A Posteriori Error Majorants of the Modeling Errors for Elliptic Homogenization Problems

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Dedicated to Dietrich Braess on the occation of his 75th birthday

Abstract

In this paper, we derive new a posteriori estimates of modeling errors for linear elliptic boundary value problems with periodic coefficients solved by homogenization. Our approach is based on the concept of functional a posteriori error estimation. The estimates are obtained for the energy norm and use solely the global flux of the non-oscillatory solution of the *homogenized model* and solution of a boundary value problem on the cell of periodicity.

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French Abstract:

Dans cette note nous obtenons de nouvelles estimations des erreurs de modélisations pour des problèmes elliptiques linéaires d'homogénéisation à coefficients périodiques. Notre approche est fondée sur le concept d'estimation à posteriori fonctionnelle. Nos estimations sont obtenues pour la norme d'énergie et utilisent seulement le flux de la solution non oscillante du problème homogénéisé et la solution d'un problème aux limites sur la cellule de périodicité.

Abridged French Version.

Dans cette note on considère des problèmes d'homogénéisation elliptiques du type $\operatorname{div}(\mathbf{A}_\varepsilon \nabla u_\varepsilon) + f = 0$ dans un domaine à frontière Lipschitzienne où la matrice $\mathbf{A}_\varepsilon$ est définie par (1.2), $\varepsilon > 0$ est un petit paramètre et $\Pi_{\mathbf{i}}^{\varepsilon} := \mathbf{x_i} + \varepsilon\widehat{\Pi}$ est la cellule définie par $\mathbf{x_i}$, la translation et dilatation de la cellule de référence $\widehat{\Pi}$ (dans toute la note, $\mathbf{x}$ désigne le système de cordonnées globales dans Ω et $\mathbf{y}$ le système de cordonnées locales dans la cellule de référence $\widehat{\Pi}$).

Il est bien connu (cf., e.g., [1], [2]) que pour $\varepsilon > 0$ petit une bonne approximation peut être obtenue sous la forme $w_\varepsilon^1 = u_0 - \varepsilon\psi^\varepsilon \mathbf{N}^{\mathrm{per}} \cdot \nabla u_0$, où u_0 est solution du problème homogénéisé (1.4), $\mathbf{N}^{\mathrm{per}}$ est défini par (2.2) et (2.4), et ψ^ε est une fonction de troncature (cf., e.g., [2]).

L'objectif principal de ce papier est d'obtenir une borne supérieure entièrement calculable de la différence entre u_ε et w_ε^1 pour la norme d'énergie (2.5) qui représente l'erreur occasionnée par la solution homogénéisée.

Théorème. *Soit $\widehat{\Pi}$ un domaine convexe, $f \in L^2(\Omega)$, et $u_0 \in H^2(\Omega)$, alors:*

$$\left\|\nabla\left(u_\varepsilon - w_\varepsilon^1\right)\right\|_{\mathbf{A}_\varepsilon} \le \mathcal{M}_\oplus\left(w_\varepsilon^1,\, \widehat{\boldsymbol{\eta}}, \boldsymbol{\lambda},\, s\right) := \mathcal{F}^{1/2}\left(w_\varepsilon^1; \widehat{\boldsymbol{\eta}}, \boldsymbol{\lambda},\, s\right) + \varepsilon^s \widetilde{C}\, \|\operatorname{div}\widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}}, \tag{0.1}$$

où la constante $\widetilde{C}$ est définie par Théorème 2.1 et $\mathcal{F}$ est défini par (2.7) et (2.8). L'estimation est vraie pour tout $\widehat{\boldsymbol{\eta}} \in \mathbf{H}_0\left(\widehat{\Pi}, \operatorname{div}\right)$, $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^d \in \mathbb{R}^d_{>0}$, et $s \in \mathbb{R}$.

Il est facile de voir que le membre de droite de (0.1) est la somme de deux termes positifs qui incluent une fonction arbitraire $\boldsymbol{\eta}$ définie sur $\widehat{\Pi}$. Il s'en suit que le calcul de ce majorant repose sur le flux de la solution homogénéisée et un choix approprié de la fonction $\boldsymbol{\eta}$. Les paramètres scalaires λ_i et la puissance

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s peuvent être choisis de façon à minimiser ce majorant. On remarquera que ce majorant ne fait pas appel à une approximation du flux associé au problème périodique initial.

Remarque. *Si la structure périodiques est grossière et consiste en relativement peu de cellules (25-100) et/ou les coefficients de la matrice* $\widehat{\mathbf{A}}$ *ont des sauts, des oscillations, etc alors le terme* $\varepsilon^s \boldsymbol{\eta}$ *est susceptible d'augmenter de façon significative le flux homogénéisé. Si la structure périodique est fine alors le terme correctif est moins important et son influence peut être diminuée en augmentant la valeur de* s*. Dans le cas limite, i.e.* $s \to +\infty$*, on obtient la version suivante simplifiée du majorant de l'erreur:*

$$\left\|\nabla\left(u_\varepsilon - w_\varepsilon^1\right)\right\|_{\mathbf{A}_\varepsilon} \leq \overline{\mathcal{M}}_\oplus\left(u_0,\, \varepsilon\right) := \|\mathbf{G}\|_{\mathbf{A}_\varepsilon},$$

où $\mathbf{G}$ *est défini par (2.7). Le majorant ne contient alors aucune constante dépendant du domaine ni une fonction auxiliaire et peut être calculer à partir de* $\widehat{N}_k$ *et* u_0*.*

1 Introduction

We will consider boundary value problems with periodic structures which arise in various applications such as composite materials. Within the framework of the homogenization theory (see, e.g., [2]), the behaviour of a heterogeneous media is described with the help of a certain homogenized problem, which is typically a boundary value problem with smooth coefficients, and the solution of a specially constructed problem with periodic boundary conditions. In this paper, our goal is to derive an a posteriori estimate of the modeling error generated by homogenization. The error majorant employs the solution of the homogenized problem and, thus, is an a posteriori estimate.

The method is based on functional a posteriori estimates which allow to treat modelling as well as numerical errors within a unified concept; for a comprehensive introduction and overview we refer to [4] - [5].

We are concerned with homogenized models of an elliptic boundary value problem with periodical coefficients. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$ such that, for a small scale parameter $\varepsilon > 0$ we have $\Omega = \bigcup_{\mathbf{i}} \Pi_{\mathbf{i}}^\varepsilon$, where

$$\Pi_{\mathbf{i}}^\varepsilon = \mathbf{x}_{\mathbf{i}} + \varepsilon\,\widehat{\Pi} = \left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; \frac{\mathbf{x} - \mathbf{x}_{\mathbf{i}}}{\varepsilon} \in \widehat{\Pi}\right\}$$

denote the dilation and translation of the basic "cell" $\hat{\Pi}$; $\mathbf{x}_{\mathbf{i}}$ is the *reference* point of $\Pi_{\mathbf{i}}^\varepsilon$. By $\mathbf{x}$ we denote the global (Cartesian) coordinate system in $\mathbb{R}^d$ and by $\mathbf{i} = (i_1,\, i_2,\, ...i_d)$ the counting multi-indices for the cells. The notations $\bigcup_{\mathbf{i}}$ and $\sum_{\mathbf{i}}$ are shorthands for the union and summation over all cells.

In the basic cell we denote the Cartesian coordinates by $\mathbf{y} \in \mathbb{R}^d$. For any $\Pi_{\mathbf{i}}^\varepsilon$, local and global coordinates are related by $\mathbf{y} = \frac{\mathbf{x}-\mathbf{x}_{\mathbf{i}}}{\varepsilon} \in \widehat{\Pi}$ for all $\mathbf{x} \in \Pi_{\mathbf{i}}^\varepsilon$ and all $\mathbf{i}$.

The diffusion matrix in the periodic setting is given via the cell matrix function $\widehat{\mathbf{A}} \in L^\infty(\widehat{\Pi},\, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, where $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the set of symmetric $d \times d-$ matrices. We assume that

$$c_1|\xi|^2 \;\leq\; \widehat{\mathbf{A}}(\mathbf{y})\xi\cdot\xi \leq \rho\left(\widehat{\mathbf{A}}\right)|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^d \quad \forall \mathbf{y} \in \widehat{\Pi}\,\text{a.e.}, \tag{1.1}$$

where $0 < c_1 \leq c_2 < \infty$. The global matrix $\mathbf{A}_\varepsilon(\mathbf{x})$ defines the periodic structure on Ω

$$\mathbf{A}_\varepsilon(\mathbf{x}) := \widehat{\mathbf{A}}\left(\frac{\mathbf{x} - \mathbf{x}_{\mathbf{i}}}{\varepsilon}\right) \qquad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^\varepsilon\ \forall \mathbf{i}. \tag{1.2}$$

For $f \in L^2(\Omega)$ we consider the second-order elliptic equation with homogeneous Dirichlet boundary conditions in variational formulation: Find $u_\varepsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega \mathbf{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla w = \int_\Omega f w \qquad \forall w \in H_0^1(\Omega). \tag{1.3}$$

For any $\varepsilon > 0$, the solution $u_\varepsilon \in H_0^1(\Omega)$ exists and is unique. It is known (see, e.g., [1], [2]) that there exists a (constant) uniformly elliptic *homogenized matrix* $\mathbf{A}_0 \in \mathbb{R}^{d\times d}_{sym}$ such the solution of the homogenized variational problem

$$\text{find } u_0 \in H_0^1(\Omega) \qquad \int_\Omega \mathbf{A}_0 \nabla u_0 \cdot \nabla w = \int_\Omega f w \qquad \forall w \in H_0^1(\Omega) \tag{1.4}$$

satisfies

$$u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega) \qquad \text{and} \qquad u_\varepsilon \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega) \quad \text{for } \varepsilon \to 0.$$

Problem (1.4) is well studied in the context of asymptotic analysis (see, e.g., [1], [2]). It was shown that the a priori estimate $\left\|u_\varepsilon - u_\varepsilon^1\right\|_{H^1(\Omega)} \leq c\sqrt{\varepsilon}$ holds, where

$$u_\varepsilon^1(\mathbf{x}) := u_0(\mathbf{x}) + \varepsilon u_1\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{x}_i}{\varepsilon}\right) \qquad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^\varepsilon \ \forall \mathbf{i} \tag{1.5}$$

and $u_1(\mathbf{x},\cdot)$ is some $\widehat{\Pi}-$periodic function.

Our *error majorants* will reflect the decomposition as in (1.5). The majorant is based on the homogenized problem and its solution and, in addition, depends on free functions defined on the cell of periodicity. They should be chosen such that the majorant for the energy norm becomes as small as possible and can either be computed as the solution of a certain boundary value problem with periodic boundary conditions on the basic cell or by minimizing the error majorant which will depend on the solution of (1.4), the small parameter ε, and some other functions, defined on $\widehat{\Pi}$.

2 Homogenization of second order elliptic operators

First we introduce some notation. The *periodification* of a sufficiently smooth function $\widehat{v} : \widehat{\Pi} \to \mathbb{R}$ is denoted by

$$v^{\mathrm{per}}(\mathbf{x}) := \widehat{v}\left(\frac{\mathbf{x}-\mathbf{x}_\mathbf{i}}{\varepsilon}\right) \qquad \forall \mathbf{x} \in \Pi_i^\varepsilon \quad \forall \mathbf{i}. \tag{2.1}$$

For a measurable subset ω and a function $\zeta \in L^1(\omega)$ the integral mean is given by $\langle\zeta\rangle_\omega := \frac{1}{|\omega|}\int\limits_\omega \zeta$. If we write $\int_\omega \langle\zeta\rangle_\omega$ we consider this average as a constant function on ω (for vector-valued functions, we apply this definition componentwise). We denote the error caused by the integral average by $\delta_\omega\zeta := \|\zeta - \langle\zeta\rangle_\omega\|_\omega$ with $\|\cdot\|_\omega$ denoting the $L^2(\omega)$ norm. For vector-valued functions $\boldsymbol{\zeta} = (\zeta_k)_{k=1}^d \in \mathbf{L}^1(\omega) := L^1\left(\omega, \mathbb{R}^d\right)$ and $\boldsymbol{\phi} = (\phi_k)_{k=1}^d \in \mathbf{L}^1(\Omega)$ we define the local and piecewise averages by

$$\delta_\omega\boldsymbol{\zeta} := \left(\|\zeta_k - \langle\zeta_k\rangle_\omega\|_\omega\right)_{k=1}^d, \qquad \delta_\Omega^{\mathrm{pw}}\boldsymbol{\phi} := \varepsilon^{d/2}\left(\sum_{\mathbf{i}}\left\|\phi_k - \langle\phi_k\rangle_{\Pi_\mathbf{i}^\varepsilon}\right\|_{\Pi_\mathbf{i}^\varepsilon}\right)_{k=1}^d.$$

For a $d \times d$ matrix (-valued function) $\mathbf{A}$ we denote by $\mathbf{a}_k$ its k-th column vector.

We summarize the three steps for computing the boundary corrected approximation of u_ε below which is well known in the theory of homogenization.

i) The solutions $\widehat{\mathbf{N}} = \left(\widehat{N}_k\right)_{k=1}^d \in \mathbf{H}^1_{\mathrm{per}}\left(\widehat{\Pi}\right)/\mathbb{R}$ of the cell problems

$$\operatorname{div}\left(\widehat{\mathbf{A}}\nabla\widehat{N}_k\right) = \operatorname{div}\widehat{\mathbf{a}}_k \quad \text{in } \widehat{\Pi} \tag{2.2}$$

allow to define the homogenized matrix $\mathbf{A}_0 := \left\langle\widehat{\mathbf{A}}\left(\mathbf{I} - \nabla\widehat{\mathbf{N}}\right)\right\rangle_{\widehat{\Pi}}$.

ii) The homogenized problem reads: Find $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega \mathbf{A}_0\nabla u_0 \cdot \nabla w = \int_\Omega f\, w \qquad \forall w \in H_0^1(\Omega). \tag{2.3}$$

iii) With the help of u_0 and $\widehat{N}_k$, we obtain the approximation w_ε^1 of u_ε via (cf. (2.1))

$$w_\varepsilon^1 := u_0 - \varepsilon\psi^\varepsilon\mathbf{N}^{\mathrm{per}} \cdot \nabla u_0 \tag{2.4}$$

with the cutoff function $\psi^\varepsilon := \min\{1, \frac{1}{\varepsilon}\operatorname{dist}(\cdot, \partial\Omega)$.

Next, we will deduce a guaranteed a posteriori error majorant of

$$\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{\mathbf{A}_\varepsilon} \quad \text{with} \quad \|\boldsymbol{q}\|_{\mathbf{A}_\varepsilon} := \left(\int_\Omega \mathbf{A}_\varepsilon\, \boldsymbol{q}\cdot\boldsymbol{q}\right)^{1/2} \tag{2.5}$$

which employs the functions $\widehat{N}_k$ as well as the homogenized solution u_0. We note that in practice the problems (2.2) and (2.3) are solved numerically so that the corresponding approximation errors should be also taken into account. This can be done by known methods (see, e.g., [4, 5] and the references therein). Therefore, in this concise note we do not focus attention on these questions (leaving them for a proper consideration in a full scale forthcoming paper).

2.1 Error estimate of the modeling error

In this section, we first prove a subsidiary result which states an upper bound of the L^2-scalar product of a globally defined function with a cell-periodic function. For a vector $\boldsymbol{\mu} = (\mu_i)_{i=1}^d \in (\mathbb{R}_{>0})^d$ and $s \in \mathbb{R}$ we denote by $\boldsymbol{\mu}^s$ the componentwise application of the power s, i.e., $\boldsymbol{\mu}^s = (\mu_i^s)_{i=1}^d$.

Lemma 2.1 *For all $\mathbf{g} \in \mathbf{L}^2(\Omega)$, $\widehat{\boldsymbol{\eta}} \in \mathbf{L}^2(\widehat{\Pi})$, and all $\boldsymbol{\lambda} = (\lambda_d)_{k=1}^d \in (\mathbb{R}_{>0})^d$ it holds*

$$(\mathbf{g}, \boldsymbol{\eta}^{\mathrm{per}})_\Omega \le |\Omega|\, \langle \mathbf{g}\rangle_\Omega \cdot \langle \widehat{\boldsymbol{\eta}}\rangle_{\widehat{\Pi}} \; + \frac{\boldsymbol{\lambda}}{2}\cdot(\delta_\Omega^{\mathrm{pw}}\mathbf{g})^2 + \frac{\boldsymbol{\lambda}^{-1}}{2}\cdot\left(\delta_{\widehat{\Pi}}\widehat{\boldsymbol{\eta}}\right)^2. \tag{2.6}$$

For a proof we refer to [7].

Let $\nabla\nabla^{\intercal} u_0\,(\mathbf{x})$ denote the Hessian matrix of u_0 and let $\nabla\widehat{\boldsymbol{N}}^{\intercal}$ be the Jacobi matrix of the vector-valued function $\widehat{\boldsymbol{N}}$. In order to present the main estimate in a transparent form, we define the function

$$\mathbf{G} := \nabla w_\varepsilon^1 - \mathbf{A}_\varepsilon^{-1}\mathbf{A}_0\nabla u_0 \tag{2.7}$$

with w_ε^1 as in (2.4) which allows to define part of the error majorant

$$\mathcal{F}\left(w_\varepsilon^1; \widehat{\boldsymbol{\eta}}, \boldsymbol{\lambda}, s\right) := \|\mathbf{G}\|_{\mathbf{A}_\varepsilon}^2 + 2\,\varepsilon^s\,|\Omega|\,\langle\mathbf{G}\rangle_\Omega\cdot\langle\widehat{\boldsymbol{\eta}}\rangle_{\widehat{\Pi}} + \varepsilon^s\left(\boldsymbol{\lambda}^{-1}\cdot(\delta_{\widehat{\Pi}}\widehat{\boldsymbol{\eta}})^2 \;+\; \boldsymbol{\lambda}\cdot(\delta_\Omega^{\mathrm{pw}}\mathbf{G})^2\right) + c_0\varepsilon^{2s}\|\widehat{\boldsymbol{\eta}}\|_{\widehat{\mathbf{A}}^{-1}}^2, \tag{2.8}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^d_{>0},\ s \in \mathbb{R}$,

$$\widehat{\boldsymbol{\eta}} \in \mathbf{H}_0\left(\widehat{\Pi},\,\mathrm{div}\right) := \left\{\widehat{\boldsymbol{\vartheta}} \in \mathbf{H}(\widehat{\Pi},\,\mathrm{div}),\ \langle \mathrm{div}\,\widehat{\boldsymbol{\vartheta}}\rangle_{\widehat{\Pi}} = 0\right\}$$

and

$$\mathbf{H}\left(\widehat{\Pi},\,\mathrm{div}\right) := \left\{\widehat{\boldsymbol{\vartheta}} \in \mathbf{L}^2\left(\widehat{\Pi}\right),\, \mathrm{div}\,\widehat{\boldsymbol{\vartheta}} \in L^2\left(\widehat{\Pi}\right)\right\}.$$

Now, we formulate of our main result.

Theorem 2.1 *Let $\mathbf{A}_\varepsilon$ be defined by (1.2). We assume that (1.1) is satisfied and that the number of cells $\Pi_{\mathbf{i}}^\varepsilon$ in Ω equals $c_0\varepsilon^{-d}$ with $c_0 = \mathcal{O}(1)$. Let the reference cell $\widehat{\Pi}$ be convex. We assume that the right-hand side in (1.3) satisfies $f \in L^2(\Omega)$ with exact solution u_ε. The solution u_0 of the homogenized problem is required to be in $H^2(\Omega)$. Let w_ε^1 be as in (2.4). Then, the error $u_\varepsilon - w_\varepsilon^1$ can be estimated by*

$$\left\|\nabla\left(u_\varepsilon - w_\varepsilon^1\right)\right\|_{\mathbf{A}_\varepsilon} \le \mathcal{M}_\oplus\left(w_\varepsilon^1,\, \widehat{\boldsymbol{\eta}}, \boldsymbol{\lambda},\, s\right) := \mathcal{F}^{1/2}\left(w_\varepsilon^1; \widehat{\boldsymbol{\eta}}, \boldsymbol{\lambda},\, s\right) + \varepsilon^s\widetilde{C}\,\|\,\mathrm{div}\,\widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}}, \tag{2.9}$$

where $\mathcal{F}$ is defined by (2.8). The quantities $\widehat{\boldsymbol{\eta}} \in \mathbf{H}_0\left(\widehat{\Pi},\,\mathrm{div}\right)$, $\boldsymbol{\lambda} \in \mathbb{R}^d_{>0}$, and $s \in \mathbb{R}$ are free parameters and $\widetilde{C} := \frac{\varrho}{\pi}\sqrt{\frac{c_0}{c_1}}$.

Proof. For any $v, w \in H_0^1(\Omega)$ and $\boldsymbol{\tau} \in \mathbf{H}(\Omega, \mathrm{div})$, we have

$$\int_\Omega \mathbf{A}_\varepsilon\nabla(u_\varepsilon - v)\cdot\nabla w = \int_\Omega(-\mathbf{A}_\varepsilon\nabla v\cdot\nabla w + f\,w) = \int_\Omega(\boldsymbol{\tau} - \mathbf{A}_\varepsilon\nabla v)\cdot\nabla w \;+ \int_\Omega(\mathrm{div}\,\boldsymbol{\tau} + f)\,w. \tag{2.10}$$

We set $w = u_\varepsilon - v$ and estimate the first term in (2.10) as follows:

$$\int_\Omega (\boldsymbol{\tau} - \mathbf{A}_\varepsilon \nabla v) \cdot \nabla (u_\varepsilon - v) \le \|\nabla (u_\varepsilon - v)\|_{\mathbf{A}_\varepsilon} \|\mathbf{A}_\varepsilon \nabla v - \boldsymbol{\tau}\|_{\mathbf{A}_\varepsilon^{-1}} . \tag{2.11}$$

We assume that $\boldsymbol{\tau}$ is of the form

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 - \varepsilon^s \boldsymbol{\eta}^{\mathrm{per}} \quad \text{with} \quad \operatorname{div} \boldsymbol{\tau}_0 = -f \quad \text{and} \quad \widehat{\boldsymbol{\eta}} \in \mathbf{H}_0 \left(\widehat{\Pi}, \operatorname{div}\right). \tag{2.12}$$

Since

$$\operatorname{div} \boldsymbol{\tau} = \operatorname{div} \boldsymbol{\tau}_0 - \varepsilon^{s-1} (\operatorname{div} \widehat{\boldsymbol{\eta}})^{\mathrm{per}} = -f - \varepsilon^{s-1} (\operatorname{div} \widehat{\boldsymbol{\eta}})^{\mathrm{per}}$$

and

$$\langle (\operatorname{div} \widehat{\boldsymbol{\eta}})^{\mathrm{per}} \rangle_{\Pi_{\mathbf{i}}^\varepsilon} = \varepsilon^d \langle \operatorname{div} \widehat{\boldsymbol{\eta}} \rangle_{\widehat{\Pi}} = 0$$

we obtain for $c_{\mathbf{i}} = \langle u_\varepsilon - v \rangle_{\Pi_{\mathbf{i}}^\varepsilon}$

$$\begin{aligned}\int_\Omega (\operatorname{div} \boldsymbol{\tau} + f)(u_\varepsilon - v) &= -\varepsilon^s \int_\Omega \operatorname{div} (\boldsymbol{\eta}^{\mathrm{per}})(u_\varepsilon - v) = -\varepsilon^s \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \operatorname{div} (\boldsymbol{\eta}^{\mathrm{per}})(u_\varepsilon - v - c_{\mathbf{i}}) \\ &= -\varepsilon^s \varepsilon^{d/2-1} \sum_{\mathbf{i}} \|\operatorname{div} \widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}} \|u_\varepsilon - v - c_{\mathbf{i}}\|_{\Pi_i^\varepsilon} \le \varepsilon^s \varepsilon^{d/2-1} \|\operatorname{div} \widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}} C_\varepsilon \sum_{\mathbf{i}} \|\nabla (u_\varepsilon - v)\|_{\Pi_{\mathbf{i}}^\varepsilon},\end{aligned}$$

where C_ε is the constant in the Poincare's inequality for the domains $\Pi_{\mathbf{i}}^\varepsilon$. If $\Pi_{\mathbf{i}}^\varepsilon$ is convex, then for any d we have $C_\varepsilon \le \frac{\operatorname{diam}\Pi_{\mathbf{i}}^\varepsilon}{\pi}$ (see[3]). We assume that $\operatorname{diam}\Pi_{\mathbf{i}}^\varepsilon = \varrho\varepsilon$, where $\varrho = O(1)$ is a certain parameter depending on d and geometric properties of the basic cell. For convex cells the corresponding ϱ is easy to find. For example, if $\Pi_{\mathbf{i}}^\varepsilon$ is a cube, then $\varrho = \sqrt{d}$.

By using our assumption on the number of cells we obtain

$$\int_\Omega (\operatorname{div} \boldsymbol{\tau} + f)(u_\varepsilon - v) \le \varepsilon^s \varepsilon^{\frac{d}{2}-1} \|\operatorname{div} \widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}} \sqrt{c_0} \varepsilon^{-\frac{d}{2}} \varepsilon \frac{\varrho}{\pi} \|\nabla (u_\varepsilon - v)\|_\Omega = \varepsilon^s \frac{\varrho}{\pi} \sqrt{c_0} \|\operatorname{div} \widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}} \|\nabla (u_\varepsilon - v)\|_\Omega .$$

In view of (1.1), we obtain

$$\int_\Omega (\operatorname{div} \boldsymbol{\tau} + f)(u_\varepsilon - v) \le \varepsilon^s \widetilde{C} \|\operatorname{div} \widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}} \|\nabla (u_\varepsilon - v)\|_{\mathbf{A}_\varepsilon} . \tag{2.13}$$

Now (2.10), (2.11), and (2.13) with the choice $v := w_\varepsilon^1$ imply the estimate

$$\|\nabla (u_\varepsilon - v)\|_{\mathbf{A}_\varepsilon} \le \|\mathbf{A}_\varepsilon \nabla v - \boldsymbol{\tau}\|_{\mathbf{A}_\varepsilon^{-1}} + \varepsilon^s \widetilde{C} \|\operatorname{div} \widehat{\boldsymbol{\eta}}\|_{\widehat{\Pi}} . \tag{2.14}$$

Consider the first term in the right-hand side of the estimate (2.14) and set

$$\boldsymbol{\tau}_0 := \mathbf{A}_0 \nabla u_0 \tag{2.15}$$

By using $\mathbf{G}$ as in (2.7), the definition of $\boldsymbol{\tau}$ as in (2.12), and (2.15) we get

$$\mathbf{A}_\varepsilon \nabla w_\varepsilon^1 - \boldsymbol{\tau} = \mathbf{A}_\varepsilon \mathbf{G} + \mathbf{A}_0 \nabla u_0 - (\boldsymbol{\tau}_0 - \varepsilon^s \boldsymbol{\eta}^{\mathrm{per}}) = \mathbf{A}_\varepsilon \mathbf{G} + \varepsilon^s \boldsymbol{\eta}^{\mathrm{per}}.$$

This leads to

$$\begin{aligned}\left\|\mathbf{A}_\varepsilon \left(\nabla w_\varepsilon^1 - \boldsymbol{\tau}\right)\right\|_{\mathbf{A}_\varepsilon^{-1}}^2 &= \|\mathbf{A}_\varepsilon \mathbf{G} + \varepsilon^s \boldsymbol{\eta}^{\mathrm{per}}\|_{\mathbf{A}_\varepsilon^{-1}}^2 = \|\mathbf{G}\|_{\mathbf{A}_\varepsilon}^2 + 2\varepsilon^s (\mathbf{G}, \boldsymbol{\eta}^{\mathrm{per}})_\Omega + \varepsilon^{2s} \|\boldsymbol{\eta}^{\mathrm{per}}\|_{\mathbf{A}_\varepsilon^{-1}}^2 \\ &= \|\mathbf{G}\|_{\mathbf{A}_\varepsilon}^2 + 2\varepsilon^s (\mathbf{G}, \boldsymbol{\eta}^{\mathrm{per}})_\Omega + c_0 \varepsilon^{2s} \|\widehat{\boldsymbol{\eta}}\|_{\widehat{\mathbf{A}}^{-1}}^2 .\end{aligned}$$

The result now follows from Lemma 2.1. ■

Remark 2.1 *The right-hand side of the majorant (2.9) is the sum of two non-negative terms, which include a "free function" $\boldsymbol{\eta}$ defined on the cell of periodicity. Hence, the computation of the majorant is based on the flux of the homogenized solution and a proper selection of the function $\boldsymbol{\eta}$ defined on the cell of periodicity. The scalar parameters λ_i and the power s can be selected in order to minimize the overall value of the majorant. We emphasize that the majorant does not require an approximation of the flux associated with the original periodic problem.*

The following remark concerns the effect of the term $\varepsilon^s \boldsymbol{\eta}$ in the ansatz for $\boldsymbol{\tau}$ in (2.12).

Remark 2.2 *If a periodic structure is coarse and consists of relatively few cells (e.g., 25-100) and/or the coefficients of the matrix* $\widehat{\mathbf{A}}$ *have jumps, oscillations, etc. then the term* $\varepsilon^s \boldsymbol{\eta}$ *may augment the homogenized flux substantially. If the periodic structure is fine, then the correction term is less significant and its influence can be diminished by increasing values of* s*. In the limit case, i.e.,* $s \to +\infty$*, we obtain the following simplified version of the error majorant*

$$\left\|\nabla \left(u_\varepsilon - w_\varepsilon^1\right)\right\|_{\mathbf{A}_\varepsilon} \leq \overline{\mathcal{M}}_\oplus \left(u_0, \varepsilon\right) := \|\mathbf{G}\|_{\mathbf{A}_\varepsilon}, \tag{2.16}$$

where $\mathbf{G}$ *is defined by (2.7). This majorant does not include any domain dependent constants or auxiliary functions and, hence, can be computed from* $\widehat{N}_k$ *and* u_0*.*

Remark 2.3 *In certain cases, we may know only numerical approximations to the solutions* $\widehat{N}_k$ *and* u_0 *of the cell problem (cf. (2.2)) and of the homogenized equation (cf. (2.3)). The corresponding approximation errors can be estimated by error majorants of similar types (see [4] - [6] and references therein). Then, the overall error majorant will include both, approximation and modeling errors. A combined modeling-discretization strategy is suggested in [6] (where the modeling error is generated by defeaturing of a complicated structure) and should be used in this case. This topic deserves a separate investigation and lies beyond the framework of this paper which is focused on the principal structure of the guaranteed error bound for homogenized problems.*

References

[1] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North-Holland, Amsterdam, 1978.

[2] V. Jikov, S. Kozlov, and O. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer, Berlin, 1994.

[3] L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.*, 5:286–292, 1960.

[4] S. Repin. A posteriori error estimation for nonlinear variational problems by duality theory. *Zapiski Nauchn. Semin. POMI*, 243:201–214, 1997.

[5] S. Repin, S. Sauter, and A. Smolianski. A Posteriori Error Estimation for the Dirichlet Problem with Account of the Error in Boundary Conditions. *Computing*, 70:205–233, 2003.

[6] S. I. Repin, T. Samrowski, and S. Sauter. Combined A Posteriori Modelling-Discretization Error Eestimate for Elliptic Problems with Variable Coefficients. *ESAIM: Math.Model.Numer.Anal.*, 46:1389–1405, 2012.

[7] S. I. Repin, T. Samrowski, and S. Sauter. Two-sided Estimates of the Modeling Error for Elliptic Homogenization Problems. Technical Report 12-2012, Institut für Mathematik, Univ. Zürich, 2012. http://www.math.uzh.ch/compmath/fileadmin/math/preprints/12_12.pdf.