A Formulation of the Panel Clustering Method for Three Dimensional Elastostatics

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1 Introduction

Although the Boundary Element Method (BEM) enjoys the boundary only discretization, a serious computational difficulty arises due to its dense matrix formulation, particularly for large scale three dimensional problems arising in engineering. This is because the method requires $O(N^2)$ memory and $O(N^2 \sim N^3)$ computation using the conventional approach, where N is the number of unknowns.

The situation is even worse for the three dimensional elastostatic problem, where the number of unknowns is three times that of the potential problem, since the unknowns at each node is a vector instead of a scalar.

Rokhlin[1] and Hackbusch and Nowak[2] independently proposed the multipole method and the panel clustering method, respectively, in order to overcome this difficulty. The main idea is to approximate the far field using multipole or polynomial expansions around a centre of a cluster of panels or boundary elements, thus reducing the $O(N^2)$ dense matrix vector multiplication to a $O(N(\log N)^{d+2})$ sparse matrix vector multiplication for each iteration of the iterative linear solver, where d is dimension of the space.

Yamada and Hayami[3] proposed a multipole boundary element method for two dimensional elastostatics. In this paper, we will present a formulation of the panel clustering method for the three dimensional elastostatic problem. The formulation is also applicable to the multipole method.

2 Boundary Element Formulation of 3-D Elastostatics

The boundary integral equation for the three dimensional (linear, isotropic) elastostatic problem is given by

$$c_{lk}(\mathbf{x})u_k(\mathbf{x}) + \int_\Gamma p^*_{lk}(\mathbf{x},\mathbf{y})u_k(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \int_\Gamma u^*_{lk}(\mathbf{x},\mathbf{y})p_k(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}), \tag{1}$$

where Γ is the boundary of the domain under consideration, u_k, p_k are the displacement and traction, respectively, $c_{lk}(\mathbf{x}) = \frac{1}{2}\delta_{lk}$ when Γ is smooth at $\mathbf{x}$, and we have neglected the body force term. $u^*_{lk}(\mathbf{x},\mathbf{y})$ is the fundamental solution corresponding to the k-th component of the displacement at $\mathbf{x}$ due to a unit point load in the l-direction at $\mathbf{y}$, which is usually given by

$$u^*_{lk} = \frac{1}{16\pi\mu(1-\nu)r}\{(3-4\nu)\delta_{lk} + r_{,l}r_{,k}\}, \tag{2}$$

where μ is the shear modulus, ν is the Poisson's ratio, $r \equiv \mid \mathbf{y} - \mathbf{x} \mid$ and $r_{,k} \equiv \partial r/\partial y_k$, $\mathbf{y} = (y_1, y_2, y_3)^T$.

For the sake of the expansions for the panel clustering method, it will prove convenient (cf. [4]) to rewrite equation (2) using

$$r_{,lk} \equiv \frac{\partial^2 r}{\partial y_l \partial y_k} = (r_{,l})_{,k} = \left(\frac{y_l - x_l}{r}\right)_{,k} = \frac{\delta_{lk} - r_{,l}r_{,k}}{r}$$

or

$$r_{,l}r_{,k} = \delta_{lk} - r(r_{,lk})$$

as

$$u^*_{lk} = \frac{1}{4\pi\mu}\left\{\delta_{lk}\frac{1}{r} - \frac{1}{4(1-\nu)}r_{,lk}\right\}. \tag{3}$$

Since the strain corresponding to u^*_{lk} is

$$\epsilon^*_{lkj} = \frac{1}{2}(u^*_{lk,j} + u^*_{lj,k}),$$

Hooke's law:

$$\sigma^*_{lkj} = \lambda\delta_{kj}\epsilon^*_{lmm} + 2\mu\epsilon^*_{lkj},$$

where $\lambda = \frac{2\mu\nu}{1-2\nu}$ is the volumetric component, gives

$$\sigma^*_{lkj} = \frac{1}{4\pi}\left\{\frac{2\nu}{1-2\nu}\delta_{kj}\left(\frac{1}{r}\right)_{,l} + \delta_{lk}\left(\frac{1}{r}\right)_{,j} + \delta_{lj}\left(\frac{1}{r}\right)_{,k}\right\} - \frac{1}{8\pi(1-\nu)}\left(\frac{\nu}{1-2\nu}\delta_{kj}r_{,lmm} + r_{,lkj}\right).$$

Then, the traction $p^*_{lk} = \sigma^*_{lkj}n_j$ corresponding to u^*_{lk}, where n_j is the unit outward normal vector at $\mathbf{y} \in \Gamma$, is given by

$$p^*_{lk} = \frac{1}{4\pi}\left\{\frac{2\nu}{1-2\nu}n_k\left(\frac{1}{r}\right)_{,l} + \delta_{lk}n_j\left(\frac{1}{r}\right)_{,j} + n_l\left(\frac{1}{r}\right)_{,k}\right\} - \frac{1}{8\pi(1-\nu)}\left(\frac{\nu}{1-2\nu}n_k r_{,lmm} + n_j r_{,lkj}\right). \tag{4}$$

3 Polynomial Expansions for the Panel Clustering Method

The panel clustering method[2, 4] makes use of polynomial expansions of the integral kernels of equation (1) for cluster of elements which are sufficiently far from the observation point $\mathbf{x}$, in order to reduce the amount of computation and required memory. This can be achieved by first considering Taylor expansions of r and $\frac{1}{r}$ around the centre of a cluster $\mathbf{y}^0 = (y_1^0, y_2^0, y_3^0)^T$, e.g.

$$r(\mathbf{y}) = r(\mathbf{y}^0 + \mathbf{h}) = \sum_{s=0}^{m-1}\sum_{\nu_1=0}^{s}\sum_{\nu_2=0}^{s-\nu_1}\frac{1}{\nu_1!\nu_2!\nu_3!}h_1^{\nu_1}h_2^{\nu_2}h_3^{\nu_3}\frac{\partial^s r(\mathbf{y}^0)}{\partial y_1^{\nu_1}\partial y_2^{\nu_2}\partial y_3^{\nu_3}} + R_m(\mathbf{y}^0, \mathbf{h}), \tag{5}$$

where $\mathbf{h} = (h_1, h_2, h_3)^T$, $s = \nu_1 + \nu_2 + \nu_3$.

Next, the expansion of equation (5) is expressed in terms of polynomial expansions in y_i instead of $h_i = y_i - y_i^0$.

Similar expansions are formed for $\frac{1}{r}$. Then, the expansions for the spatial derivatives of r and $\frac{1}{r}$ appearing in equations (3) and (4) can be easily derived using the expansions obtained for r and $\frac{1}{r}$.

We conclude by pointing out that equations (3) and (4) are also convenient for deriving multipole expansions using spherical harmonics in the multipole method.

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