frac{\alpha_l - \alpha_i}{m+p} \right] &= \left[ \frac{\beta_l - \beta_j}{m+p} \right] - \left[ \frac{\beta_i - \beta_j}{m+p} \right] \end{aligned} \quad (4.13)$$

for all  $i \in (j, l)$

which implies also

$$\left[ \frac{\beta_i - \beta_j}{m+p} \right] = \left[ \frac{\alpha_i - \alpha_j}{m+p} \right] \text{ for all } i \in (j, l). \quad (4.14)$$

Therefore by the formula (3.2),

$$d_\beta - d_\alpha = \left[ \frac{\alpha_l - \alpha_j}{m+p} \right] - \left[ \frac{\beta_l - \beta_j}{m+p} \right] = 1.$$

■

Lemma 4.7 can be written in terms of Kronecker and pivot indices:

**Corollary 4.9** *Let  $\nu(\alpha)$  and  $\mu(\alpha)$  be the ordered Kronecker and pivot indices of  $\alpha$ . If  $\beta$  covers  $\alpha$ , then the ordered Kronecker indices and pivot indices of  $\alpha$  and  $\beta$  must take one of the following forms:*

1.  $\nu_i(\alpha) = \nu_i(\beta)$  for all  $i$  and there exists a  $j$  such that  $\mu_i(\alpha) = \mu_i(\beta)$  for all  $i \neq j$ ,

$$\mu_j(\alpha) = \max\{r \in [1, \mu_j(\beta)) \mid r \neq \mu_i(\beta), i < j\}.$$

2. There exists a  $j$  such that  $\nu_i(\alpha) = \nu_i(\beta)$  and  $\mu_i(\alpha) = \mu_i(\beta)$  for all  $i \neq j$ ,

$$\nu_j(\alpha) = \nu_j(\beta) - 1,$$

$$\mu_j(\beta) = \min\{r \in [1, m+p] \mid r \neq \mu_i(\beta), i < j\}$$

and

$$\mu_j(\alpha) = \max\{r \in [1, m+p] \mid r \neq \mu_i(\beta), i < j\}.$$

3.  $\nu_i(\alpha) = \nu_i(\beta)$  for all  $i$  and there exist  $j < l$ ,  $\nu_j(\beta) < \nu_l(\beta)$ ,  $\mu_j(\beta) > \mu_l(\beta)$  and  $\mu_i(\beta) \notin (\mu_l(\beta), \mu_j(\beta))$  for all  $i \in (j, l)$ , such that  $\mu_i(\alpha) = \mu_i(\beta)$  for all  $i \neq j, l$ ,

$$\mu_j(\alpha) = \mu_l(\beta)$$

and

$$\mu_l(\alpha) = \mu_j(\beta).$$

4. There exist  $j < l$ ,  $\nu_j(\beta) \leq \nu_l(\beta)$ ,  $\mu_j(\beta) < \mu_l(\beta)$ ,  $\mu_i(\beta) \notin [1, \mu_j(\beta)) \cup (\mu_l(\beta), m+p]$  for all  $i \in (j, l)$ , such that  $\nu_i(\alpha) = \nu_i(\beta)$  and  $\mu_i(\alpha) = \mu_i(\beta)$  for all  $i \neq j, l$ ,

$$\nu_j(\alpha) = \nu_j(\beta) - 1,$$

$$\nu_l(\alpha) = \nu_l(\beta) + 1,$$

$$\mu_j(\alpha) = \mu_l(\beta)$$

and

$$\mu_l(\alpha) = \mu_j(\beta).$$

The goal of the next two lemmas is to show that the partial order introduced in Definition 4.3 corresponds on the topological side to the closure inclusion of the cells.

**Lemma 4.10** *If  $\alpha \leq \beta$  then*

$$C_\alpha \subset \overline{C}_\beta \tag{4.15}$$

*Proof:* We only need to prove this for the  $\beta$  which covers  $\alpha$ . The lemma is obvious for case 1), 2) and 3) of the Corollary 4.9. So we only consider case 4).

Let  $\alpha < \beta$  satisfy the conditions of case 4) of Corollary 4.9,  $\nu = \nu(\alpha)$ ,  $\mu = \mu(\alpha)$  and

$$P(s) = \begin{bmatrix} a_1(s) \\ \vdots \\ a_p(s) \end{bmatrix} \in C_\alpha$$

be a polynomial matrix in echelon form with

$$a_i(s) = a_{0i} + a_{1i}s + \cdots + a_{\nu_i i} s^{\nu_i}.$$

Define

$$P_t(s) = \begin{bmatrix} b_1(s) \\ \vdots \\ b_p(s) \end{bmatrix}$$

where

$$b_j(s) = a_{0j} + a_{1j}s + \cdots + (a_{\nu_j j} - ta_{\nu_l - 1l})s^{\nu_j} - ta_{\nu_l l} s^{\nu_j + 1}$$

and

$$b_i(s) = a_i(s), \quad i \neq j.$$

Then

$$\lim_{t \rightarrow 0} P_t(s) = P(s).$$

For  $t \neq 0$ ,  $P_t(s)$  is equivalent to

$$Q_t(s) = \begin{bmatrix} c_1(s) \\ \vdots \\ c_p(s) \end{bmatrix}$$

where

$$c_l(s) = c_{0l} + c_{1l}s + \cdots + \frac{1}{t} a_{\nu_j j} s^{\nu_l - 1}.$$

and

$$c_i(s) = b_i(s), \quad i \neq l.$$

Finally the ordered Kronecker and pivot indices of  $Q_t(s)$  are  $(\nu_1, \dots, \nu_j + 1, \dots, \nu_l - 1, \dots, \nu_p)$  and  $(\mu_1, \dots, \mu_l, \dots, \mu_j, \dots, \mu_p)$ , i.e. the equivalent class of  $P_t(s)$  belongs to  $C_\beta$  for any  $t \neq 0$ . ■

**Lemma 4.11** *The union*

$$\bigcup_{\alpha \leq \beta} C_\alpha \tag{4.16}$$

is a closed subset of  $A_{p,m}^{\leq n}$ .

*Proof:* It is enough to prove that

$$\bigcup_{\alpha \not\leq \beta} C_\alpha$$

is open, i.e. we need to prove that the set formed by all the polynomial matrices in  $P_{p,m}^{\leq n,q}$  of indices  $\alpha \not\leq \beta$  is open in  $P_{p,m}^{\leq n,q}$ . For this, it is sufficient to prove that there is no sequence of polynomial matrices of indices  $\beta$  in  $P_{p,m}^{\leq n,q}$  which approaches to a polynomial matrix of indices  $\alpha$  if  $\alpha \not\leq \beta$ .

Assume that there are  $\alpha$  and  $\beta$ ,  $f_l(\alpha) > f_l(\beta)$  for some  $l$ , and there are polynomial matrices  $\{Q_i(s) \mid i = 1, 2, \dots\} \subset P_{p,m}^{\leq n,q}$  of indices  $\beta$  and  $P(s) \in P_{p,m}^{\leq n,q}$  of indices  $\alpha$  such that

$$Q_i(s) \rightarrow P(s),$$

Change  $P(s)$  into the echelon form defined in Proposition 3.5 by unimodular row operation and change  $Q_i(s)$  by the same operation. Then some of the  $Q_i(s)$  may not belong to  $P_{p,m}^{\leq n,q}$ , but the degrees of the entries of  $Q_i(s)$ ,  $i = 1, 2, \dots$ , are still uniformly bounded.

Let  $P(s) = P_0 + P_1s + \dots + P_qs^q$  and consider the infinite matrix

$$\begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_q & 0 & 0 & \cdots \\ 0 & P_0 & P_1 & \cdots & P_{q-1} & P_q & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}. \tag{4.17}$$

Let  $P$  be the infinite matrix obtained from the above matrix by rearranging the rows such that the  $i$ -th row belongs to  $V_{f_i(\alpha)}$ , where  $V_r$  is the vector space consisting of all

$$x = (x_1, x_2, \dots, x_r, 0, 0, \dots).$$

Then the elementary unimodular polynomial row operations on  $P(s)$  correspond to elementary row operations on  $P$ . For any  $j > l$ , let  $P^j$  be the submatrix of  $P$  formed by the first  $j$  rows and  $Q_i^j$  be the corresponding matrices obtained from  $Q_i(s)$ . Then  $Q_i^j \rightarrow P^j$  and

$$\text{row sp}(P^j) \in C(f_1(\alpha), f_2(\alpha), \dots, f_j(\alpha))$$

where

$$C(f_1, f_2, \dots, f_j) = \{W \in \text{Grass}(j, \infty) \mid \dim(W \cap V_{f_k}) = k, \\ \dim(W \cap V_r) < k \text{ for all } r < f_k, k = 1, \dots, j\} \tag{4.18}$$



is a Schubert cell in the infinite Grassmannian  $\text{Grass}(j, \infty)$ . So

$$\dim((\text{row sp}(Q_i^j)) \cap V_{f_i(\beta)}) < l$$

when  $i$  is large enough.

On the other hand, since the degrees of the entries of  $\{Q_i(s)\}$  are uniformly bounded,  $\{Q_i(s)\}$  can be changed into the echelon forms by multiplying from the left by unimodular polynomial matrices whose degrees of entries are uniformly bounded. So when  $j$  is large enough,

$$\dim((\text{row sp}(Q_i^j)) \cap V_{f_i(\beta)}) = l$$

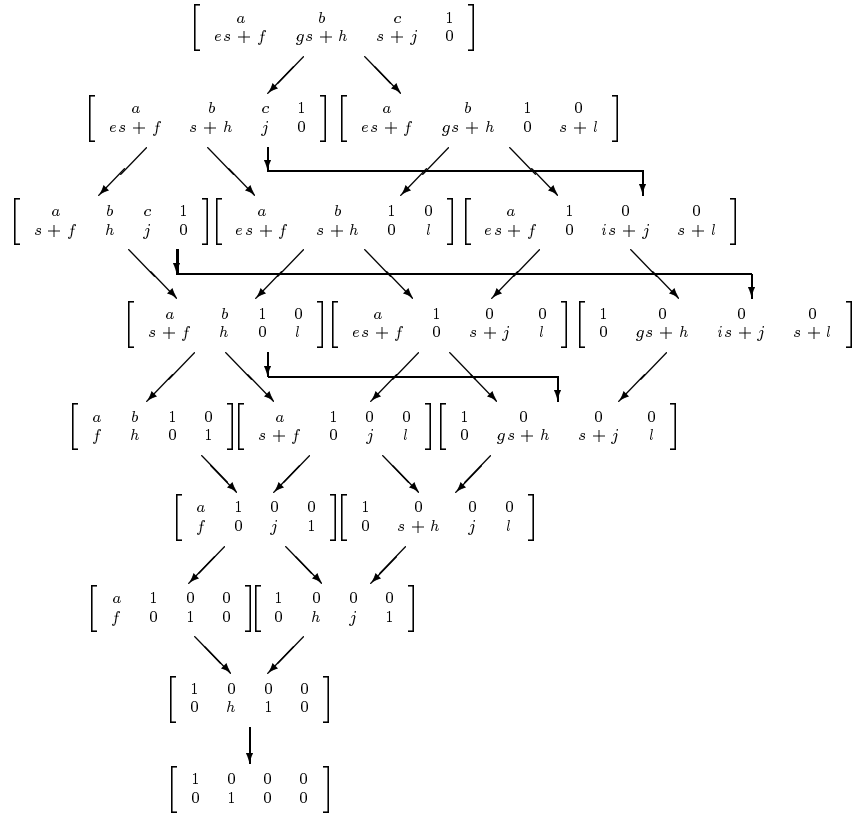
for all  $i$ , which is a contradiction. ■

Combining Lemma 4.11 and 4.10 we have the following theorem:

**Theorem 4.12** *The closure of the cell  $C_\beta$  in  $A_{p,m}^{\leq n}$  is given by:*

$$\overline{C}_\beta = \bigcup_{\alpha \leq \beta} C_\alpha. \quad (4.19)$$

**Example 4.13** The cell decomposition of  $A_{2,2}^{\leq 1}$  is given by:



## 5 Finite CW-Complex

In this section we will show that the cell decomposition of the space  $A_{p,m}^{\leq n}$  considered in the previous sections is actually of the type of a finite CW-complex. For the convenience of the reader we repeat the relevant definition from topology. More information can be found e.g. in [18].

Let

$$D^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid \sum_l x_l^2 \leq 1\} \quad (5.1)$$

be the unit disk and

$$O^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid \sum_l x_l^2 < 1\} \quad (5.2)$$

be the interior of  $D^d$ . Any space homeomorphic to  $O^d$  is called an open  $d$ -cell. In particular,  $\mathbb{R}^d$  is an open  $d$ -cell.

**Definition 5.1** [22, Definition 6.1] A finite CW-complex consists of a Hausdorff space  $X$  together with a partition of  $X$  into a finite collection  $\{C_\alpha\}$  of disjoint subsets such that

1. Each  $C_\alpha$  is topologically an open cell of dimension  $d_\alpha \geq 0$ . Furthermore for each cell  $C_\alpha$  there exists a continuous map

$$\chi : D^{d_\alpha} \rightarrow X \quad (5.3)$$

which carries  $O^{d_\alpha}$  homeomorphically onto  $C_\alpha$ . The map  $\chi$  is called a characteristic map for the cell  $C_\alpha$ .

2. Each point  $x$  which belongs to the closure  $\overline{C_\alpha}$ , but not to  $C_\alpha$  itself, must lie in a cell  $C_\beta$  of lower dimension.

We want to remark at this point that a finite CW-complex is necessarily a compact topological space. Before we establish the result that  $A_{p,m}^{\leq n}$  is a finite CW-complex we would like to make the connection to the paper [23] of Ravi and the second author.

In this paper the space of homogeneous autoregressive systems is considered. An autoregressive system  $P(s, t) \in \mathbb{K}^{p \times (m+p)}[s, t]$  is called homogeneous if each entry  $f_{ij}(s, t)$ ,  $i = 1, \dots, p, j = 1, \dots, m + p$  of  $P(s, t)$  is a homogeneous polynomial of degree  $\nu_i$  and at least one principal minor, necessarily a homogeneous polynomial of degree  $d = \sum_{i=1}^p \nu_i$  is nonzero. In [23] it is then shown that the set of all homogeneous  $p \times (m + p)$  autoregressive systems of degree  $d$ , which we like to denote by  $\tilde{K}_{p,m}^d$ , has the structure of a smooth projective variety.

Based on earlier work by Stromme [27] an explicit embedding of  $\tilde{K}_{p,m}^n$  into the Grassmann manifold  $\text{Grass}(np + p - n, (n + 1)(m + p))$  was constructed. Over  $\mathbb{C}$  (or over  $\mathbb{R}$ ) one can equip  $\tilde{K}_{p,m}^n$  with the subset topology coming from the complex (real) Grassmannian. One has a natural projection  $\pi : \tilde{K}_{p,m}^n \rightarrow A_{p,m}^{\prec n}$  given through the dehomogenization  $P(s, t) \mapsto P(s, 1)$  and this projection is generically one-one.

The following Lemma relates the topology of  $A_{p,m}^{\prec n}$  as introduced in Definition 3.7 with the topology of  $\tilde{K}_{p,m}^n$ .

**Lemma 5.2** *If  $d \leq n$  the map*

$$\begin{aligned} \pi_{d,n} : \tilde{K}_{p,m}^d &\longrightarrow A_{p,m}^{\prec n} \\ P(s, t) &\longmapsto P(s, 1) \end{aligned} \quad (5.4)$$

*is continuous.*

*Proof:* For any closed set  $S \in A_{p,m}^{\prec n}$ , let  $x$  be a point in the closure of  $\pi^{-1}(S)$ . Then there is a homogeneous polynomial matrix  $P(s, t)$  representing the equivalence class of  $x$  and a sequence of polynomial matrices  $P_n(s, t)$  in  $\pi^{-1}(S) \subset \tilde{K}_{p,m}^d$  such that  $\lim P_n(s, t) = P(s, t)$ . So  $\lim P_n(s, 1) = P(s, 1)$  which means that  $x \in \pi^{-1}(S)$ . ■

Based on the facts that  $\tilde{K}_{p,m}^d$  is a compact topological space [23] and  $\pi = \pi_{n,n}$  is onto we have:

**Corollary 5.3**  *$A_{p,m}^{\prec n}$  is a compact topological space.*

We are now in a position to state the main theorem of this section:

**Theorem 5.4**  *$A_{p,m}^{\prec n}$  is a finite CW-complex.*

*Proof:* From Lemma 3.8 we already know that the set  $\{C_\alpha\}$  partitions  $A_{p,m}^{\prec n}$  into a finite collection of disjoint cells which we will therefore identify with the open balls  $O^{d_\alpha}$ . Assume that the degree of the systems in  $C_\alpha$  are equal to  $d$ . Then

$$C_\alpha \subset A_{p,m}^d \subset A_{p,m}^{\prec n}.$$

Let  $P(s) = P_0 + P_1s + \dots + P_d s^d \in C_\alpha$  be the matrices in the echelon form defined by Proposition 3.5 and consider the infinite matrix

$$P = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_d & 0 & 0 & \cdots \\ 0 & P_0 & P_1 & \cdots & P_{d-1} & P_d & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}.$$

Recall the definition of the ordered sequence  $f_1(\alpha) < f_1(\alpha) < \dots$  as introduced in (4.2). Notice that

$$\#\{i \mid f_i(\alpha) \leq (d+1)(m+p)\} = dp + p - d. \quad (5.5)$$

In particular there are exactly  $dp + p - d$  rows of  $P$  which are elements of the vector space  $V_{(d+1)(m+p)}$ . (Compare with (4.17)). The subspace spanned by these rows is a point in  $\text{Grass}(dp + p - d, (d+1)(m+p))$ , and if we write these rows into a matrix, each element has the following particular row reduced echelon form:

$$\left[ \begin{array}{cccccccccccc} & & & f_1 & & & f_2 & \cdots & & f_{dp+p-d} & & & & \\ * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & * & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 \end{array} \right] \quad (5.6)$$

So we have natural embeddings:

$$C_\alpha \xrightarrow{i_1} C(f_1(\alpha), \dots, f_{dp+p-d}(\alpha)) \xrightarrow{i_2} \text{Grass}(dp + p - d, (d+1)(m+p)). \quad (5.7)$$

where the Schubert cell  $C(f_1, \dots, f_{dp+p-d})$  was earlier defined in (4.18). Let

$$k = \sum_{l=1}^{dp+p-d} (f_l - l) \quad (5.8)$$

be the dimension of this cell and let  $O^k$  be the corresponding homeomorphic open ball. Furthermore let  $D^k := \overline{O^k}$  and  $D^{d_\alpha} := \overline{O^{d_\alpha}}$ . The following diagram explains the interrelation between the different spaces and maps defined so far:

$$\begin{array}{ccccc} C_\alpha & \xrightarrow{i_1} & C(f_1, \dots, f_{dp+p-d}) & & \\ \downarrow & & \downarrow & \searrow^{i_2} & \\ O^{d_\alpha} & \xrightarrow{i_3} & O^k & \xrightarrow{i_4} & \text{Grass}(dp+p-d, (d+1)(m+p)) \\ \downarrow^{i_5} & & \downarrow^{i_6} & \nearrow^\phi & \uparrow^{i_8} \\ D^{d_\alpha} & \xrightarrow{i_7} & D^k & & \tilde{K}_{p,m}^d \\ & & & & \downarrow^{\pi_{d,n}} \\ & & & & A_{p,m}^{<n} \end{array} \quad (5.9)$$

In this commutative diagram  $i_1, i_2$  are the inclusions defined in (5.7) and  $i_3, i_4$  are the maps induced by  $i_1, i_2$ . The maps  $i_5, i_6, i_7$  denote the natural inclusion maps and  $i_8$  denotes the embedding of the compact manifold  $\tilde{K}_{p,m}^d$  as defined in [23]. It is our goal to show the existence of a characteristic map from  $D^{d_\alpha}$  to  $A_{p,m}^{\prec n}$ .

Since  $\text{Grass}(dp + p - d, (d + 1)(m + p))$  is a CW-complex and the Schubert cells of the form (5.6) define a cell decomposition of  $\text{Grass}(dp + p - d, (d + 1)(m + p))$  (see [18, Ch.I, Ex. 2.5.] or [22, §6]), there exists characteristic map  $\phi$ , which carries  $O^k$  homeomorphically onto  $C(f_1, \dots, f_{dp+p-d})$  when viewed as a subset of the Grassmann manifold.

Crucial for the proof is the observation that by definition  $(i_4 \circ i_3)(O^{d_\alpha}) \subset i_8(\tilde{K}_{p,m}^d)$ . Since  $i_8(\tilde{K}_{p,m}^d)$  is closed and the characteristic map  $\phi$  is continuous we conclude that  $(\phi \circ i_7)(D^{d_\alpha}) \subset i_8(\tilde{K}_{p,m}^d)$ . But then the map

$$\chi := \pi_{d,n} \circ (i_8)^{-1} \circ \phi \circ i_7 \quad (5.10)$$

is well defined and continuous. By continuity  $\chi(D^{d_\alpha}) \subset \overline{\chi(O^{d_\alpha})} = \overline{C_\alpha}$ . By Theorem 4.12  $\chi$  is therefore a characteristic map.  $\blacksquare$

**Remark 5.5** The embedding of  $C_\alpha$  into  $\text{Grass}(dp + p - d, (d + 1)(p + m))$  gives us another way to describe the projective manifold  $\tilde{K}_{p,m}^d$  introduced in [23]. For this consider in (5.9) the situation when  $d = n$ . Then the closure of  $(i_2 \circ i_1)(C_\alpha)$  is necessarily isomorphic to the manifold  $\tilde{K}_{p,m}^n$ .

In conclusion of this section we describe the singular homology groups for the set  $A_{p,m}^{\prec n}(\mathbb{C})$ . Note that the cells  $C_\alpha(\mathbb{C})$  have real dimension  $2d_\alpha$  in particular there are no cells of odd real dimension. Define

$$b_k = \begin{cases} \#\{\alpha \mid d_\alpha = k/2\} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \quad (5.11)$$

By the properties of finite CW-complex [18] defined over  $\mathbb{C}$ , we deduce

**Theorem 5.6** *The singular homology  $H_*(A_{p,m}^{\prec n}; \mathbf{Z})$  of  $A_{p,m}^{\prec n}(\mathbb{C})$  has no torsion,*

$$H_k(A_{p,m}^{\prec n}; \mathbf{Z}) = \mathbf{Z}^{b_k}$$

and the  $k$ -th Betti number of  $A_{p,m}^{\prec n}(\mathbb{C})$  is  $b_k$ .

**Acknowledgments.** We would like to take the opportunity to thank U.Helmke, M.S.Ravi and the anonymous referee for several helpful comments and discussions.

## References

- [1] J.Ball, I.Gohberg and L.Rodman, Realization and interpolation of rational matrix functions, *Topics in Interpolation Theory of Rational Matrix Valued Functions* (I.Gohberg ed.) Birkhäuser Verlag, Basel, 1–72 (1988).
- [2] H.Blomberg and R.Ylinen, *Algebraic Theory for Multivariable Linear Systems*, Academic Press, London, 1983.
- [3] C.I.Byrnes and N.E.Hurt, On the Moduli of Linear Dynamical Systems, *Adv. in Math.*, Suppl. Series vol.4, 83–122 (1978).
- [4] C.I.Byrnes and T.Duncan, On Certain Topological Invariants arising in Systems Theory, *New Directions in Applied Mathematics*, (P.Hilton and G.Young eds.), Springer-Verlag, New York, 29–71 (1981).
- [5] J.M.Clark, The consistent selection of local coordinates in linear system identification, *Proc. Joint Automatic Control Conference*, 576–580 (1976).
- [6] D.F.Delchamps, “Global Structure of Families of Multivariable Linear Systems with an Application to Identification,” *Math. Systems Theory* 18, 329–380 (1985).
- [7] G.D.Forney, Minimal bases of rational vector spaces, with applications to multivariable linear systems, *SIAM J. Control Optim.*, 13: 493–520 (1975).
- [8] P.A.Fuhrmann and P.S.Krishnaprasad, Towards a cell decomposition for rational functions, *IMA J. Math. Control Inform.* 3: 137–150 (1986).
- [9] H.GlÜsing-Lüerßen, Gruppenaktionen in der Theorie Singulärer Systeme, Ph.D. Thesis, Universität Bremen, Germany, 1991.
- [10] M.Hazewinkel, Moduli and canonical forms for linear dynamical systems III: the algebraic geometric case, *Proc. of the 1976 Ames Research Center (NASA) Conference on Geometric Control Theory* (C.F.Martin and R.Hermann eds.), Math. Sci. Press, 291–336 (1977).
- [11] M.Hazewinkel and C.Martin, Representations of the symmetric group, the specialization order, systems and Grassmann manifolds, *L'Enseignement Mathématique*, 29: 53–87 (1983).
- [12] U.Helmke, The topology of a moduli space for linear dynamical systems, *Comment. Math. Helv.*, 60: 630–655 (1985).

- [13] U.Helmke, The Cohomology of Moduli Spaces of Controllable Linear Systems, Habilitationsschrift, Universität Regensburg, 1989.
- [14] U.Helmke, D.Hinrichsen and W.Manthey, A cell decomposition of the space of real Hankel matrices of rank  $\leq n$  and some applications, *Linear Algebra Appl.*, 122/123/124: 331–355 (1989).
- [15] U.Helmke and M.A.Shayman, Topology of the Orbit Space of Generalized Linear Systems, *Mathematics of Control, Signals and Systems*, vol.4, no.4, 411-437 (1991).
- [16] P.Janssen, General results on the McMillan degree and the Kronecker indices of ARMA and MFD Models, *Internat. J. Control*, 48: 591–608 (1988).
- [17] M.Kuijper and J.M.Schumacher, Realization of autoregressive equations in pencil and descriptor form, *SIAM J. Control Optim.*, 28: 1162–1189 (1990).
- [18] A.T.Lundell and S.Weingram, *The Topology of CW Complexes*, Van Nostrand Reinhold Company, 1969.
- [19] B.M.Mann and R.J.Milgram, Some spaces of holomorphic maps to complex Grassmann manifolds, *J. Differential Geom.*, 33: 301–324 (1991).
- [20] W.Manthey, Die Bruhat-Zellzerlegung von Räumen rationaler Funktionen und Hankel Matrizen, Ph.D. Thesis, Universität Bremen, Germany, 1991.
- [21] C.F.Martin and R.Hermann, Applications of algebraic geometry to system theory: the McMillan degree and Kronecker indices as topological and holomorphic invariants, *SIAM J. Control*, 16: 743–755 (1978).
- [22] J.W.Milnor and J.D.Stasheff, *Characteristic Classes*, Annals of Mathematics Studies No.76, Princeton University Press, 1974.
- [23] M.S.Ravi and J.Rosenthal, A smooth compactification of the space of transfer functions with fixed McMillan degree, preprint.
- [24] J.Rosenthal, On dynamic feedback compensation and compactification of systems, *SIAM J. Control Optim.*, to appear.
- [25] J.Rosenthal, M.Sain and X.Wang, Topological considerations for autoregressive systems with fixed Kronecker indices, *Circuits Systems Signal Process.*, to appear.

- [26] J.M.Schumacher, Transformations of linear systems under external equivalence, *Linear Algebra Appl.*, 102: 1–33 (1988).
- [27] S.A.Stromme, On parameterized rational curves in Grassmann varieties, *Lecture Notes in Mathematics* no. 1266, Springer-Verlag, Berlin, New York, 251–272 (1987).
- [28] X.Wang, On output feedback via Grassmannians, *SIAM J. Control Optim.*, 29: 926–935 (1991).
- [29] J.C.Willems, From time series to linear systems – part I. finite dimensional linear time invariant systems, *Automatica*, 22: 561–580 (1986).
- [30] J.C.Willems, Models for dynamics, *Dynamics Reported*, 2: 171–269 (1989).
- [31] J.C.Willems, Paradigms and puzzles in the theory of dynamical systems, *IEEE Trans. Automat. Control*, 36: 259–294 (1991).