On Extension Theorems For Domains Having Small Geometric Details

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Abstract

In this paper, we will consider the problem whether the extension of functions in Sobolev spaces onto larger domains depend on small geometric details of the domain.

We will use generalizations of the Whitney extension and prove that the norm of these operators essentially depends only on the global Lipschitz constant of the domain and not on the size of the micro structures.

1 The Whitney Extension

In the following we recapitulate the main properties of the Whitney extension (cf. [5]). Here and in the following, Ω will always denote an open subset of $\mathbf{R}^d$. In order to introduce the notation of *minimally smooth boundaries* as described in the book of Stein [4], we first have to define special Lipschitz domains as follows.

Definition 1 *Let $\varphi : \mathbf{R}^{d-1} \to \mathbf{R}^1$ be a function which satisfies the Lipschitz condition:*

$$|\varphi(x) - \varphi(y)| \leq M\,|x - y|\,, \qquad \forall x, y \in \mathbf{R}^{d-1}. \tag{1}$$

The smallest constant M for which (1) is true is denoted by C_φ. In terms of this function we can define the special Lipschitz domain. It determines to be the set of points lying above the hypersurface $y = \varphi(x)$ in $\mathbf{R}^d$, i.e.,

$$\Omega = \left\{ x \in \mathbf{R}^d : x_d > \varphi(x_1, x_2, \ldots, x_{d-1}) \right\}.$$

The Lipschitz constant of Ω is defined by $C_\Omega := C_\varphi$.

The definition of a domain with a minimally smooth boundary is given below.

Definition 2 *The boundary $\partial\Omega$ of Ω is said to be minimally smooth, if there exist an $\epsilon > 0$, an integer N, an $M > 0$, and a sequence $\{U_i\}_{i\in\mathbf{N}}$ of open sets so that:*

1. *If $x \in \partial\Omega$, then $B_x(\epsilon) \subset U_i$, for suitable i, where $B_x(\epsilon)$ denote the ball with radius ϵ centred at x.*
2. *No point $x \in \mathbf{R}^d$ is contained in more than N of the U_i's.*
3. *For each i there exists a special Lipschitz domain $\tilde{\Omega}_i$ with $C_{\Omega_i} \leq M$ such that*
$$U_i \cap \Omega = U_i \cap \Phi\left(\tilde{\Omega}_i\right)$$
with a suitable rotation Φ.

We want to investigate extension theorems for Sobolev spaces. We recall the basic definitions and some notations. Let $s \in \mathbf{N_0}$ and Ω be a minimally smooth domain. Let S be the space of all infinitely differentiable functions with compact support. Consider $f \in L^1_{loc}(\Omega)$. Let $\alpha \in \mathbf{N}_0^d$ and

$$D_x^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}}.$$

The αth (weak) derivative of f exists if there is $g_\alpha \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f D_x^\alpha v dx = (-1)^{|\alpha|}\int_\Omega g_\alpha v dx, \qquad \forall v \in S.$$

The αth derivative of f is then denoted by $D^\alpha f := g_\alpha$. The space $H^k(\Omega)$ is defined by

$$H^k(\Omega) := \left\{ f \in L^2(\Omega) \mid \|D^\alpha f\|_{L^2(\Omega)} < \infty, \qquad \forall\, |\alpha| \leq k \right\}.$$

These spaces are Hilbert spaces equipped with the scalar product

$$(u, v)_k := \sum_{|\alpha|\leq k} \int_\omega D^\alpha u D^\alpha v dx.$$

The norm is given by $\|u\|_k := \sqrt{(u,u)_k}$. We will also need the corresponding expressions which only contain the highest derivatives. We define

$$(u,v)_{=k} := \sum_{|\alpha|=k} \int_\Omega D^\alpha u D^\alpha v dx$$

and $|u|_k := \sqrt{(u,u)_{=k}}$. The space $\overset{\circ}{H}^k(\Omega)$ is given by

$$\overset{\circ}{H}^k(\Omega) := \overline{\{f \in H^k(\Omega) \mid D^\alpha f = 0 \text{ in a neighborhood of } \partial\Omega, \text{ for all } |\alpha| \leq k-1\}}.$$

We are now able to formulate the theorem concerning the Whitney extension of functions in Sobolev spaces on minimally smooth domains Ω.

Theorem 3 *Let Ω be a domain with minimally smooth boundary. Then, there exists a linear extension operator $\mathfrak{E}$ mapping functions on Ω to functions on $\mathbf{R}^d$ with the properties*

1. $\mathfrak{E}(f)\,|_\Omega = f$.
2. *$\mathfrak{E}$ maps $H^k(\Omega)$ continuously into $H^k(\mathbf{R}^d)$ for all $k \in \mathbf{N_0}$:*

$$\|\mathfrak{E}u\|_{k,\mathbf{R}^d} \leq C_{ext} \|u\|_{k,\Omega}, \qquad \forall u \in H^k(\Omega),$$

with a constant depending only on k and Ω, i.e., the constants ϵ, N, M defined in Definition (2).

Proof. This theorem is stated and proved in [4, Theorem 5, pp. 181]. ■

Remark 1 *Generalizations to the case of $W^{k,p}-$spaces and the case of fractional derivatives are possible but not of main interest of this investigation.*

The assumptions of Theorem 3 cannot be weakened in general (cf. [4, p.189]). However, we will impose the additional assumption that domain Ω is simply connected and derive the result under weaker assumptions.

We are interested in the case that a domain is of proper shape but might contain micro-scales. As an example consider the annular region:

$$\Omega_\delta := \left\{x \in \mathbf{R}^d \mid \delta < \|x\| < 1\right\}.$$

For δ small, the first condition of the definition of minimally smooth domains causes problem because the radius ϵ of the balls $B_x(\epsilon)$ has to be chosen smaller with decreasing δ. Hence, the bound of the (Whitney) extension operator given above will depend on δ. In the next chapter we will define an extension operator where the norm is independent of the (small) size of the micro-structures.

2 Extension Operators for Domains Containing Small Geometric Details

Let ω, Ω be open bounded subsets of $\mathbf{R}^d$ with minimally smooth boundary and $\omega^c := \Omega \backslash \bar{\omega}$. The domain ω has to be simply connected. Furthermore, we assume that ω^c is simply connected, too, otherwise one has to apply the construction below to each simply connected subdomain. Let $k \in \mathbf{N_0}$ and $u \in H^k(\omega)$. Then u can be extended by the extension operator $\mathfrak{E}$ of Theorem 3 to a function on $\mathbf{R}^d$. The restriction to the domain Ω defines $u^\star$:

$$\begin{aligned} u^\star &\in H^k(\Omega) \\ u^\star &: = \mathfrak{E}(u)\,|_\Omega\,. \end{aligned}$$

From the extension theorem above, we know that

$$\|u^\star\|_{k,\Omega} \leq \|u^\star\|_{k,\mathbf{R}^d} \leq C_\omega \|u\|_{k,\omega}\,. \tag{2}$$

In the following we will define an extension of u which satisfies (2) with the norm on the right hand side above replaced by the corresponding semi norm.

Consider the problem of finding $z \in \overset{\circ}{H}{}^k(\omega^c)$ such that

$$(z, w)_{=k} = -(u^\star, w)_{=k}\,, \qquad \forall w \in \overset{\circ}{H}{}^k(\omega^c)\,. \tag{3}$$

Putting $w = z$, we obtain

$$|z|_k^2 = -(u^\star, z)_{=k} \leq |u^\star|_k\,|z|_k\,.$$

Using the Poincaré-Friedrichs inequality (cf. [1, Theorem 1.7], we obtain that

$$\|v\|_{k,\omega^c} \leq \gamma\,|v|_{k,\omega^c}\,, \qquad \forall v \in \overset{\circ}{H}{}^k(\omega^c)$$

with γ depending only on k and $\operatorname{diam}(\omega^c)$:

$$\gamma = (1 + \operatorname{diam}(\omega_c))^k\,.$$

Using the Lax-Milgram Lemma we get that (3) has a unique solution which satisfies

$$\|z\|_{k,\omega^c} \leq \gamma\,|u^\star|_{k,\omega^c}\,.$$

For $u \in H^s(\Omega)$, we define the extension operator P by

$$Pu = \begin{cases} u & \text{in } \omega, \\ u^\star + z & \text{in } \omega^c \text{ with } z \text{ denoting the solution of (3).} \end{cases} \tag{4}$$

Combining the previous estimates, we obtain

$$\begin{aligned} \|Pu\|_{k,\Omega} &\leq \|Pu\|_{k,\omega} + \|Pu\|_{k,\omega^c} = \|u\|_{k,\omega} + \|u^\star + z\|_{k,\omega^c} \\ &\leq \|u\|_{k,\omega} + \|u^\star\|_{k,\omega^c} + \|z\|_{k,\omega^c} \\ &\leq \|u\|_{k,\omega} + (1+\gamma)\|u^\star\|_{k,\omega^c} \leq \|u\|_{k,\omega} + (1+\gamma)\|u^\star\|_{k,\Omega} \\ &\leq \|u\|_{k,\omega} + C_\omega(1+\gamma)\|u\|_{k,\omega} \\ &= (1 + C_\omega(1+\gamma))\|u\|_{k,\omega}. \end{aligned} \tag{5}$$

In the following, we will show that the estimate above also holds if the norms are replaced by the semi-norms.

Lemma 4 *Let p be a polynomial of degree $k-1$, i.e., $p \in P_{k-1}$.*
Then $Pp = p$.

Proof. Equation (3) can be rewritten as

$$(Pp, w)_{=k} = 0, \qquad \forall w \in \overset{\circ}{H}{}^k(\omega^c).$$

Since $Pp = p$ satisfies the equation above, we only have to show that $z = p - u^\star \in \overset{\circ}{H}{}^k(\omega^c)$. We know that $p - u^\star \in \overset{\circ}{H}{}^k(\Omega)$ and $p - u^\star = 0$ in ω. This implies that $p - u^\star \in \overset{\circ}{H}{}^k(\omega^c)$.

■

Let now $u \in H^k(\omega)$ be given and choose a polynomial of degree $k-1$, i.e., $p \in P_{k-1}$ such that

$$\int_\omega D^\alpha(u-p)\,dx = 0 \qquad \forall |\alpha| \leq k-1.$$

Using [2, Theorem 1.5], we obtain that

$$\|u-p\|_{k,\omega} \leq \hat{C}_\omega |u-p|_{k,\omega} = \hat{C}_\omega |u|_{k,\omega}.$$

Replacing u by $u-p$ in (5) we obtain

$$|Pu|_{k,\Omega} = |Pu-p|_{k,\Omega} \leq \|P(u-p)\|_{k,\Omega} \leq (1 + C_\omega(1+\gamma))\hat{C}_\omega |u|_{k,\omega} \tag{6}$$

We are now ready to define the extension operator for domains containing micro-structures. We use the same technique as introduced in [3, pp. 42] for the extension of H^1-functions. Let ω be a domain which contains small holes, i.e.,

$$\omega = \omega^\star \backslash \bigcup_{j=1}^{q} \omega_j^c, \tag{7}$$

where the domains ω_j^c and $\omega^\star$ are null-homotopic and $\operatorname{diam} \omega_j^c =: \epsilon_j$. Further assumptions will be imposed in the sequel. The extension of $u \in H^k(\omega)$ onto $u \in H^k(\omega^\star)$ is done locally in the following way. We define domains $\omega_j^\star$ having the property that

$$\begin{aligned} &\omega_j^c \subset \omega_j^\star \subset \omega^\star \\ &\omega_j^\star \cap \omega_i^\star = \emptyset \text{ for } i \neq j \\ &\omega_j := \omega_j^\star \backslash \bar{\omega}_j^c \text{ is simply connected} \\ &\omega_0^\star := \omega^\star \backslash \bigcup_{j=1}^{q} \omega_j^\star \end{aligned} \tag{8}$$

We further have to assume that the holes are properly separated from each other such that

$$C_1 \epsilon_j \leq \operatorname{dist}\left(\partial \omega_j^\star, \partial \omega_j^c\right) \leq C_2 \epsilon_j \tag{9}$$

is true for $1 \leq j \leq q$. We state that these assumptions can be weakened but then the extension operators have to be modified according to the situation under consideration. Here, we focus our interest to the question what happens if the size of the hole goes to zero. Let $u \in H^k(\omega)$. This implies that $u_j := u \left|_{\omega_j} \in H^k(\omega_j)\right.$. In the following we will extend u_j to a function on $\omega_j^\star$ and estimate the norm $u_j^\star$ by the norm of u_j. The global extension then is defined by

$$u^\star(x) = \begin{cases} u(x), & x \in \omega, \\ u_j^\star(x) & x \in \omega_j^c. \end{cases}$$

We need the following assumption on the subdomains ω_j.

Assumption 5 *There exist domains $\omega_j^\star$ satisfying (8) and local coordinate systems such that the mapping (in local coordinates) $\chi(x) := \frac{x}{\epsilon}$ has the property that the image $\hat{\omega}_j := \chi(\omega_j)$ has a minimally smooth boundary.*

The extension now goes as follows. Let $\hat{\omega}_j^\star := \chi\left(\omega_j^\star\right)$ and $\hat{\omega}_j^c := \chi\left(\omega_j^c\right)$. Let $u \in H^k(\omega)$. Hence, $\hat{u}_j := u_j \circ \chi^{-1} \in H^k(\hat{\omega}_j)$. Let $\hat{u}_j^\star := P\hat{u}_j$ with P defined in (4). The pull back

$$u^\star(x) := Eu := \begin{cases} \hat{u}_j^\star \circ \chi(x) & x \in \omega_j^c, \\ u(x) & x \in \omega, \end{cases}$$

defines the extension of u onto $\omega^\star$. In the following we will estimate the norm of $u^\star$. The semi norms of $u_j^\star$ can be expressed in terms of the semi norm of $\hat{u}_j^\star$. We first consider the semi-norm of order k. We obtain

$$|u^\star|_{k,\omega_j^\star}^2 = \sum_{|\alpha|=k} \int_{\omega_j^\star} (D_x^\alpha u^\star)^2\, dx = \epsilon^{d-2k} \sum_{|\alpha|=k} \int_{\hat{\omega}_j^\star} (D_{\hat{x}}^\alpha \hat{u}^\star)^2\, d\hat{x} = \epsilon^{d-2k} |\hat{u}^\star|_{k,\hat{\omega}_j^\star}^2 .$$

Using (6) for the domains $\hat{\omega}_j \subset \hat{\omega}_j^\star$, we obtain

$$|\hat{u}^\star|_{k,\hat{\omega}_j^\star}^2 \leq \hat{C}\, |\hat{u}|_{k,\hat{\omega}_j}^2$$

with a constant $\hat{C}$ dependent only on k and $\hat{\omega}_j$. Transforming the semi norm on $\hat{\omega}_j$ back onto ω_j we obtain

$$|u^\star|_{k,\omega_j^\star}^2 \leq \epsilon^{d-2k} |\hat{u}^\star|_{k,\hat{\omega}_j^\star}^2 \leq \hat{C}\epsilon^{d-2k} |\hat{u}|_{k,\hat{\omega}_j}^2 \leq \hat{C}\, |u|_{k,\omega_j}^2 .$$

Next, we investigate the L^2-norm.

$$|u^\star|_{0,\omega_j^\star}^2 = \int_{\omega_j^\star} (u^\star)^2\, dx = \epsilon^d \int_{\hat{\omega}_j^\star} (\hat{u}^\star)^2\, d\hat{x} = \epsilon^d\, |\hat{u}^\star|_{0,\hat{\omega}_j^\star}^2 .$$

Using $|\hat{u}^\star|_{0,\hat{\omega}_j^\star}^2 \leq \|\hat{u}^\star\|_{k,\hat{\omega}_j^\star}^2$ and (5), we get

$$|u^\star|_{0,\omega_j^\star}^2 \leq \hat{C}\epsilon^d \, \|\hat{u}\|_{k,\hat{\omega}_j}^2 \leq \hat{C}\, \|u\|_{k,\omega_j}^2 .$$

An interpolation argument yields the same results for all intermediate semi-norms.

Summarizing the above considerations we have proven the following theorem.

Theorem 6 *Let ω a domain with holes satisfying conditions (7, 9) and Assumption 5. Let $\omega^\star$ be the extended domain defined by (7). Then, there exists* *an extension operator E which maps the space $H^k(\omega)$ onto the space $H^k(\omega^\star)$ for all $k \in \mathbf{N}_0$:*

$$\|Eu\|_{k,\omega^\star} \le C \|u\|_{k,\omega}, \qquad \forall u \in H^k(\omega)$$

where C depends only on $k, \operatorname{diam}(\omega^\star)$ and the constant in the estimate (6) applied to the normalized domains $\hat{\omega}_j$ defined above. This means, the constant C is independent of the diameter of the holes.

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